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FUZZY NATURAL TRANSFORM TO SOLVE VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce the fuzzy Natural transform (FNT) method to solve the fuzzy Volterra integro-differential equation (VFIDE) of the first kind with a convolution kernel under Hukuhara differentiability. The FNT method is a simple and reliable approach to solving such equations analytically. Finally, an illustrative example with crisp convolution kernels is given to show the ability of the proposed method.

1. Introduction

In recent years, the area of fuzzy integro-differential equations (FIDE) has developed a lot and plays a key role in the field of engineering. Furthermore, FIDEs in a fuzzy setting are a natural way to model the ambiguity of dynamic systems. Consequently, different scientific fields, such as physics, geography, medicine, and biology, attach great importance to the solution of different FIDE. Solutions to these equations can be utilized in different engineering problems. The working concept is based on the main presumption that the fuzzy events are to be measured by fuzzy numbers. The technique to realize this

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concept has been independently introduced by Dubois and Prade [3] for fuzzy domains of the real line and by Yager [25] in a more general framework. One of the most important fields of fuzzy theory is fuzzy differential equations [2, 4, 11], fuzzy integral equations [9] and fuzzy integro-differential equations [6, 5, 8, 16].

Integral transforms constitute fundamental tools in operational calculus. They are mathematical operators that have been used widely in solving many practical problems in applied mathematics, physics, and engineering. The precursor of integral transforms is the Fourier transform, which is used to express functions in a finite interval. There are a number of works on the theories and applications of integral transforms, some of which are Laplace, Mellin and Hankel transforms [13, 15, 17, 20, 19]. Subsequently, the concept of integral transforms was expanded to remove the necessity of finite intervals.

The Natural transform evolved from the Fourier integral, and it converges into the Laplace transform and the Sumudu transform given a unit value to each transform parameters, respectively. The definition of the Natural transform and the study of their properties and applications were first made by Khan and Khan [12]. Several research works in connection with Natural transform properties and applications are published in [1].

The subject of this paper is to apply the fuzzy Natural transform for solving the convolution type fuzzy Volterra integro-differential equation

$$\int_0^x k_1(x-s) \odot w(s))ds \oplus \int_0^x k_2(x-s) \odot w^{(m)}(s)ds = g(x), \tag{1.1}$$

with the initial conditions

$$w^{(i)}(0) = b_i, \quad i = 0, 1, 2, ..., m-1, \tag{1.2}$$

where $k_2(x-s) \neq 0$, $k_1, \ k_2 : [a,b] \times [a,b] \to \mathbb{R}$, are continuous functions and $g, w : [a,b] \to E^1$ are continuous fuzzy-number valued functions and b_i, $(i = 0, 1, ..., m-1)$ are fuzzy numbers. The set E^1 is the set of all fuzzy numbers.

By using fuzzy Natural transform method, we directly convert fuzzy Volterra integro-differential equation into an algebraic equation. Solving this algebraic equation and applying fuzzy inverse Natural transform, we obtain the exact solution. This method is illustrated by an example.

2. Preliminaries

In this section, we give some basic definitions and theorems for fuzzy numbers, fuzzy-valued functions, and derivatives of fuzzy-valued functions.

Definition 2.1 ([7])**.** A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ that satisfies the following properties

(1) u is upper semi-continuous on $\mathbb{R}$,
(2) $u(x) = 0$ outside of some interval $[c, d]$,
(3) there are real numbers a and b with $c \leq a \leq b \leq d$, such that u increasing on $[c, a]$, decreasing on $[b, d]$ and $u(x) = 1$ for each $x \in [a, b]$,
(4) u is fuzzy convex set i. e. that is $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.

The set of all fuzzy numbers is denoted by E^1. Any real number $a \in \mathbb{R}$ can be interpreted as a fuzzy number $\tilde{a} = \chi_{[a]}$, where

$$\chi_{[a]}(x) = \begin{cases} a, & x = a \\ 0, & x \neq a. \end{cases}$$

Therefore $\mathbb{R} \subset E^1$.

Definition 2.2 ([10])**.** Let $u \in E^1$ and $r \in (0, 1]$. The set of the r-level u is the crisp set $\{x \in \mathbb{R} : u(x) \geq r\}$.

Denote $[u]^r$ the set of the r-level of fuzzy number u.

It can be concluded that any set of r-levels is bounded and closed interval and denoted by $[\underline{u}(r), \overline{u}(r)]$ for all $r \in [0, 1]$, where the functions $\underline{u}, \overline{u} : [0, 1] \to \mathbb{R}$ are the lower and upper bound of $[u]^r$, respectively.

