

A FIXED POINT APPROACH TO THE STACKELBERG MODEL IN OLIGOPOLY MARKETS

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ABSTRACT. This paper investigates the connection between the Stackelberg equilibrium in duopoly markets and the theory of coupled fixed points. Instead of relying on differentiability and explicit maximization of payoff functions, we model the leader–follower interaction through response mappings satisfying a mixed contractive condition. By extending the Banach contraction principle to ordered pairs of maps, we prove a general theorem guaranteeing the existence and uniqueness of a coupled fixed point, together with a priori and a posteriori error estimates, and convergence rates. The result provides a unified fixed-point framework for equilibrium analysis in oligopoly markets. A linear Stackelberg duopoly is presented to illustrate that the fixed-point formulation reproduces the classical equilibrium and confirms the leader’s informational advantage.

1. INTRODUCTION

Oligopoly markets play a central role in micro-economic theory, as they describe industries dominated by a few firms whose strategic decisions are mutually interdependent. Unlike perfect competition, where no firm influences

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price, or monopoly, where one firm dictates market conditions, oligopolies exhibit strategic interdependence, meaning each firm's output or pricing decision both affects and is affected by its rivals [18, 26].

A central problem is the determination of market equilibrium - the outcome of firms' optimal reactions to one another. Such an equilibrium typically lies between the extremes of perfect competition and monopoly, generating intermediate levels of price, quantity, and welfare [27].

Two classical models capture these strategic interactions. In the Cournot model [7, 8], firms make their decisions simultaneously, treating the competitor's output as fixed; the resulting Cournot equilibrium represents mutual best responses. In contrast, the Stackelberg model [24] introduces a sequential structure: the leader commits first, and the follower reacts optimally. This framework highlights how the sequence of actions and information asymmetry shape market outcomes [17, 24].

Most analyses rely on maximizing profit functions [10, 23], often under differentiability and concavity assumptions [9, 13]. Although analytically convenient, these assumptions may be restrictive: real firms face technological, informational, and contractual limitations that constrain their ability to optimize fully. Even so, firms adjust their behavior through response functions that link optimal output to competitors' choices [9, 10]. Studies such as [9, 13, 14] show that with differentiable profit functions, the Cournot model yields consistent equilibrium-existence results.

Our goal is to demonstrate that a similar connection exists within the Stackelberg model as well.

2. PRELIMINARIES

Following [10, 23], consider two firms producing quantities x and y for a common market with total output $Z = x + y$. The inverse demand is $P(Z) = P(x + y)$, and cost functions are $c(x)$ and $C(y)$, respectively.

2.1. Cournot Model. If both firms act rationally, their payoff functions are

$$\Phi(x, y) = xP(x + y) - c(x), \quad \Psi(x, y) = yP(x + y) - C(y).$$

Each firm maximizes its own profit:

$$\max_x \Phi(x, y), \quad \max_y \Psi(x, y).$$

Assuming differentiability, the first-order conditions are

$$\begin{cases} \frac{\partial \Phi}{\partial x}(x, y) = P(x + y) + xP'(x + y) - c'(x) = 0, \\ \frac{\partial \Psi}{\partial y}(x, y) = P(x + y) + yP'(x + y) - C'(y) = 0. \end{cases} \quad (1)$$

The solution of (1) defines the Cournot equilibrium where neither firm can increase profit by unilateral deviation [10, 23]. Often, (1) can be written via reaction functions $x = \varphi(y)$ and $y = \psi(x)$. Uniqueness is ensured if either the payoff functions are concave or the second-order conditions

$$\frac{\partial^2 \Phi}{\partial x^2}(\xi, \eta) < 0, \quad \frac{\partial^2 \Psi}{\partial y^2}(\xi, \eta) < 0 \quad (2)$$

are satisfied, where (ξ, η) is the solution of (1).

2.2. Stackelberg Model. In the Stackelberg framework [24], decisions are sequential. The follower solves

$$\frac{\partial \Psi}{\partial y}(x, y) = P(x + y) + yP'(x + y) - C'(y) = 0,$$

yielding a reaction function $y = \psi(x)$. Knowing $\psi(x)$, the leader maximizes his profit

$$\Phi(x, \psi(x)) = xP(x + \psi(x)) - c(x),$$

knowing the response of the follower, which leads to the first-order system

$$\begin{cases} \frac{\partial \Psi}{\partial y}(x, y)P(x + y) + yP'(x + y) - C'(y) = 0, \\ \frac{d}{dx}\Phi(x, \psi(x)) = P(x + \psi(x)) + xP'(x + \psi(x))(1 + \psi'(x)) - c'(x) = 0. \end{cases}$$

This sequential mechanism gives the leader informational advantage and typically a slightly higher equilibrium profit [17].

