# THE IMPACT OF DIFFERENT REPRESENTATIONS IN SOLVING NON-PROPORTIONAL STEREOMETRY PROBLEMS

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## ABSTRACT

*The aim of the present study was to investigate which type of representation of three-dimension figures influences students in the 6th grade of the elementary school to use proportional reasoning in volume tasks. Thus, the use of four different representations (informational picture, solid net, decorative picture and verbal representation) is examined, in an effort to identify differentiation among students' responses. The results of the research revealed that students use proportional reasoning even in situations where linearity is not applicable. The extent to which students used proportional reasoning appropriately was dependent on the type of representation. Specifically, the problems accompanied by an informative picture or nets enhanced the illusion of linearity, in contrast with decorative pictures which inhibited the application of linear relations.*

## INTRODUCTION

Several researches indicated that students of different ages have a strong tendency to apply linear or proportional models, even in situations where linearity is not applicable (De Bock Verschaffel & Janssens, 1998; Van Dooren, De Bock, Hessels, Janssens & Verschaffel, 2005; Modestou & Gagatsis, 2004; Modestou, Gagatsis & Pitta-Pantazi, 2004). According to De Bock and his colleagues (1998), this tension has been defined in several ways, as illusion of linearity, linear trap, linear obstacle or linear misconception.

The phenomenon of illusion of linearity appears in tasks of area and volume, where learners tend to apply the linear model in non-proportional situations, as in the case of enlargement or reduction of a figure's size (De Bock et al, 1998; 2002b). Indeed, recent studies (De Bock et al, 1998; Modestou & Gagatsis, 2007; 2006) revealed a deep-rooted tendency among students to improperly apply the linear model in word problems which involve the concept of length in combination with the concepts of areas and volumes, respectively. While previous studies reported the extent to which different characteristics of the task affect students' improper use of linear reasoning, it is unclear which types of representations strengthen this phenomenon. Thus, the aim of the present study was to investigate whether different representations of rectangular parallelepiped impact on the use of proportionality in volume tasks. Specifically, verbal representation, net, informational picture and decorative picture are exploited.

## THEORETICAL FRAMEWORK

### *Proportional or non-proportional reasoning*

Proportional relations are widely applicable and useful for the understanding numerous everyday life situations and furthermore many mathematical problems (Van Dooren et al., 2004). Both from a psychological and a mathematical point of view, the idea of linearity comes first, since linear functions appear immediately in human's mind (Rouche, 1989).

However, the wide application of linearity at numerous occasions in school mathematics may lead students– and even adults– in a tendency to see and apply the linear model 'everywhere'. The "illusion of linearity" is a recurrent phenomenon that seems to be universal and resistant to a variety forms of support aimed at overcoming it (De Bock et al., 2003). Proportions appear to be deeply rooted in students' intuitive knowledge and are used in a spontaneous and even unconscious way, which makes the linear approach quite natural, unquestionable and to certain extend inaccessible for introspection or reflection (De Bock et al., 2002a). Therefore, as Verschaffel and his colleagues (2000) illustrate, it takes a radical conceptual shift to move from the uncritical application of this simple and neat mathematical formula to the modeling perspective that takes into account the reality of the situation being described.

### *Using the linear model in the concepts of area and volume*

According to Freudenthal (1983), "Linearity is a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as if it were linear" (p. 267). The tendency to overgeneralise the linear model is repeatedly mentioned in mathematics education's literature and in recent years it has been in the focus of systematic empirical research. For example, the abovementioned phenomenon had been studied in elementary arithmetic (Van Dooren et al., 2005), probability (Van Dooren et al., 2003), algebra and calculus (Esteley, Villareal & Alagia, 2004). Moreover, geometry is a particular mathematical area where the illusion of linearity is widely applicable. Concretely, students of different ages believe in a linear relation between the lengths, areas and volumes of similarly enlarged geometrical figures, thinking that if a figure is enlarged *k* times, the area and volume of that figure are enlarged *k* times as well (De Bock, Verschaffel, & Janssens, 1998; De Bock et al., 2002a; 2002b; Freudenthal, 1983; Modestou et al, 2004).

De Bock, Verschaffel and Janssens (2002b) pointed out, that students' experience from real life situations with enlarging and reducing operations do not necessarily make them aware of the different growth rates of length, area and volume. Therefore, students strongly tend to see the relations between length and area or between length and volume as linear instead of quadratic and cubic, respectively. As a consequence, students apply linear relations instead of square or cube relations to determine the area or volume of an enlarged or a reduced figure.