Definition 2.3 ([10])**.** A fuzzy number in parametric form is given as an order pair of the form $u = (\underline{u}(r), \overline{u}(r))$, where $0 \leq r \leq 1$ satisfying the following conditions

(1) $\underline{u}(r)$ is a bounded left continuous monotonic increasing function in $[0, 1]$,
(2) $\overline{u}(r)$ is a bounded left continuous monotonic decreasing function in $[0, 1]$,
(3) $\underline{u}(r) \leq \overline{u}(r)$.

For an arbitrary fuzzy number $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and an arbitrary crisp number $k \in \mathbb{R}$ the addition and the scalar multiplication are defined by $[u \oplus v]^r = [u]^r + [v]^r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$ and

$$[k \odot u]^r = k.[u]^r = \begin{cases} [k\underline{u}(r), k\overline{u}(r)], & k \geq 0 \\ [k\overline{u}(r), k\underline{u}(r)], & k < 0. \end{cases}$$

The neutral element with respect to $\oplus$ in E^1, denoted by $\tilde{0} = \chi_{[0]}$. The algebraic properties of addition and scalar multiplication of fuzzy numbers are given in [23].

Definition 2.4 ([18])**.** Let $x, y \in E^1$ and there exists an element $z \in E^1$, such that $x = y \oplus z$. Then z is called the H-difference of x and y and is given by $x \ominus y$.

We use the Hausdorff metric as the distance between fuzzy numbers.

Definition 2.5 ([7]). For arbitrary fuzzy numbers

$$u = (\underline{u}(r), \overline{u}(r))$$

and

$$v = (\underline{v}(r), \overline{v}(r)),$$

the quantity

$$d(u, v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$$

is the distance between u, v.

The space $(E^1,\ d)$ is a complete metric space.

Let $I \subset \mathbb{R}$. Then for any fuzzy-number-valued function $f : I \to E^1$ we can define the functions $\underline{f}(.,r)$, $\overline{f}(.,r) : I \subset \mathbb{R} \to \mathbb{R}$, by $\underline{f}(t,r) = \underline{f(t,r)}$, $\overline{f}(t,r) = \overline{f(t,r)}$ for each $t \in I$, for each $r \in [0,1]$. These functions are called the left and right $r-$level functions of f.

Theorem 2.6 ([24]). *Let for all $r \in [0,1]$ the functions $\underline{f}(x,r)$ and $\overline{f}(x,r)$ be Riemann-integrable in $[0,b]$ for every $b \geq 0$ and there are constants $\underline{M}(r) > 0$ and $\overline{M}(r) > 0$, such that*

$$\int_0^b |\underline{f}(x,r)|dx \leq \underline{M}(r), \quad \int_0^b |\overline{f}(x,r)|dx \leq \overline{M}(r), \quad for\ every\ b \geq 0.$$

Then the function $f(x)$ is improper fuzzy Riemann-integrable on $[0,\infty)$ and

$$(FR)\int_0^\infty f(x)dx = \left(\int_0^\infty \underline{f}(x,r)dxdy,\ \int_0^\infty \overline{f}(x,r)dx\right).$$

For fuzzy valued functions $w = w(x)$ we define the H-derivatives as given in [21, 22].

Definition 2.7. Let $w : (a,b) \to E^1$. Then w is said to be a strongly generalized H-differentiable function at $x_0 \in (a,b)$, if there exists an element $w'(x_0) \in E^1$ such that

(1) for all $h > 0$ sufficiently small, the H-differences

$$w(x_0 + h) \ominus w(x_0), \quad w(x_0) \ominus w(x_0 - h),$$

exist and the following limits hold

$$\lim_{h\to 0} \frac{w(x_0 + h) \ominus w(x_0)}{h} = \lim_{h\to 0} \frac{w(x_0) \ominus w(x_0 - h)}{h} = w'(x_0);$$

(2) for all $h > 0$ sufficiently small, the H-differences

$$w(x_0) \ominus w(x_0 + h), \;\; w(x_0 - h) \ominus w(x_0),$$

exist and the following limits hold

$$\lim_{h \to 0} \frac{w(x_0) \ominus w(x_0 + h)}{-h} = \lim_{h \to 0} \frac{w(x_0 - h) \ominus w(x_0)}{-h} = w'(x_0).$$

The first type of differentiability as in Definition 2.7 is referred to as (i)-differentiable, while the second type is referred to as (ii)-differentiable.