2.3. Illustrative Example. Consider linear demand $P(Z) = 10 - 0.1Z = 10 - 0.1(x + y)$ and identical quadratic costs $c(t) = C(t) = t^2/2$.

- **Cournot equilibrium:**

$$\begin{cases} 10 - 1.2x - 0.1y = 0, \\ 10 - 0.1x - 1.2y = 0, \end{cases}$$

giving $\xi = \eta \approx 7.69$, price $P(\xi, \eta) \approx 8.46$, and profit $\Phi(\xi, \eta) = \Psi(\xi, \eta) \approx 35.5$.

- **Stackelberg equilibrium:** The follower's response is $y = \psi(x) = \frac{25}{3} - \frac{x}{12}$; the leader solves $\frac{d}{dx}\Phi(x, \psi(x)) = 0$, obtaining $\xi \approx 7.74$, $\eta \approx 7.69$, confirming a leader's advantage.

These observations show that the leader, having additional information about the follower's behavior, achieves a higher equilibrium profit, even though both firms sell at the same market price.

TABLE 2.1. Comparison at the equilibrium points of both models

Model		Leader	Follower	Sum
Stackelberg model	Volumes	7.74	7.69	15.43
	Profit	35.50	35.46	70.96
	Market price	8.45	8.45	
Cournot model	Volumes	7.69	7.69	15.38
	Profit	35.50	35.50	71.00
	Market price	8.46	8.46	

These findings illustrate the strategic implications of early commitment. The leader produces marginally more than in the Cournot setting, shifting the follower to a less favorable portion of the reaction curve. Although the individual payoff advantage is clear, the overall market quantity and price remain close to the Cournot equilibrium, suggesting that the distribution of payoffs changes more significantly than total welfare in this simplified linear environment.

2.4. Model Limitations. Analytical solutions may fail for nonlinear price or cost functions (e.g., $P = 10 - \arctan(x + y)$, $C(x) = xe^x$). In such cases, implicit or iterative formulations are used [9]. If output adapts dynamically as

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = G(x_n, y_n),$$

the resulting system can be analyzed using coupled fixed-point techniques [4, 14, 9, 13].

2.5. Coupled Fixed Points for Ordered Pairs of Maps. The Banach fixed point concept [3], despite being over 100 years old, has numerous extensions and implementations. Various directions of generalization exist; here, we focus on the generalization in [11], which reformulates the fixed-point notion through coupled fixed points.

By considering a map of two variables $F : A \times A \rightarrow A$ ([5, 11]) the notion of a coupled fixed point $(x, y) \in A \times A$ of F in A is defined as $x = F(x, y)$ and $y = F(y, x)$.

The concept of exploring fixed points for their existence and uniqueness in cyclic maps, i.e., $T : A \rightarrow B$ and $T : B \rightarrow A$, versus self-maps was initially presented in [16].

By combining the ideas of cyclic maps and coupled fixed points the meaning of fixed points was broadened to coupled fixed points for cyclic maps of two variables in [25] by investigating of maps $F : A \times A \rightarrow B$, $F : B \times B \rightarrow A$, and the search of sufficient conditions that will ensure the existence of an ordered pair (x, y) so that $x = F(x, y)$ and $y = F(y, x)$.

However, the equations $x = F(x, y)$ and $y = F(y, x)$ that define the linked fixed points [5, 11] frequently result in an ordered pair (x, y) such that $x = y$.

This limitation was solved in [29], wherein a modified coupled fixed point for an ordered pair of maps (F, G) was developed. It is suggested in [29] to combine two maps $F : A \times B \rightarrow A$, $G : A \times B \rightarrow B$ and defining an ordered pair (x, y) as a coupled fixed point for (F, G) if $x = F(x, y)$ and $y = G(x, y)$.

The definition from [5, 11] is obtained when $G(x, y) = F(y, x)$, and $A = B$.

