TWIN CIRCLES IN A TRIANGLE AND ITS APPLICATION TO AN ARBELOS

Hiroshi Okumura, Masayuki Watanabe

ABSTRACT

In this paper, we will show the existence of new pair of congruent circles in a triangle and use it to generalize the results in (OKUMURA, 2008) and (OKUMURA, 2009) which showed the existence of non-Archimedean twins of an arbelos. The new results can be used also in Informatics education by providing a learning material for computer programming.

Keywords: arbelos, twins circles, isogonal conjugate.

1. Twin Circles in a Triangle

For a triangle ABC, let δ_A and δ'_A denote the circles passing through the vertex A and touching the side BC at B and C respectively. The circles δ_B, δ'_B, δ_C and δ'_C are defined similarly. Then, we have

Theorem 1. Three circles δ_A, δ_B and δ_C meet in a single point.

Let S denote the intersection point of the circles. Three circles δ'_A, δ'_B and δ'_C also meet in a single point, which we denote by T. Then,

Theorem 2. The point T is an isogonal conjugate of S.

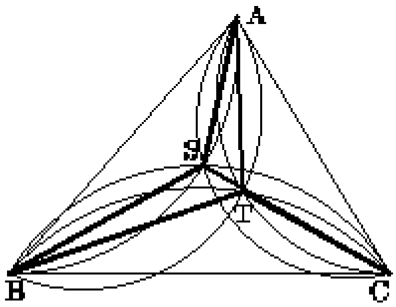

Fig. 1.

Now, we show the existence of new pairs of congruent circles which exist in a triangle ABC.

Theorem 3. Let C_{A} be a circle in a triangle ABC touching the sides AC and BC and touching the circle δ_{A} externally. Let C'_{A} be a circle touching the sides AB and BC and touching the circle δ'_{A} externally. Then the circles C_{A} and C'_{A} are congruent with each other and their radii are given by $2ar/(a+b+c)$, where r is an inradius of the triangle ABC. The radius is expressed also by

$$\frac{h}{2}\left(\frac{\cos \mathrm{B}+\cos \mathrm{C}+\cos \mathrm{A}-1}{\sin \mathrm{B}\cdot \sin \mathrm{C}}\right)^2,$$

where h is the distance from the vertex A to the line BC.

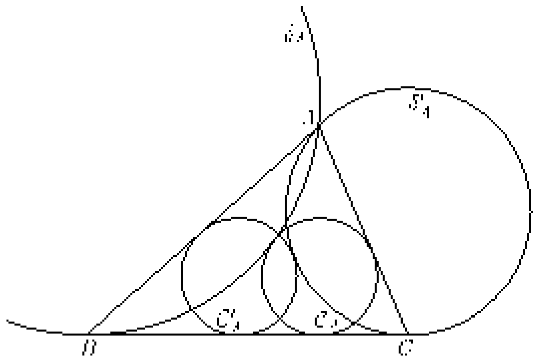

Fig. 2.

Similarly, define C_{B} and C_{C} for the vertices B and C respectively. Then

Corollary It holds that $r_{\mathrm{A}}+r_{\mathrm{B}}+r_{\mathrm{C}}=2r$, where r_{A}, r_{B} and r_{C} are the radii of C_{A}, C_{B} and C_{C}.

2. An application to an arbelos

First, we consider an usual arbelos formed by three circles α, β and γ, where α and β touches externally at a point O, and γ touches the circles α and β internally. We denote by A the tangent point of α and γ, and by B the tangent point of β and γ. Let J be a point different from O such that the line OJ touches both α and β at O, and let E and F be points on the lines AJ and BJ respectively such that OEJF form a parallelogram. In (OKUMURA, 2008) and (OKUMURA, 2009), we showed

Theorem 4. Let α_{J} and β_{J} be circles touching the line OJ at O and passing through the points E and F respectively. Then the inscribed circle in a curvilinear triangle formed by the segments EJ, OJ and the circle α_{J} is congruent to the inscribed circle in a curvilinear triangle formed by the segments FJ, OJ and the circle β_{J}. If we denote the distance between the points O and J by $2d$, the radii of the congruent circles are given by

$$\frac{\left(\left(\sqrt{a^2+d^2}-d\right)\left(\sqrt{b^2+d^2}-d\right)-ab\right)^2}{ab(a+b)},$$

where a and b are the radii of α and β respectively.