A considerable number of research studies (De Bock et al., 1998; 2002a; 2003; Van Dooren et al, 2005; Modestou & Gagatsis, 2004; Modestou et al, 2004) examines and tries to overcome students' tendency to deal with non-proportional tasks concerning area and volume as if they were proportional. In particular, De Bock and his partners (1998) revealed a strong tendency among 12-13 year old students to apply proportional reasoning in problem situations concerning area. In an effort to overcome students' linear obstacle De Bock and his colleagues increased the authenticity of the problem context (De Bock et al., 2003) and used visual and metacognitive scaffolds (De Bock et al., 2002b), without leading students to the desirable result. Specifically, in the case of visual support at the non-proportional problems, students relied on formal strategies such as using formulas instead of their own or a given drawing (De Bock et al., 1998). Students in some cases discarded the results given from mathematical formulas for measuring the area and volume of a figure, in favour of the application of the linear model (Modestou & Gagatsis, 2007).

*Different representations*

There is a strong support in the mathematics education community that students reinforce their mathematical conceptual understanding by experiencing multiple representations (e.g., Janvier, 1987; Sierpinska, 1992). The term "representations" is interpreted as the tools used for mathematical ideas' exhibition such as tables, graphs, and equations (Confrey & Smith, 1991). A representation is defined as any configuration of characters, images and concrete objects that can symbolize or "represent" something else (Kaput 1985; Goldin, 1998; DeWindt-King & Goldin, 2003).

Carney and Levin (2002) proposed five functions that pictures serve in text processing– decorative, representational, organizational, interpretational and transformational. Thus, the studies presented in this section suggest four functions of pictures in mathematics problem solving: (a) decorative, (b) representational, (c) organizational and (d) informational. Decorative pictures do not give any actual information concerning the solution of the problem. Representational pictures represent the whole or a part of the content of the problem, while organizational pictures provide directions for drawing or written work that support the solution procedure. Finally, informational pictures provide information that is essential for the solution of the problem.

## METHOD

Data were collected through an anonymous test that was administered to 87 sixth-grade students of elementary schools in Cyprus, who were randomly selected.

The test was consisted of four tasks asking students to measure the volume of a rectangular parallelepiped and then to estimate the new volume while one or two dimensions of the solid were increased. Concretely, the test included four tasks (see Appendix), each of them presented in different representation: informational picture (task 1), solid net (task 2), decorative picture (task 3) and verbal representation (task 4). Figure 1 shows the encoding of the variables.

*Vi*: Volume in the task accompanied with informative picture
*Vip*: Volume while the dimension/s of the figure presented in task Vi is/are increased
*SVipr*: Use of proportional reasoning in the task Vip
*Vn*: Volume in the task accompanied with solid net
*Vnp*: Volume while the dimension/s of the figure presented in task Vn is/are increased
*SVnpr*: Use of proportional reasoning in the task Vnp
*Vd*: Volume in the task accompanied with decorative picture
*Vdp*: Volume while the dimension/s of the figure presented in task Vd is/are increased
*SVdpr*: Use of proportional reasoning in the task Vdp
*Vv*: Volume in the task accompanied with verbal representation
*Vvp:* Volume while the dimension/s of the figure presented in task Vv is/are increased

*Figure 1: Variables*

Students' answers were coded as 0, 0.5 and 1. More specific, correct responses in volume measurement tasks were marked with 1 and incorrect responses with 0. However, 0.5 was used in the cases that only the solution appeared, without any explanation for the process followed is giving.

Data were analyzed using the statistical package CHIC which produced three diagrams: the similarity diagram, the implication graph and the hierarchical tree (Bodin, Coutourier, & Gras, 2000). The former diagram represents groups of variables which are based on the similarity of students' responses to these variables. The implication graph shows implications of the form A→B, meaning that success in question A implies success in question B. Finally, the hierarchical tree shows the implication between sets of variables. In this study we use only the first two diagrams.

## RESULTS

The basic aim of the study was to examine whether different types of representation affect sixth grades students' performance in solving volume tasks. Table 1 shows students' performance on the four tasks of the questionnaire. The highest percentage in measuring the volume of the rectangular parallelepiped (71.84%) is observed when the task was accompanied with decorative picture and with verbal representation (70.69%), while the lowest percentage (37.93%) refers to the task with informational picture (37.93%). Although, is observed that only in the case of decorative picture, students used less proportional reasoning (14.94%) and gave the right answer (48.28%). It is important to mention that the task with verbal representation was the only task of the test, which requires proportional reasoning for the right answer and for this reason the percentage of using proportional reasoning is considerable (47,7%).