2.6. Banach Contraction Fixed Point Theorem.

Definition 2.1 ([3]). Let (X, ρ) be a metric space. A map $T : X \rightarrow X$ is called a Banach contraction map if there exist a non-negative constants a , such that for all $x, y \in X$ the following inequality holds:

$$\rho(Tx, Ty) \leq a\rho(x, y). \tag{2.1}$$

Theorem 2.2 ([3]). Let (X, ρ) be a complete metric space and $T : X \rightarrow X$ be a Banach contraction map, then:

- (1) there is a unique fixed point $\xi \in X$ of T and, moreover, for any initial guess $x_0 \in X$, the iterated sequence $x_n = Tx_{n-1}$ for $n = 1, 2, \dots$ converges to the fixed point ξ
- (2) there holds a priori error estimate: $\rho(\xi, x_n) \leq \frac{a^n}{1-a} \rho(x_0, x_1)$
- (3) there holds a posteriori error estimate: $\rho(\xi, x_n) \leq \frac{a}{1-a} \rho(x_{n-1}, x_n)$
- (4) the rate of convergence is: $\rho(\xi, x_n) \leq a\rho(\xi, x_{n-1})$.

3. MAIN RESULT

3.1. Coupled Fixed Points for Ordered Pairs of Maps Satisfying a Mixed Contractive Condition. Let (X_i, d_i) , $i = 1, 2$ be two metric spaces, $F_i : X_1 \times X_2 \rightarrow X_i$ for $i = 1, 2$ be two mappings and $z = (x, y), w = (u, v) \in$

$X_1 \times X_2$. Let us denote

$$M(x, y, u, v) = d_1(x, u) + d_2(y, v).$$

Definition 3.1. Let X_i , $i = 1, 2$ be two non-empty sets and (F_1, F_2) be a semi-cyclic map. The sequence $\{(x_k, y_k)\}_{k=0}^{\infty}$, define inductively by $(x_0, y_0) \in X_1 \times X_2$ and $x_{n+1} = F_1(x_n, F_2(x_n, y_n))$, $y_{n+1} = F_2(x_n, y_n)$ for $n \geq 0$ will be called a mixed iterative sequence.

Definition 3.2. Let X be a set and $F_i : X \times X \rightarrow X$ for $i = 1, 2$. We say that the ordered pair of maps (F_1, F_2) has local double cross symmetry property at a point $(x^*, y^*) \in X \times X$ if

$$F_2(x^*, y^*) = F_1(y^*, F_2(y^*, x^*)) \quad \text{and} \quad F_2(y^*, x^*) = F_1(x^*, F_2(x^*, y^*)).$$

Following [19, 20] for any arbitrary chosen initial guess point $(x_0, y_0) \in X_1 \times X_2$ the mixed iterated sequence $\{(x_k, y_k)\}_{k=0}^{\infty}$ can be written as $(x_{n+1}, y_{n+1}) = T(x_n, y_n)$, where

$$T(x, y) = (F_1(x, F_2(x, y)), F_2(x, y)) : X_1 \times X_2 \rightarrow X_1 \times X_2.$$

Theorem 3.3. Let (X_i, d_i) , $i = 1, 2$ be two complete metric spaces, $F_i : X_1 \times X_2 \rightarrow X_i$ for $i = 1, 2$ be two mappings. Assume there exist a non-negative constant $k \in (0, 1)$ and the following inequality holds for all $(x, y), (u, v) \in X_1 \times X_2$:

$$d_1(F_1(x, F_2(x, y)), F_1(u, F_2(u, v))) + d_2(F_2(x, y), F_2(u, v)) \leq kM(x, y, u, v).$$

Then:

- (1) there exists a unique coupled fixed point $(x^*, y^*) \in X_1 \times X_2$ for the ordered pair of maps (F_1, F_2)
- (2) for any initial start point $(x_0, y_0) \in X_1 \times X_2$, the elements of the mixed iterative sequence $\{(x_k, y_k)\}_{k=0}^{\infty}$ converge to x^* and y^* , respectively
- (3) the following holds:

- A priori estimate:

$$d_1(x_n, x^*) + d_2(y_n, y^*) \leq \frac{k^n}{1-k} (d_1(x_1, x_0) + d_2(y_1, y_0))$$

- A posteriori estimate:

$$d_1(x_n, x^*) + d_2(y_n, y^*) \leq \frac{k}{1-k} (d_1(x_n, x_{n-1}) + d_2(y_n, y_{n-1}))$$

- convergence rate:

$$d_1(x_n, x^*) + d_2(y_n, y^*) \leq k (d_1(x_{n-1}, x^*) + d_2(y_{n-1}, y^*)).$$

If in addition $X_1 = X_2 = X$, $d_1 = d_2 = d$ and the ordered pair (F_1, F_2) satisfies the local double cross symmetry property at the unique coupled fixed point (x^*, y^*) , then $x^* = y^*$.