Thorem4 is proved again by using Theorem3, and the similar results hold when the circles α and β touches internally. In this case, we consider that the circle γ is the one touching one of α and β internally and the other externally. Define O, A, B, J, E, F, α_{J}, β_{J} and d as above.

Theorem 5. There exists a pair of congruent circles in the parallelogram OEJF as in the previous Theorem, and the radii of the congruent circles are given by

$$\frac{\left(\left(\sqrt{a^2+d^2}-d\right)\left(\sqrt{b^2+d^2}+d\right)-ab\right)^2}{ab(b-a)} \quad \text{if } \alpha \text{ lies inside } \beta,$$

$$\frac{\left(\left(\sqrt{a^2+d^2}+d\right)\left(\sqrt{b^2+d^2}-d\right)-ab\right)^2}{ab(a-b)} \quad \text{if } \beta \text{ lies inside } \alpha.$$

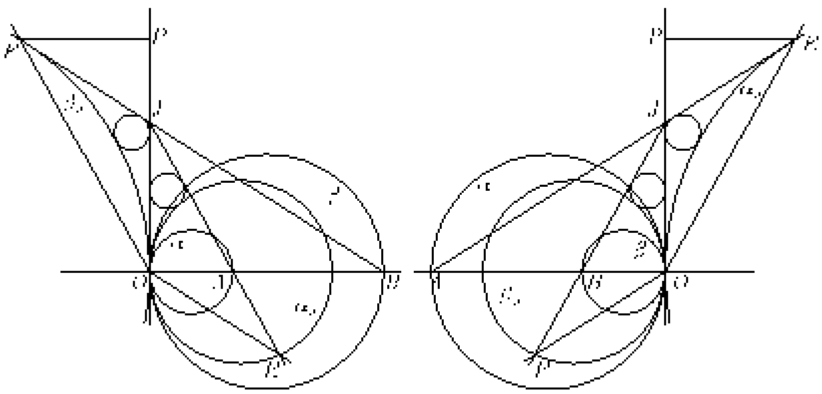

Fig. 3.

We have the similar results even when one of α and β is a line, which is also the consequence form Theorem3. When β is a line, it holds that A=E, $\alpha = \alpha_J$ and γ is a tangent line of α at A. When α is a line, it holds that B=F, $\beta = \beta_J$ and γ is a tangent line of β at B. Then,

Theorem 6. There exists a pair of congruent circles in the parallelogram OEJF as in the previous Theorem, and the radii of the congruent circles are given by

$$\left(\sqrt{a^2+d^2}-d-a\right)^2 \Big/ a \quad \text{if } \beta \text{ is a line,}$$
$$\left(\sqrt{a^2+d^2}-d-b\right)^2 \Big/ b \quad \text{if } \alpha \text{ is a line.}$$

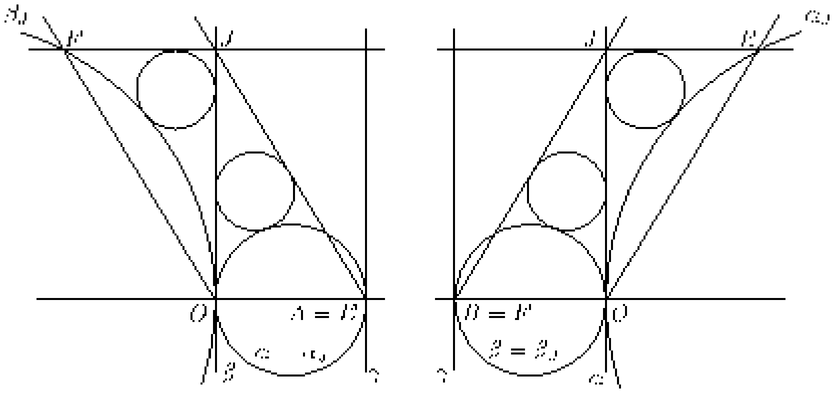

Fig. 4.