*Table 1: Students' performance on problem tasks.*

| | **Measurement of Volume** | **Use of non-proportional reasoning (Right answer)** | **Use of proportional reasoning** |
|---|---|---|---|
| Informational picture | 37.93% | 15.52% | 37.93% |
| Net | 63.22% | 23.56% | 33.33% |
| Decorative picture | 71.84% | 48.28% | 14.94% |
| Verbal representation | 70.69% | - | 47.7% |

To examine the relationships between students' performance to solve volume measurement tasks were given in different representations and their tendency to use proportional reasoning we employed the statistical implicative analysis. The analysis gave us the similarity diagram (see figure 2), which allowed for the grouping of the tasks and the statements based on the homogeneity by which they were handled by students.

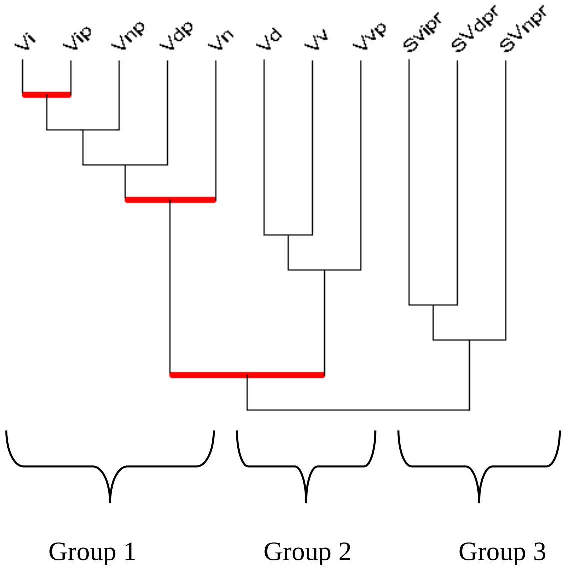



*Figure 2: Similarity diagram of students' responses to the four-part of test.*

The similarity diagram showed a formation of three linked groups of tasks. The first group (Vi, Vip, Vnp, Vdp, Vn) include the right answers in non linearity tasks and also the volume measuring tasks with informational picture and net (Vi and Vn). These tasks were the most difficult for students. The second group (Vd, Vv, Vvp) was comprised by the volume measuring task with decorative picture and the tasks with verbal description. The third group (SVipr, SVdpr, SVnpr) consisted of the use of proportional reasoning to solve the volume tasks with informational picture, decorative picture and net.

The implication graph in figure 3 shows significant implicative relations between the tasks of the test. Specifically, it suggests that success in non proportional problems with informational picture (Vip) and net (Vnp) implies success in measuring the volume of rectangular parallelepiped with informational picture (Vi), find the right answer in non proportional task with decorative picture (Vdp) and in measuring the volume of the solids with net (Vn), with verbal description (Vv) and with decorative picture (Vd).

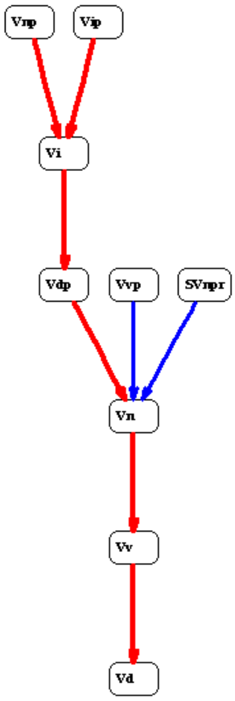



*Figure 3: Implication graph illustrating relations among the students' answers*

As figure 3 shows, the easiest task of the test was to find the volume of the solid with decorative picture in contrast with the non proportional tasks with net and informational picture which considered as the most difficult. Also, it is observed that the use of proportional reasoning appears only in the case of the net (SVnpr) and implies the measuring the volume of the solid of the net.

## DISCUSSION

The aim of the present study was to examine the influence of four different representations (informational picture, net, decorative picture, verbal representation) on the application of proportional relations in the cases of non proportional tasks. More concretely, the effect of verbal representation, net, informational picture and decorative picture was examined, in the measurement of the volume of rectangular parallelepiped when while one or more dimensions of the solid are increased.