Proof. Define the product space $X_1 \times X_2$ equipped with the metric

$$\rho((x, y), (u, v)) := d_1(x, u) + d_2(y, v)$$

which makes $(X_1 \times X_2, \rho)$ a complete metric space since both (X_1, d_1) and (X_2, d_2) are complete.

Now define the map $G : X_1 \times X_2 \rightarrow X_1 \times X_2$ by:

$$G(x, y) := (F_1(x, F_2(x, y)), F_2(x, y)).$$

We aim to prove that G has a unique fixed point in $X_1 \times X_2$ by verifying that G is a Banach contraction map in $(X_1 \times X_2, \rho)$.

Let $(x, y), (u, v) \in X_1 \times X_2$ and let us denote $\bar{x} = (x, y)$, $\bar{u} = (u, v)$. Then

$$\begin{aligned} \rho(G(\bar{x}), G(\bar{u})) &= d_1(F_1(x, F_2(\bar{x})), F_1(u, F_2(\bar{u}))) + d_2(F_2(\bar{x}), F_2(\bar{u})) \\ &\leq k(d_1(x, u) + d_2(y, v)) = k\rho(\bar{x}, \bar{u}). \end{aligned}$$

This shows that G is a Banach contraction map in the complete metric space $X_1 \times X_2, \rho$. Now, we can apply Theorem 2.2 and conclude that there exists a unique point $(x^*, y^*) \in X_1 \times X_2$ such that

$$(x^*, y^*) = G(x^*, y^*) = (F_1(x^*, F_2(x^*, y^*)), F_2(x^*, y^*)),$$

i.e.,

$$x^* = F_1(x^*, y^*) \quad \text{and} \quad y^* = F_2(x^*, y^*).$$

The error estimates follow immediately from Theorem 2.2 and the way the metric ρ is defined.

(*The symmetry case*) Assume $X_1 = X_2 = X$, $d_1 = d_2 = d$, and that the mappings $F_1, F_2 : X \times X \rightarrow X$ satisfy the local double-cross symmetry

$$F_1(y^*, F_2(y^*, x^*)) = F_2(x^*, y^*), \quad F_2(y^*, x^*) = F_1(x^*, F_2(x^*, y^*))$$

for the coupled fixed point $(x^*, y^*) \in X \times X$. We have that

$$x^* = F_1(x^*, F_2(x^*, y^*)), \quad y^* = F_2(x^*, y^*).$$

Applying the contractive inequality for (x^*, y^*) and $(u, v) = (y^*, x^*)$, we get

$$\begin{aligned} d(x^*, y^*) + d(y^*, x^*) &= d(F_1(x^*, F_2(x^*, y^*)), F_1(y^*, F_2(y^*, x^*))) \\ &\quad + d(F_2(x^*, y^*), F_2(y^*, x^*)) \\ &\leq k(d(x^*, y^*) + d(y^*, x^*)). \end{aligned}$$

We conclude that $d(x^*, y^*) = 0$, and therefore $x^* = y^*$. □

We will introduce a global double cross symmetry property, which will be the ordered pair of maps to satisfies the local one for all ordered pairs (x, y) .

Definition 3.4. Let X be a set. If the mappings $F_1, F_2 : X \times X \rightarrow X$ satisfy

$$F_1(y, F_2(y, x)) = F_2(x, y), \quad F_2(y, x) = F_1(x, F_2(x, y)),$$

for all $x, y \in X$ we will say that the ordered pair of maps (F_1, F_2) has the global double-cross symmetry property.

Remark 3.5. Although, global double cross symmetry appears more restrictive at first glance, it is useful in situations where the existence of a coupled fixed point is guaranteed, but the fixed point cannot be computed explicitly. In such cases one cannot directly verify the local double cross symmetry at that point. If a global symmetry holds, however, it automatically ensures that any coupled fixed point satisfies $x = y$.

In [14], the simpler assumption $F_2(x, y) = F_1(y, x)$ was sufficient to deduce $x = y$, since the model considered there was fully symmetric (Cournot-type). In the Stackelberg-type formulation adopted here, however, the nested dependence $F_1(x, F_2(x, y))$ introduces an intrinsic asymmetry between the leader and the follower, so the stronger double-cross symmetry is required to guarantee $x = y$.

3.2. The Stackelberg Model in Duopoly Market by Response Functions, Rather than Payoff Maximization.

3.2.1. *The Model.* Let $\Phi(x, y)$ and $\Psi(x, y)$ be the payoff functions of the two players in a duopoly market, where they compete on a single good.