Theorem 2.8 ([2]). *Let $w : (a, b) \to E^1$ be a continuous fuzzy-valued function and $w(x) = (\underline{w}(x, r), \overline{w}(x, r))$ for all $r \in [0, 1]$. Then*

(1) if $w(x)$ is (i)-differentiable, then $\underline{w}(x, r)$ and $\overline{w}(x, r)$ are differentiable and

$$w'(x) = (\underline{w}'(x, r), \;\; \overline{w}'(x, r))\,; \tag{2.1}$$

(2) if $w(x)$ is (ii)-differentiable, then $\underline{w}(x, r)$ and $\overline{w}(x, r)$ are differentiable and

$$w'(x) = (\overline{w}'(x, r), \;\; \underline{w}'(x, r))\,. \tag{2.2}$$

For fuzzy valued functions $w = w(x)$ we define the m-th order H-derivatives as given in [8].

Definition 2.9. Let $w : (a, b) \to E^1$, then w is said to be a strongly generalized H-differentiable function of the m-th order at $x_0 \in (a, b)$, if there exists an element

$$w^{(m)}(x_0) \in E^1$$

such that

(1) for all $h > 0$ sufficiently small, the H-differences $w^{(m-1)}(x_0 + h) \ominus w^{(m-1)}(x_0)$, $\;w^{(m-1)}(x_0) \ominus w^{(m-1)}(x_0 - h)$, exist and the following limits hold

$$\lim_{h \to 0} \frac{w^{(m-1)}(x_0 + h) \ominus w^{(m-1)}(x_0)}{h} = \lim_{h \to 0} \frac{w^{(m-1)}(x_0) \ominus w^{(m-1)}(x_0 - h)}{h} = w^{(m)}(x_0), \tag{2.3}$$

(2) for all $h > 0$ sufficiently small, the H-differences

$$w^{(m-1)}(x_0) \ominus w^{(m-1)}(x_0 + h), \;\; w^{(m-1)}(x_0 - h) \ominus w^{(m-1)}(x_0),$$

exist, and the following limits hold

$$\lim_{h\to 0} \frac{w^{(m-1)}(x_0) \ominus w^{(m-1)}(x_0+h)}{-h} = \lim_{h\to 0} \frac{w^{(m-1)}(x_0-h) \ominus w^{(m-1)}(x_0)}{-h} = w^{(m)}(x_0). \quad (2.4)$$

Similarly to Theorem 2.8 we have the following results from m-th order strongly generalized H-differentiability of fuzzy-valued function.

Theorem 2.10 ([2]). *Let $w(x)$, $w'(x)$, ..., $w^{(m-1)}(x)$ be differentiable fuzzy-valued functions and $w(x) = (\underline{w}(x,r), \overline{w}(x,r))$ for all $r \in [0,1]$. Then*

(1) if $w(x)$ and $w'(x)$, ..., $w^{(m-1)}(x)$ are (i)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the m-th order and

$$w^{(m)}(x) = \left(\underline{w}^{(m)}(x,r),\ \overline{w}^{(m)}(x,r)\right);$$

(2) if $w(x)$ and $w'(x)$, ..., $w^{(m-1)}(x)$ are (ii)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the m-th order and

$$w^{(m)}(x) = \left(\overline{w}^{(m)}(x,r),\ \underline{w}^{(m)}(x,r)\right) \ \ if \ \ m \text{ is even},$$

$$w^{(m)}(x) = \left(\underline{w}^{(m)}(x,r),\ \overline{w}^{(m)}(x,r)\right) \ \ if \ m \text{ is odd};$$

(3) if $w(x)$ is (i)-differentiable and $w'(x)$, ..., $w^{(m-1)}(x)$ are (ii)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the m-th order and

$$w^{(m)}(x) = \left(\overline{w}^{(m)}(x,r),\ \underline{w}^{(m)}(x,r)\right) \ \ if \ \ m \text{ is even},$$

$$w^{(m)}(x) = \left(\underline{w}^{(m)}(x,r),\ \overline{w}^{(m)}(x,r)\right) \ \ if \ \ m \text{ is odd};$$

(4) if $w(x)$ is (ii)-differentiable and $w'(x)$, ..., $w^{(m-1)}(x)$ are (i)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the m-th order and

$$w^{(m)}(x) = \left(\overline{w}^{(m)}(x,r),\ \underline{w}^{(m)}(x,r)\right).$$