With regard to the results of the study, the students confronted difficulties in stereometry tasks since low percentages of correct answers appeared. It can be assumed that students' difficulties in stereometry tasks are due to the mathematics curriculum and textbooks used in Cyprus. Specifically, in mathematics teaching, the three-dimensional figures are presented in a two-dimensional form that inhibits students' conceptual understanding (Parzysz, 1988). According to Gutierrez (1992), the abovementioned teaching approach encourages students to create in their minds many separate images and then to try to connect them.

Moreover, the present study confirmed students' tendency to handle non proportional tasks as proportional, verifying similar findings (De Bock et al., 1998; De Bock, et al, 2002a; Modestou, Gagatsis & Pitta-Pantazi, 2004). De Bock and his colleagues (2002b), in an attempt to analyze students' cognitive process leading to contradicting applications of the proportional model, reported that this process is influenced to a large extent by students' mathematical perceptions. Indeed, students' perceptions that all numerical relations between numbers have a proportional form, their intuitive knowledge, as well as their limited geometrical knowledge strengthen the mistaken application of the proportional model (De Bock, Verschaffel & Janssens, 2002a; 2002b).

With respect to the use of different representations, the results of the present study revealed that they affect the degree to which proportional relations are applied during volume's measurement. In particular, problems accompanied by informative picture or nets enhanced the illusion of linearity. This finding is conflicting due to the fact that both informative picture as nets include information useful for problem solving, that was expected to strengthen students' conceptual understanding and consequently to avoid the erroneous use of the proportional model.

On the other hand, tasks accompanied by decorative representation inhibit the application of linear relations. At this end, problems' complicated verbal description in combination with students' failure to use suitably knowledge of real world for the solution of the problem leads to the use of proportional reasoning in verbal problems. A similar outcome would be expected also in the case of problems accompanied by a decorative picture since the picture does not provide any additional information to the problem and so the students grounded on the verbal form of the problem. Nevertheless, decorative pictures helped students to conceive better the data of problem, the situation that was described and consequently deduced the application of the proportional model.

Summing up, it can be deduced that the results of the present study offer important instructive practices. Since students apply proportional relations even in cases that are not applicable, it is essential to develop a suitable teaching environment capable of reducing students' tendency to overuse proportional relations. This instructive intervention has to affect students' conceptual comprehension for the proportional reasoning in the specific socio-cultural frame where teaching takes place (De Bock et al., 2003).

## REFERENCES

Bodin, A., Coutourier, R., & Gras, R. CHIC : Classification Hierarchique Implicative et Cohesitive-Version sous Windows – CHIC 1.2. Rennes: Association pour le Recherche en Didactique des Mathematiques, (2000).

Carney, R. N., & Levin, J. R. Pictorial illustrations still improve students' learning from text. *Educational Psychology Review, 14* (1), 101-120, (2002).

Confrey, J., & Smith, E. A framework for functions: Prototypes, multiple representations and transformations. In R. G. Underhill (Ed.), *Proceedings of the 13th annual meeting of the North American Chapter of The International Group for the Psychology of Mathematics Education* (pp. 57-63). Blacksburg: Virginia Polytechnic Institute and State University, (1991).

De Bock, D., Verschaffel, L., & Janssens, D. The predominance of the linear model in secondary school students' solution of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics, 35*, 65-85, (1998).

De Bock, D., Van Dooren, W., Verschaffel, L., & Janssens, D. Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*, 311-314, (2002a).

De Bock, D., Verschaffel, L., & Janssens, D. The effects of different problem presentations and formulations on the illusion of linearity in secondary school students. *Mathematical Thinking and Learning*, *4*(1), 65-89, (2002b).

De Bock, D., Verschaffel, K., Janssens, D., Van Dooren., W., & Claes, K. Do realistic contexts and graphical representations always have a beneficial impact on students; performance? Negative evidence from a study on modeling non-linear geometry problems. *Learning and Instruction, 13* (4), 441-463, (2003).

DeWindt-King, A.M., & Goldin, G. A. Children's visual imagery: Aspects of cognitive representation in solving problems with fractions. *Mediterranean Journal for Research in Mathematics Education, 2* (1), 1-42, (2003).

Esteley, C., Villarreal, M., & Alagia, H. Extending linear models to non-linear contexts: An in-depth study about two university students' mathematical productions. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 343–350). Bergen, Norway, (2004).

Freudenthal, H. *Mathematics as an educational task.* Dordrecht: Reidel, (1983).

Gutierrez, A. Exploring the links between Van Hiele levels and 3-dimensional geometry. *Structural Topology, 18*, 31-48, (1992).

Goldin, G.A. Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior*, 17(2), 137-165, (1998).

Hart, K. M. *Children's understanding of mathematics*. London: John Murray, (1981).

Janvier, C. Representation and understanding: The notion of functions as an example. In C. Janvier (ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 67-72). Hillsdale, N.J.: Lawrence Erlbaum Associates, (1987).

Kaput, J.J. Representation and problem solving: methodological issues related to modelling. In E.A. Silver (Ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* (pp. 381-398). Hillsdale, NJ: Erlbaum, (1985).

Klein, A. S., Beishuisen, M., & Treffers, A. The empty number line in Dutch second grade: Realistic versus gradual program design. *Journal for Research in Mathematics Education, 29*(4), 443-464, (1998).

Modestou, M. and Gagatsis, A. Students' improper proportional reasoning: A result of the epistemological obstacle of "linearity". *Educational Psychology 27*(1), 75-92 (2007).

Modestou, M. and Gagatsis, A. Can the spontaneous and uncritical application of the linear model be questioned?, in J. Novotna, H. Moraova, M. Kratka and N. Stehlikova (eds.). *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, Prague, Czech Republic, 169-176, (2006).

Modestou, M., & Gagatsis, A. Linear or not linear? Students' improper proportional reasoning. In G. Makrides, A. Gagatsis & K. Nicolaou (Eds.), *Proceedings of the CASTME International and CASTME Europe Conference (21-33).* Nicosia: Cyprus, (2004).

Modestou, M., Gagatsis, A. & Pitta-Pantazi, D. Students' improper proportional reasoning: the case of area and volume of rectangular figures. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 3*, 345-352. Bergen: Norway, (2004).

Parzysz, B. Knowing vs seeing. Problems of the plane representation of space geometry figures, *Educational Studies in Mathematics* 19, 79-92, (1988).

Rouche, N. Prouver. Amener à l'évidence ou contrôler des implications?, *Actes du 7ème Colloque inter-IREM Epistémologie et Histoire des Mathématiques*, Besançon, France, 8–38, (1989).

Sierpinska, A. On understanding the notion of function. In E. Dubinsky, & G. Harel (Ed.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25-58). USA: Mathematical Association of America, (1992).

Van Dooren, W., De Bock, D., Hessels, A., Janssens D. & Verschaffel, L. Ups and downs' is students' over-reliance on proportional methods. *Cognition and Instruction, 23*(1), 57-86, (2005).

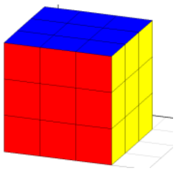
Van Dooren, W., De Bock, D. Hessels, A. Janssens, D. Verschaffel, L. Remedying Secondary School Students' Illusion of Linearity: A Teaching Experiment Aiming at Conceptual Change. *Learning and Instruction, 14* (5), 485-501, (2004).

Van Dooren, W., De Bock, D., Depaepe, F., Janssens, D., & Verschaffel, L. The illusion of linearity: Expanding the evidence towards probabilistic reasoning. *Educational Studies in Mathematics, 53,* 113–138, (2003).

Verschaffel, L., Greer, B. & de Corte, E. *Making Sense of Word Problems*. Lisse: Swets & Zeitlinger. (2000).

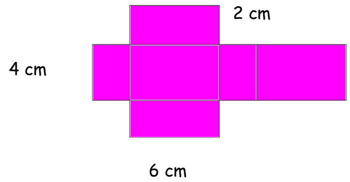
## APPENDIX

1. a) Calculate the volume of the cube.



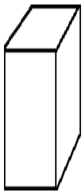
   b) Calculate the volume of the cube, if all its dimensions are tripled.

2. a) Calculate the volume of the box when the shape folds.




   b) Calculate the volume of the box if any two of the three dimensions are doubled.

3. a) The small cornflakes pack has dimensions 10 cm height, 5 cm length and 4 cm width. Calculate the volume of the small pack?



   b) The big cornflakes pack has double dimensions in comparison with the small pack. Calculate the volume of the big pack?

4. a) Michael's swimming-pool is 2 m deep, 3 m wide and 5 m long. How many liters of water Michael needs in order to fill the swimming-pool?

   b) If the length of the swimming-pool is doubled, how many times the volume of water will increase in order to fill the swimming-pool?