Following [9], instead of solving the first order equation for the follower, i.e., the second player

$$\frac{\partial \Psi}{\partial y}(x, y) = 0 \tag{3.1}$$

we consider the function $F_2(x, y) = \frac{\partial \Psi}{\partial y} + y$ and we alter (3.1) to a fixed point one, i.e., $F_2(x, y) = y$. Thus if not possible to solve (3.1), one may regard the equation (3.1) as implicitly defining a function $y = \psi(x)$ under suitable regularity conditions, and we can find the values of its derivative by implicit differentiation, when some sufficient conditions are satisfied.

The first order equation for the leader will be

$$\frac{d}{dx} \Phi(x, F_2(x, \psi(x))) = 0.$$

We can alter the above equation into a fixed point one as

$$F_1(x, F_2(x, \psi(x))) = \frac{d}{dx} \Phi(x, F_2(x, \psi(x))) + x = x.$$

Therefore we get that in the case of rational players, with differentiable payoff functions, and playing according to Stackelberg’s model the maximization problem can be viewed as a problem of coupled fixed points. Define $\mathcal{G} : X_1 \times X_2 \rightarrow X_1 \times X_2$ by:

$$\mathcal{G}(x, y) := (F_1(x, F_2(x, y)), F_2(x, y)). \tag{3.2}$$

A Stackelberg equilibrium (ξ, η) satisfies $\mathcal{G}(\xi, \eta) = (\xi, \eta)$.

By converting the payoff maximization in the Stackelberg model with the coupled fixed point notion, i.e., a solution of the system of equations:

$$\begin{cases} F_1(x, F_2(x, y)) = x \\ F_2(x, y) = y, \end{cases}$$

we will avoid the need to justify the existence of the implicit function ψ ; equilibrium emerges from fixed-point convergence.

If the responses of the two players satisfy the assumption in Theorem 3.3 then there is an unique market equilibrium in the Stackalberg’s model.

3.3. Stackelberg’s model.

Corollary 3.6. *Let us consider a duopoly market, satisfying:*

- (1) *the two players are competing on a single good*
- (2) *both goods are homogeneous and perfect substitutes*
- (3) *the follower can produce quantities from the set X_2 , and the leader can produce quantities from the set X_1 , where X_1 and X_2 are closed, nonempty subsets of a complete metric space $(\mathbb{R}, |\cdot - \cdot|)$.*
- (4) *let there be a closed subset $D \subseteq X_1 \times X_2$ and maps $F_i : D \rightarrow X_i$, $i = 1, 2$, so that:*

$$(F_1(x, F_2(x, y)), F_2(x, y)) \in D$$

for every $(x, y) \in D$, where the maps (F_1, F_2) be the response functions for the leader and the follower, respectively.

- (5) *there is an $\alpha < 1$, so that the inequality:*

$$\begin{aligned} |F_1(x, F_2(x, y)) - F_1(u, F_2(u, v))| + |F_2(x, y) - F_2(u, v)| \\ \leq \alpha(|x - u| + |y - v|) \end{aligned} \tag{3.3}$$

holds for all $(x, y), (u, v) \in X_1 \times X_2$.

Then there is a unique market equilibrium pair (ξ, η) in D , i.e., $\xi = F_1(\xi, \eta)$ and $\eta = F_2(\xi, \eta)$.

The error estimates from Theorem 3.3 hold for $d_1(x, y) = d_2(x, y) = |x - y|$.

Suppose $X_1 = X_2 = X$. If the unique mixed coupled fixed point (ξ, η) satisfies the double cross symmetry at the fixed point:

$$F_2(\xi, \eta) = F_1(\eta, F_2(\eta, \xi)) \quad \text{and} \quad F_2(\eta, \xi) = F_1(\xi, F_2(\xi, \eta)),$$

then $\xi = \eta$.

3.3.1. The Second Order Conditions.

Proposition 3.7. *If the responses are obtained by the consecutive maximization of the follower and leader payoff functions in the Stackelberg's model, assuming differentiability of the payoff functions and the assumption in Corollary 3.6 are satisfied then coupled fixed points satisfy and the second order conditions in the Stackelberg's model.*

Proof. Fix $x \in X_1$, consider $u = x$, $v = y + \Delta y$, and substitute in (3.3) we get $|F_1(x, F_2(x, y + \Delta y)) - F_1(x, F_2(x, y))| + |F_2(x, y + \Delta y) - F_2(x, y)| \leq \alpha |\Delta y|$.