Note that the results in Theorem6 is the limit of the results in Theorem5 when $b \to \infty$ or $a \to \infty$.

3. An animation of twin circles of an arbelos programmed by Java

The above theorems provide a new learning material for a computer programming in a Informatics Education. We gave the following assignment to a student in a informatics course of a university as a part of his graduation thesis.

Consider an usual arbelos formed by α, β and γ. For <u>*any*</u> *point J not lying on the line AB, take points E and F on the lines JA and JB respectively such that OEJF make a parallelogram, and let α_J and β_J be circles touching the line OJ at O and passing through the points E and F respectively. Make a program to describe twin circles in a parallelogram OEJF inscribed to curvilinear triangles*

OEJ and OFJ, and to move them in accordance with the change of the radii of α *and* β.

It was realized by JAVA programming under the following consideration.

Let l *a line perpendicular to OJ passing through O, A' be the intersection point of* l *and the line JA, B' be the intersection point of* l *and the line JB, and let* α' *and* β' *be the circles with diameters OA' and OB' respectively. Note that the points A' and B' are on different sides of the line OJ when the point J is outside* α *and* β, *and are on the same side when the point J is inside* α *or* β. *When the point J is on* β, *the line JB is parallel to* l, *so we consider B' to be a point at infinity and* β' *to be a line. Similarly, we consider A' to be a point at infinity and* α' *to be a line when the point J is on* α. *Then the required twin circles can be considered the congruent circles with respect to* α' *and* β' *in Theorem4 if the point J is outside* α *and* β, *those in Theorem5 if the point J is inside* α *or* β, *and those in Theorem6 if the point J lies on* α *or* β. *Then the results in Theorem4, 5 and 6 gave us the sufficient information to describe the circles.*

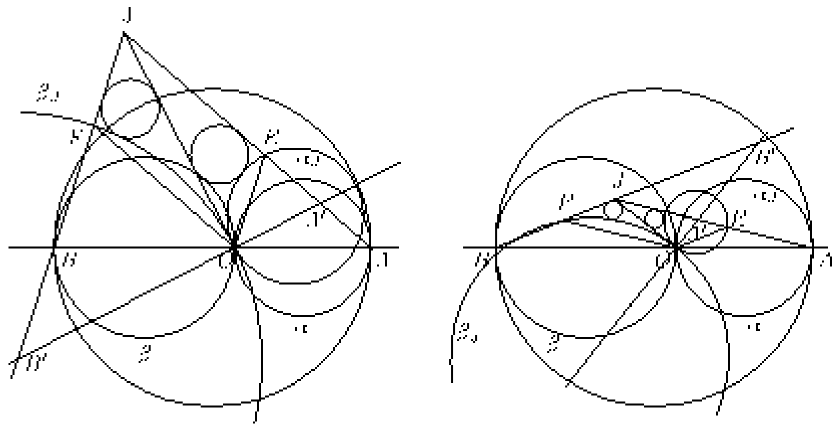

Fig. 5.

The student could develop his ability of programming through this assignment, and conversely, to make the program helped him understand the geometric background of the theorems better. It can be concluded that the learning of arbelos, especially, to find new properties, and a programming to realize them in computer graphics motivate students to study both geometry and programming synergistically, and make their understanding deeper.

REFERENCES

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Hiroshi Okumura
Department of Life Science and Information
Maebashi Institute of Technology
460-1Kamisadori Maebashi Gunma 371-0816, Japan
ok@maebashi-it.ac.jp:

Masayuki Watanabe
Department of Integrated Design Engineering
Maebashi Institute of Technology
460-1Kamisadori Maebashi Gunma 371-0816, Japan
mwatanabe@maebashi-it.ac.jp: