THE SECRETS OF THE M-CONFIGURATION OF A TRIANGLE

Svetoslav Jordanov Bilchev
Assoc. Prof. Dr., Department of Algebra and Geometry
Faculty of Natural Sciences and Education
University of Rousse, Rousse, Bulgaria

ABSTRACT

*This paper considers an easy construction of points A_a, B_a, C_a on the sides of a triangle ABC, such that the configuration M of the path $BC_aA_aB_aC$ consists of 4 segments of equal lengths. With the help of the M - configuration consisting of the three figures M of a triangle, the new geometric **"M–Conf–T"** transformation is constructed.*

1. INTRODUCTION

In an arbitrary triangle *ABC* we consider the point A_a on the line *BC*, the point B_a on the half line *CA*, and the point C_a on the half line *BA* such that $BC_a = C_aA_a = A_aB_a = B_aC$. We denote the figure $BC_aA_aB_aC$ as **configuration M_a**, because the figure looks like the letter M when triangle *ABC* is acute-angled (see the Figure 1.1). Figure 1.2 gives the case of the figure $BC_aA_aB_aC$ when the triangle *ABC* is obtuse-angled. Analogically, we can construct **configurations M_b and M_c** as well (see [4]). The three **configurations M_a, M_b, M_c** form the so named **M-configuration** of triangle *ABC* (see red M, blue M and green M on the Figure 2).

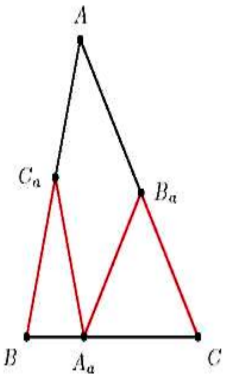

Figure1.1

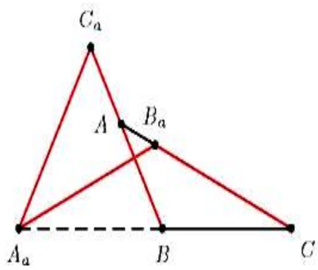

Figure1.2

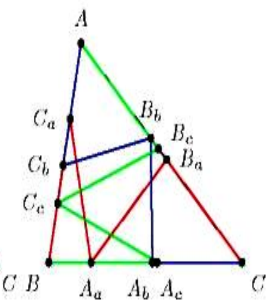

Figure 2

Corollary 1. *The lines AA_a, BB_a, CC_a have a common point with homogeneous barycentric coordinates*

(1) $$\left(\frac{1}{\cos A},\frac{1}{\cos B},\frac{1}{\cos C}\right),$$

where A, B, C are the angles of the triangle ABC.

Proof. Let $q_a = BC_a = C_aA_a = A_aB_a = B_aC$. Then the segments $BA_a = 2q_a \cos B$ and $A_aC = 2q_a \cos C$. Now it is possible to find the ratio $BA_a : A_aC = \cos B : \cos C$. Analogically : $CB_b : B_bA = \cos C : \cos A$ and $AC_c : C_cB = \cos A : \cos B$. Hence, by the Ceva's theorem it follows that the lines AA_a, BB_a, CC_a have a common point with homogeneous barycentric coordinates (1). This point appears in [3] as $X92$.

Remark. Since $2q_a \cos B + 2q_a \cos C = a = 2R \sin A$, where R is the circumradius of the triangle ABC, then

(2) $$q_a = \frac{a}{2(\cos B + \cos C)} = \frac{R\sin A}{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}} = R\frac{\cos\frac{A}{2}}{\cos\frac{B-C}{2}}.$$

It is not so difficult to compute the absolute barycentric coordinates of A_a, B_a, C_a by using of q_a:

(3) $$A_a = \frac{2q_a}{a}(\cos C.B + \cos B.C),$$
$$B_a = \frac{1}{b}\left(q_a.A + (b - q_a).C\right),$$
$$C_a = \frac{1}{c}\left(q_a.A + (c - q_a).B\right).$$

2. CONSTRUCTION OF THE CONFIGURATION M_a

Corollary 2. *Let A' be the intersection point of the bisector of angle A with the circumcircle of the triangle ABC. Then:*

(a) *A_a is the intersection of BC with the parallel to AA' through the orthocenter H.*

(b) *B_a is the intersection of CA with the parallel to CA' through the circumcenter O.*

(c) *C_a is the intersection of BA with the parallel to BA' through the circumcenter O.*

Proof. **(a)** The straight line through the point $Aa = (0 : \cos C : \cos B)$ and the point $H = \left(\frac{a}{\cos A} : \frac{b}{\cos B} : \frac{c}{\cos C}\right)$ has the following equation:

$$\begin{vmatrix} x & y & z \\ 0 & \cos C & \cos B \\ \frac{a}{\cos A} & \frac{b}{\cos B} & \frac{c}{\cos C} \end{vmatrix} = 0,$$

from where we obtain (see Figure 3) :

$$HA_a \rightarrow \quad -(b-c)x\cos A + a(y\cos B - z\cos C) = 0.$$

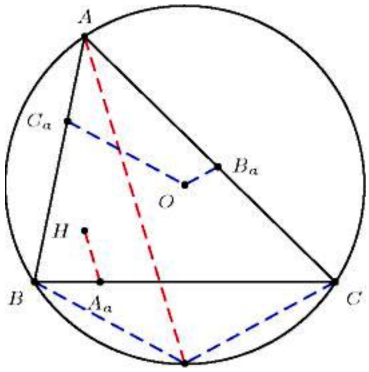

Figure 3

The line HA_a has the infinite point

$$[-a(\cos B+\cos C) : a\cos C-(b-c)\cos A : (b-c)\cos A + a\cos B] =$$
$$= [-a(\cos B+\cos C) : b(1-\cos A) : c(1-\cos A)].$$

The above infinite point is the same point as the infinite point $[-(b+c) : b : c]$ lying on the bisector of the angle A, i.e. on the line through the point A and the incentre I of the triangle *ABC*.

(b) Let M be the midpoint of *BC*, and *Y*, *Z* are the pedals of B_a, C_a on *BC*, respectively (see Figure 4). We have

$$OM = \frac{a}{2}\cot gA = q_a(\cos B+\cos C)\cot A,$$
$$C_aZ = q_a \sin B,$$
$$MZ = \frac{a}{2} - q_a\cos B = q_a(\cos B + \cos C) - q_a\cos B = q_a\cos C,$$

from where

$$\frac{C_aZ-OM}{MZ}=\frac{\sin B-(\cos B+\cos C)\cot A}{\cos C}=$$

$$=\frac{\sin B\sin A-(\cos B+\cos C)\cos A}{\cos C\sin A}=$$

$$=\frac{-\cos(A+B)-\cos C\cos A}{\cos C\sin A}=\frac{\cos C(1-\cos A)}{\cos C\sin A}=\frac{1-\cos A}{\sin A}=\tan\frac{A}{2}\ .$$

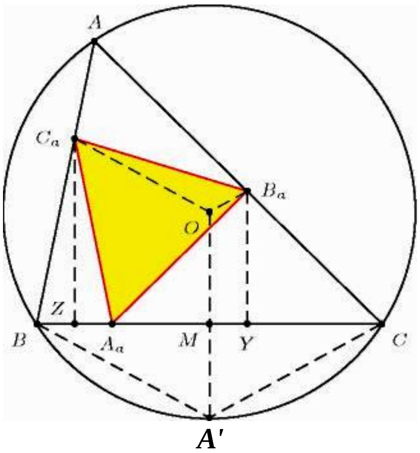

Figure 4

The obtained result means that the acute angle between the lines C_aO and BC is equal to the angle $\frac{A}{2}$, i.e. the line C_aO is parallel to the line BA'.
Analogically, we prove that the line B_aO is parallel to the line CA'.

3. SOME CALCULATIONS AND THE "M-CONF-T" TRANSFORMATION.

From Figure 2 and (2) by using of the Cosine Law, it is very easy to compute the segments BA_a and CA_a in terms of the sides a,b,c of the triangle ABC:

(4) $$BA_a=\frac{a\cos B}{\cos B+\cos C}=\frac{ab\left(c^2+a^2-b^2\right)}{4(s-b)(s-c)(b+c)}\ ,$$

(5) $$CA_a=\frac{ac\left(a^2+b^2-c^2\right)}{4(s-b)(s-c)(b+c)}\ .$$

Analogically,

(6) $$AC_c = \frac{ca\left(b^2+c^2-a^2\right)}{4(s-a)(s-b)(a+b)}\ , \quad BC_c = \frac{cb\left(c^2+a^2-b^2\right)}{4(s-a)(s-b)(a+b)}\ ,$$

(7) $$AB_b = \frac{ba\left(b^2+c^2-a^2\right)}{4(s-c)(s-a)(c+a)}\ , \quad CB_b = \frac{bc\left(a^2+b^2-c^2\right)}{4(s-c)(s-a)(c+a)}\ .$$

Now by using of (6), (7) and the Sine Law, we can obtain formulae for the area of the triangle AB_bC_c :

(8) $$F_{AB_bC_c} = \frac{a^2\left(b^2+c^2-a^2\right)^2}{16r(s-a)(a+b)(a+c)}\ .$$

Further:

(9) $$F_{A_aBC_c} = \frac{b^2\left(c^2+a^2-b^2\right)^2}{16r(s-b)(b+c)(b+a)}\ ,$$

(10) $$F_{A_aB_bC} = \frac{c^2\left(a^2+b^2-c^2\right)^2}{16r(s-c)(c+a)(c+b)}\ .$$

We are interested about the area of the triangle $A_aB_bC_c$. So, from (8), (9) and (10) we get:

(11) $$F_{A_aB_bC_c} = F_{ABC} - \frac{1}{16r\Pi(a+b)}\sum\frac{a^2\left(b^2+c^2-a^2\right)^2(b+c)}{s-a}$$

where the sum $\sum$ and the product Π are cyclic.

From the triangle AB_bC_c and (6), (7) it follows:

(12) $$B_bC_c = \frac{a\left|b^2+c^2-a^2\right|}{4(s-a)}\cdot\left[\frac{4}{(a+b)(a+c)} + \left(\frac{(b-c)\left(b^2+c^2-a^2\right)}{2(s-b)(s-c)(a+b)(a+c)}\right)^2\right]^{\frac{1}{2}}$$

Hence, from the well-known Weitzenböck's inequality for the sides and the area of an arbitrary triangle (see [5], [6]):

$$A_aB_b^{\,2}+B_bC_c^{\,2}+C_cA_a^{\,2}\geq 4\sqrt{3}.F_{A_aB_bC_c}$$

and from (11) and (12) we get a new geometric inequality for any triangle:

$$(13)\quad \sum\frac{a^2\left(b^2+c^2-a^2\right)^2}{16(s-a)^2}\cdot\left[\frac{4}{(a+b)(a+c)}+\left(\frac{(b-c)\left(b^2+c^2-a^2\right)}{2(s-b)(s-c)(a+b)(a+c)}\right)^2\right]+$$

$$+\frac{\sqrt{3}}{4r\Pi(a+b)}.\sum\frac{a^2\left(b^2+c^2-a^2\right)^2(b+c)}{s-a}\geq 4\sqrt{3}F\ ,$$

where $F=F_{ABC}$.

Of course, it is possible to transform the inequality (13) to an easier form. But here it was important to demonstrate that the so constructed M-configuration can be associated with a new geometric transformation, named **M-Conf-T.** The main equalities of the transformation **M-Conf-T** are (11) and (12).

REFERENCES

[1] Hatzipolakis A. P., Hyacinthos message 5321, April 30, 2002.

[2] Kimberling C., Triangle centers and central triangles, *Congressus Numerantium*, 129,1998, 1 - 285.

[3] Kimberling C., *Encyclopedia of Triangle Centers,* May 23, 2003 edition, (http://faculty.evansville.edu/ck6/encyclopedia/ETC.html).

[4] Myakishev A., The M-Configuration of a Triangle, Forum Geometricorum, Volume 3, 2003, 135 – 144.

[5] Bilchev Svetoslav Jordanov, Multiple Connection Tasks or Methods for Solving Multiple Problems, Proceedings of the International Research Workshop of the Israel Science Foundation, Multiple Solution Connecting Tasks, 20 - 21 - 22 February, 2008, Haifa, Israel, CET - The Center for Educational Technology, Tel Aviv, 2008, 13 – 24

[6] Bilchev Svetoslav Jordanov, Activating the Interest of Gifted Students to Work in Mathematics by Obtaining New Results on High Level, Proceedings of the 5th International Conference on Creativity in Mathematics and the Education of Gifted Students, Projects and Ideas, Editor Roza Leikin, Haifa, Israel, February 24 – 28, 2008, CET - The Center for Educational Technology, Tel Aviv, 2008, 277 – 288, ISBN 965 – 354 – 006 – 8.

РЕЗЮМЕ

В работата се разглежда една конструкция от три точки A_a, B_b, C_c *върху страните на триъгълника* ABC*, такава че конфигурацията М от начупената линия* $BC_aA_aB_aC$ *съдържа 4 равни отсечки. С помощта на М - конфигурацията, съставена от трите М - фигури за триъгълника, се конструира нова геометрична* ***"M–Conf–T"*** *трансформация.*