3. Fuzzy Natural transform

Definition 3.1. Let $w : \mathbb{R} \to E^1$ be a continuous fuzzy-valued function and let the function $e^{-sx} \odot w(ux)$ is improper fuzzy Riemann-integrable on $[0, \infty)$, then

$$(FR)\int_0^\infty e^{-sx} \odot w(ux)dx$$

is called fuzzy Natural transform (FNT) and is denoted by

$$W(s;u) = N[w(x)] = (FR)\int_0^\infty e^{-sx} \odot w(ux)dx, \tag{3.1}$$

for $s > 0$, and $u \in [-\sigma_1, \sigma_2]$, where the variable u is used to factor the variable x in the argument of the fuzzy-valued function, and $\sigma_1,\ \sigma_2 > 0$.

The parametric form of FNT is as follows

$$N[w(x)] = (n[\underline{w}(x,r)], n[\overline{w}(x,r)]), \tag{3.2}$$

where

$$\begin{aligned} n[\underline{w}(x,r)] &= \underline{W}(u,r) = \int_0^\infty e^{-sx}\underline{w}(ux,r)dx, \\ n[\overline{w}(x,r)] &= \overline{W}(u,r) = \int_0^\infty e^{-sx}\overline{w}(ux,r)dx. \end{aligned} \tag{3.3}$$

Equation (3.1) we can rewrite in the form

$$W(s;u) = N[w(x)] = \frac{1}{u}(FR)\int_0^\infty e^{-\frac{sx}{u}} \odot w(x)dx. \tag{3.4}$$

Definition 3.2 ([6])**.** The fuzzy inverse Natural transform can be written as the formula

$$N^{-1}\left[W[s;u]\right] = w(x) = \left(n^{-1}[\underline{W}[s;u,r)]], n^{-1}[\overline{W}[s;(u,r)]]\right),$$

where

$$n^{-1}[\underline{W}[s;u,r)]] = \frac{1}{2\pi\imath}\int_{\gamma-\imath\infty}^{\gamma+\imath\infty} e^{\frac{sx}{u}}\underline{W}[s;u,r)]du,$$

$$n^{-1}[\overline{W}[s;u,r)]] = \frac{1}{2\pi\imath}\int_{\gamma-\imath\infty}^{\gamma+\imath\infty} e^{\frac{sx}{u}}\overline{W}[s;u,r)]du.$$

For all $r \in [0,1]$ the functions $\underline{W}[(s,p);(u,v,r)]$ and $\overline{W}[(s,p);(u,v,r)]$ must be analytic functions for all u and v in the region defined by the inequalities $Reu \geq \gamma$ and $Rev \geq \delta$, where γ and δ are real constants to be chosen in a suitable way.

In [14] classical Natural transform is applied to some special functions.

(1) Let $w(x) = 1$ for $x > 0$, then

$$n[w(x)] = \frac{1}{s}.$$

(2) Let $w(x) = x^m$, where m is a positive integer, then

$$n[w(x)] = m!\frac{u^m}{s^{m+1}}.$$

(3) Let $w(x) = e^{ax}$, where a is any constant, then

$$n[w(x)] = \frac{1}{s - au}.$$

(4) Let $w(x,y) = cos(ax)$, where a is any constant, then

$$n[cos(ax)] = \frac{(s - au)}{(s^2 + a^2u^2)}.$$

(5) Let $w(x) = sin(ax)$, where a is any constant, then

$$n[sin(ax)] = \frac{(s + au)}{(s^2 + a^2u^2)}.$$

Definition 3.3. If $k(x)$ and $w(x)$ are fuzzy Riemann integrable functions, then the fuzzy convolution of $k(x)$ and $w(x)$ is given by

$$(k * w)(x) = (FR)\int_0^x k(x - s) \odot w(s)ds \tag{3.5}$$

and the symbol $*$ denotes the fuzzy convolution.

Theorem 3.4. *Let $k : [0,\infty) \to \mathbb{R}$ and $w(x)$ be fuzzy functions. Then the FNT of the fuzzy convolution k and w, is given by*

$$N[(k * w)(x)] = un[k(x)] \odot N[w(x)]. \tag{3.6}$$

Proof. By applying the definition of FNT and fuzzy convolution, one obtains the following result

$$N[(k * w)(x)] = \frac{1}{u}(FR)\int_0^\infty e^{-\frac{sx}{u}} \odot (k * w)(x)dx.$$

The fuzzy convolution k and w can be presented parametrically as follows

$$(k * w)(x) = ((k * \underline{w})(x,r), (k * \overline{w})(x,r)) .$$

Then from Theorem 3.1 in [14], we have

$$n[(k * \underline{w})(x,r)] = \frac{1}{u} \int_0^\infty e^{-\frac{sx}{u}} \left(\int_0^x k(x-\alpha)\underline{w}(\alpha,r) d\alpha \right) dx.$$

Let $\eta = x - \alpha$ and extend the upper bound of integrals to $x \to \infty$ it implies that

$$n[(k * \underline{w})(x,r)] = \frac{1}{u} \int_0^\infty e^{-\frac{s\alpha}{u}} \underline{w}(\alpha,r) d\alpha \int_{-\eta}^\infty e^{-\frac{s\eta}{u}} k(\eta) d\eta.$$

The function $k(x)$ has zero value for $x < 0$, therefore

$$n[(k * \underline{w})(x,r)] = \frac{1}{u} \int_0^\infty e^{-\frac{s\alpha}{u}} \underline{w}(\alpha,r) d\alpha \int_0^\infty e^{-\frac{s\eta}{u}} k(\eta) d\eta.$$

Then

$$n[(k * \underline{w})(x,r)] = un[k(x)]n[\underline{w}(x,r)].$$

The proof is complete. □

We introduce results of FNT for fuzzy derivatives.

Theorem 3.5. *Let $w : \mathbb{R} \to E^1$ be a continuous fuzzy-valued function. The functions $e^{-sx} \odot w(ux)$, $e^{-sx} \odot w^{(m)}(ux)$ are improper fuzzy Riemann-integrable on $[0, \infty)$. Then*

$$N\left[w^{(m)}(x)\right] = \frac{d^m}{dx^m}[N[w(x)]], \tag{3.7}$$

where $n \in \mathbb{N}$.

Proof. Let the function $w(x)$ be (i)-differentiable. From the definition of FNT, we have

$$\begin{aligned} N\left[w^{(m)}(x)\right] &= (FR)\int_0^\infty e^{-sx} \odot w^{(m)}(ux)dx \\ &= \left(\int_0^\infty e^{-sx}\underline{w}^{(m)}(ux)dxt,\ \int_0^\infty e^{-sx}\overline{w}^{(m)}(ux)dx\right) \\ &= \frac{d^m}{dx^m}\left[\int_0^\infty e^{-sx}\underline{w}(ux)dx,\ \int_0^\infty e^{-sx}\overline{w}(ux)dx\right] \\ &= \frac{d^m}{dx^m}N[w(x)]. \end{aligned}$$

The proof is complete. □

Theorem 3.6. *Let $w : \mathbb{R} \to E^1$ and the functions $e^{-sx} \odot w(ux)$, $e^{-sx} \odot w^{(n)}(ux)$ be improper fuzzy Riemann-integrable on $[0, \infty)$ and the functions $w(x)$, $w'(x)$, ..., $w^{(m-1)}(x)$ be differentiable fuzzy-valued functions and $w(x) = (\underline{w}(x, r), \overline{w}(x, r))$ for all $r \in [0, 1]$. Then*

(1) if $w(x)$ and $w'(x)$, ..., $w^{(m-1)}(x)$ are (i)-differentiable, then $\underline{w}(x, r)$ and $\overline{w}(x, r)$ are H-differentiable of the m-th order and

$$N\left[w^{(m)}(x)\right] = \left(n\left[\underline{w}^{(m)}(x, r)\right], n\left[\overline{w}^{(m)}(x, r)\right]\right);$$

(2) if $w(x)$ and $w'(x)$, ..., $w^{(m-1)}(x)$ are (ii)-differentiable, then $\underline{w}(x, r)$ and $\overline{w}(x, r)$ are H-differentiable of the m-th order and

$$N\left[w^{(m)}(x)\right] = \left(n\left[\overline{w}^{(m)}(x, r)\right], s\left[\underline{w}^{(m)}(x, r)\right]\right) \text{ if } m \text{ is even,}$$

$$N\left[w^{(m)}(x)\right] = \left(n\left[\underline{w}^{(m)}(x, r)\right], s\left[\overline{w}^{(m)}(x, r)\right]\right) \text{ if } m \text{ is odd;}$$

(3) if $w(x)$ is (i)-differentiable and $w'(x)$, ..., $w^{(m-1)}(x)$ are (ii)-differentiable, then $\underline{w}(x, r)$ and $\overline{w}(x, r)$ are H-differentiable of the m-th order and

$$N\left[w^{(m)}(x)\right] = \left(n\left[\overline{w}^{(m)}(x, r)\right], n\left[\underline{w}^{(m)}(x, r)\right]\right) \text{ if } m \text{ is even,}$$

$$N\left[w^{(m)}(x)\right] = \left(n\left[\underline{w}^{(m)}(x, r)\right], n\left[\overline{w}^{(m)}(x, r)\right]\right) \text{ if } m \text{ is odd;}$$

(4) *if $w(x)$ is (ii)-differentiable and $w'(x)$, ..., $w^{(m-1)}(x)$ are (i)-differentiable, then $\underline{w}(x,r)$ and $\overline{w}(x,r)$ are H-differentiable of the m-th order and*

$$N\left[w^{(m)}(x)\right] = \left(n\left[\overline{w}^{(m)}(x,r)\right], n\left[\underline{w}^{(m)}(x,r)\right]\right),$$

where

$$n\left[\underline{w}^{(m)}(x,r)\right] = \left(\frac{s}{u}\right)^m n\left[\underline{w}(x,r)\right] - \sum_{j=1}^{m}\frac{s^{j-1}}{u^j}\left[\underline{w}^{(m-j)}(x,r)\right]_{x=0}, \tag{3.8}$$

$$n\left[\overline{w}^{(n)}(x,r)\right] = \left(\frac{s}{u}\right)^m n\left[\overline{w}(x,r)\right] - \sum_{j=1}^{m}\frac{s^{j-1}}{u^j}\left[\overline{w}^{(m-j)}(x,r)\right]_{x=0}. \tag{3.9}$$

Proof. Let the function $w(x)$ be (i)-differentiable. By induction, we prove the equation (3.8). For $m = 1$ from condition (2.1) we have

$$N\left[w'(x)\right] = (n[\underline{w}'(x,r)],\ \ n\left[\overline{w}'(x,r)\right]).$$

By partial integration and condition (3.3), we obtain

$$n[\underline{w}'(x,r)] = \int_0^\infty e^{-sx}\underline{w}'(ux,r)dx = \frac{1}{u}(sn[\underline{w}(x,r)] - \underline{w}(0,r)).$$

Let for $m = k$ the equation (3.8) hold. Then

$$n\left[\underline{w}^{(k)}(x,r)\right] = \left(\frac{s}{u}\right)^k n\left[\underline{w}(x,r)\right] - \sum_{j=1}^{k}\frac{s^{j-1}}{u^j}\left[\underline{w}^{(k-j)}(x,r)\right]_{x=0}.$$

Hence, for $m = k+1$ we get

$$\begin{aligned} s&\left[\underline{w}^{(k+1)}(x,r)\right] \\ &= \frac{d}{dx}n\left[\underline{w}^{(k)}(x,r)\right] = \left(\frac{s}{u}\right)^k n[\underline{w}'(x,r)] - \sum_{j=1}^{k}\frac{s^{j-1}}{u^j}\left[\underline{w}^{(k+1-j)}(x,r)\right]_{x=0} \\ &= \left(\frac{s}{u}\right)^k\left(\frac{s}{u}n[\underline{w}(x,r)] - \frac{1}{u}n[\underline{w}(0,r)]\right) - \sum_{j=1}^{k}\frac{s^{j-1}}{u^j}\left[\underline{w}^{(k+1-j)}(x,r)\right]_{x=0} \\ &= \left(\frac{s}{u}\right)^{k+1} n[\underline{w}(x,r)] - \sum_{j=1}^{k+1}\frac{s^{j-1}}{u^j}\left[\underline{w}^{(k+1-j)}(x,r)\right]_{x=0}. \end{aligned}$$

The proof is complete. □

4. Fuzzy Natural transform method for solving Volterra integro-differential equation

In this section, we introduce the parametric form of the integro-differential equation (1.1), (1.2) and then apply FNT to solve this equation.

Applying FNT on both sides of the integro-differential equation (1.1), we obtain

$$N\left[(FR)\int_0^x k_1(x-s)\odot w(x)dx\right] \oplus N\left[(FR)\int_0^x k_2(x-s)\odot w^{(m)}(x)dx\right] = S[g(x)]. \tag{4.1}$$

Using fuzzy convolution (3.6) we obtain the following formula

$$un[k_1(x)]\odot N[w(x)]\oplus un[k_2(x)]\odot N[w^{(m)}(x)] = N[g(x)]. \tag{4.2}$$

Consider the parametric form of $w(x)$ and $g(x)$ respectively

$$w(x,r) = (\underline{w}(x,r), \overline{w}(x,r))$$

and

$$g(x,r) = (\underline{g}(x,r), \overline{g}(x,r)), \;\; 0\le r\le 1, \;\; x\in[a,b].$$

Consider the functions $k_i(x)\ge 0,\ i=1,2$.

Let $w^{(m)}(x) = \left(\underline{w}^{(m)}(x,r),\ \ \overline{w}^{(m)}(x,r)\right)$. Then the parametric form of equation (4.1) is as follows

$$un[k_1(x)]n[\underline{w}(x,r)] + un[k_2(x)]n\left[\underline{w}^{(m)}(x,r)\right] = n[\underline{g}(x,r)]$$

$$un[k_1(x)]n[\overline{w}(x,r)] + un[k_2(x)]n\left[\overline{w}^{(m)}(x,r)\right] = n[\overline{g}(x,r)].$$

Then, using Theorem 3.6 and the initial conditions, we get

$$un[k_1(x)]n[\underline{w}(x,r)]+un[k_2(x)]\left[\left(\tfrac{s}{u}\right)^m n[\underline{w}(x,r)] - \sum_{j=1}^{m}\tfrac{s^{j-1}}{u^j}\underline{b}_{m-j}(r)\right] = n[\underline{g}(x,r)],$$

$$un[k_1(x)]n[\overline{w}(x,r)]+un[k_2(x)]\left[\left(\tfrac{s}{u}\right)^m n[\overline{w}(x,r)] - \sum_{j=1}^{m}\tfrac{s^{j-1}}{u^j}\overline{b}_{n-j}(r)\right] = n[\overline{g}(x,r)].$$

Hence, we obtain

$$n[\underline{w}(x,r)] = \frac{u^{m-1}n[\underline{g}(x,r)] + u^m n[k_2(x)]\sum_{j=1}^{m}\tfrac{s^{j-1}}{u^j}\underline{b}_{n-j}(r)}{u^m n[k_1(x)] + s^m n[k_2(x)]} \tag{4.3}$$

and

$$n[\overline{w}(x,r)] = \frac{u^{m-1}n[\overline{g}(x,r)] + u^m n[k_2(x)] \sum\limits_{j=1}^{m} \frac{s^{j-1}}{u^j}\overline{b}_{n-j}(r)}{u^m n[k_1(x)] + s^m n[k_2(x)]}. \tag{4.4}$$

Applying the inverse Natural transform, we find the solution of the equation (1.1), (1.2).

Case 2. Let $w^{(m)}(x) = \left(\overline{w}^{(m)}(x,r),\ \ \underline{w}^{(m)}(x,r)\right)$. Then the parametric form of equation (4.1) is as follows

$$un[k_1(x)]n[\underline{w}(x,r)] + un[k_2(x)]n\left[\overline{w}^{(m)}(x,r)\right] = n[\underline{g}(x,r)],$$
$$un[k_1(x)]n[\overline{w}(x,r)] + un[k_2(x)]n\left[\underline{w}^{(m)}(x,r)\right] = n[\overline{g}(x,r)].$$

Then, using Theorem 3.6 and the initial conditions, we get the following

$$un[k_1(x)]n[\underline{w}(x,r)]+un[k_2(x)]\left[\left(\tfrac{s}{u}\right)^m n[\overline{w}(x,r)] - \sum_{j=1}^{m} \tfrac{s^{j-1}}{u^j}\overline{b}_{m-j}(r)\right] = n[\underline{g}(x,r)],$$

$$un[k_1(x)]n[\overline{w}(x,r)]+un[k_2(x)]\left[\left(\tfrac{s}{u}\right)^m n[\underline{w}(x,r)] - \sum_{j=1}^{m} \tfrac{s^{j-1}}{u^j}\underline{b}_{m-j}(r)\right] = n[\overline{g}(x,r)].$$

Hence, we obtain

$$u^m n[k_1(x)]n[\underline{w}(x,r)] + s^m n[k_2(x)]n[\overline{w}(x,r)] = u^{m-1}n[\underline{g}(x,r)] + u^m n[k_2(x)]\sum_{j=1}^{m}\frac{s^{j-1}}{u^j}\overline{b}_{m-j}(r), \tag{4.5}$$

$$u^m n[k_1(x)]n[\overline{w}(x,r)] + s^m n[k_2(x)]n[\underline{w}(x,r)] = u^{m-1}n[\overline{g}(x,r)] + u^m n[k_2(x)]\sum_{j=1}^{m}\frac{s^{j-1}}{u^j}\underline{b}_{m-j}(r). \tag{4.6}$$

From (4.5) and (4.6) we find $n[\underline{w}(x,r)]$ and $n[\overline{w}(x,r)]$. Applying the inverse Natural transform, we obtain the solution of the equation (1.1), (1.2).

5. Numerical example

In this section, we find the solution of Volterra integro-differential equation by using FNT. Consider the following equation

$$\int_0^x w(s)ds \oplus (FR)\int_0^x (x-s)\odot w''(s)ds = g(x), \ \ x\in[0,1],$$

with initial condition $w(0) = (0, 0)$, $w'(0) = (0, 0)$ and

$$g(x) = \left(\left(\frac{x^3}{3} + x^2 \right)(1 + r), \left(\frac{x^3}{3} + x^2 \right)(3 - r) \right).$$

In this case $k_1(x - s) = 1$, $k_2(x - s) = x - s$ and $b_0 = (0, 0)$, $b_1 = (0, 0)$. Let $w(x)$ and $w'(x)$ are (i) or (ii)-differentiable. Then

$$w''(x) = \left(\underline{w}''(x, r),\ \ \overline{w}''(x, r)\right).$$

Using equations (4.3) and (4.4), we obtain

$$n\left[\underline{w}(x, r)\right] = \frac{un[\underline{g}(x, r)]}{u^2 n[k_1(x)] + s^2 n[k_2(x)]}$$

and

$$n\left[\overline{w}(x, r)\right] = \frac{un[\overline{g}(x, r)]}{u^2 n[k_1(x)] + s^2 n[k_2(x)]},$$

where

$$n[\underline{g}(x, r)] = \left(\frac{1}{3} n[x^3] + n[x^2]\right)(1 + r) = \left(2\frac{u^3}{s^4} + 2\frac{u^2}{s^3}\right)(1 + r),$$

$$n[\overline{g}(x, r)] = \left(\frac{1}{3} n[x^3] + n[x^2]\right)(3 - r) = \left(2\frac{u^3}{s^4} + 2\frac{u^2}{s^3}\right)(3 - r),$$

$$n[k_1(x)] = n[1] = \frac{1}{s},\ \ n[k_2(x)] = n[x] = \frac{u}{s^2}.$$

Hence

$$n\left[\underline{w}(x, r)\right] = \frac{u\left(2\frac{u^3}{s^4} + 2\frac{u^2}{s^3}\right)(1 + r)}{\frac{u^2}{s} + u} = \frac{2u^2(u^2 + su)(1 + r)}{s^3(u^2 + su)} = 2\frac{u^2}{s^3}(1 + r)$$

and

$$n\left[\overline{w}(x, r)\right] = \frac{u\left(2\frac{u^3}{s^4} + 2\frac{u^2}{s^3}\right)(1 + r)}{\frac{u^2}{s} + u} = \frac{2u^2(u^2 + su)(3 - r)}{s^3(u^2 + su)} = 2\frac{u^2}{s^3}(3 - r).$$

Applying the inverse Natural transform, we obtain

$$\underline{w}(x, r) = n^{-1}[2\frac{u^2}{s^3}(1 + r)] = x^2(1 + r)$$

and

$$\overline{w}(x, r) = n^{-1}[2\frac{u^2}{s^3}(3 - r)] = x^2(3 - r).$$

6. Conclusions

In this research paper, we introduce a new fuzzy integral transform, called a fuzzy Natural transform, which is defined with the help of the unitary Natural transform. We establish some of its basic properties. We prove theorems about fuzzy Hukuhara derivatives and fuzzy unit convolution. Using these new results, we successfully obtained the exact solution to a fuzzy Volterra integro-differential equation with convolution kernels. We construct a numerical example to illustrate the application of the new method.

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РАЗМИТА ТРАНСФОРМАЦИЯ НА НАТУРАЛ ЗА РЕШАВАНЕ НА ИНТЕГРО-ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ

Атанаска Георгиева, Йорданка Гудалова

Резюме. В тази статия е въведен методът на размитото преобразуване на Натурал и е използван за решаване на размитото интегро-диференциално уравнение на Волтера от първи род с конволюционно ядро при диференцируемост по Хукухара. Показано е, че предложеният метод е прост и надежден подход за аналитично решаване на такива уравнения. Накрая е даден илюстративен пример с точни конволюционни ядра, за да се покаже ефективността на предложения метод.