Divide by $|\Delta y|$ and the above inequality and taking the limit $\Delta y \rightarrow 0$ in it we get

$$\left| \frac{\partial F_1}{\partial y}(x, F_2(x, y)) \cdot \frac{\partial F_2}{\partial y}(x, y) \right| + \left| \frac{\partial F_2}{\partial y}(x, y) \right| \leq \alpha. \quad (3.4)$$

Since absolute values are non-negative, (3.4) implies:

$$\left| \frac{\partial F_2}{\partial y}(x, y) \right| \leq \alpha. \quad (3.5)$$

Recall $F_2(x, y) = \frac{\partial \Psi}{\partial y}(x, y) + y$ and differentiating it with respect to y we get

$$\frac{\partial F_2}{\partial y} = \frac{\partial^2 \Psi}{\partial y^2} + 1. \quad (3.6)$$

Substitute (3.6) into (3.5) we obtain the inequality

$$\left| \frac{\partial^2 \Psi}{\partial y^2} + 1 \right| \leq \alpha. \quad (3.7)$$

Since $\alpha < 1$ the last inequality implies the inequalities $-\alpha \leq \frac{\partial^2 \Psi}{\partial y^2} + 1 \leq \alpha$.

From the assumption that $\alpha \in (0, 1)$ it follows that $\frac{\partial^2 \Psi}{\partial y^2} < \alpha - 1 < 0$.

Therefore the follower's second-order condition is satisfied.

For the rest of the proof if we use the notation $\Phi(x, y)$ or $F_1(x, F_2(x, y))$ it refers to the function obtained by evaluating the payoff function Φ or the response function F_1 at the ordered pair $(x, F_2(x, \psi(x)))$ depending on x , i.e., $\Phi(x, \psi(x))$ or $F_1(x, F_2(x, \psi(x)))$.

Fix $y \in X_2$, consider $u = x + \Delta x$, $v = y$, and substitute in (3.3) we get $|F_1(x + \Delta x, F_2(x + \Delta x, y)) - F_1(x, F_2(x, y))| + |F_2(x + \Delta x, y) - F_2(x, y)| \leq \alpha |\Delta x|$.

Divide by $|\Delta x|$ the last inequality and take limit as $\Delta x \rightarrow 0$ in it we find

$$\left| \frac{d}{dx} F_1(x, F_2(x, y)) \right| + \left| \frac{\partial F_2}{\partial x}(x, y) \right| \leq \alpha. \tag{3.8}$$

From (3.8) it follows the inequality

$$\left| \frac{d}{dx} F_1(x, F_2(x, y)) \right| \leq \alpha. \tag{3.9}$$

Recall $F_1(x, F_2(x, y)) = \frac{d}{dx} \Phi(x, F_2(x, y)) + x$ and differentiating it we get

$$\frac{d}{dx} F_1 = \frac{d^2}{dx^2} \Phi + 1. \tag{3.10}$$

From (3.9) and (3.10) follows the inequality $\left| \frac{d^2}{dx^2} \Phi + 1 \right| \leq \alpha$. Since $\alpha < 1$ conclude that $\frac{d^2}{dx^2} \Phi < \alpha - 1 < 0$.

Thus, the leader's second-order condition is satisfied too. □

3.3.2. Illustrative Example. We consider a duopoly with linear demand from Section 2.3.

We get the response function F_1 and F_2

$$F_2(x, y) = \frac{\partial \Psi}{\partial y} + y = 10 - 0.1x - 0.2y$$

and

$$F_1(x, F_2(x, y)) = \frac{d}{dx} \Phi(x, F_2(x, y)) + x = 9 - 0.18x + 0.02y$$

from the first order equations.

Let us put

$$\mathcal{G}(x, y) = (F_1(x, F_2(x, y)), F_2(x, y)).$$

We show \mathcal{G} is a Banach contraction map under the metric

$$\rho((x, y), (u, v)) = |x - u| + |y - v|.$$

Just to fit the next chain of inequalities we will denote $\bar{x} = (x, y)$ and $\bar{u} = (u, v)$. Indeed

$$\begin{aligned} \rho(\mathcal{G}(\bar{x}), \mathcal{G}(\bar{u})) &= \rho((F_1(x, F_2(\bar{x})), F_2(\bar{x})) - (F_1(u, F_2(\bar{u})), F_2(\bar{u}))) \\ &= |F_1(x, F_2(\bar{x})) - F_1(u, F_2(\bar{u}))| + |F_2(\bar{x}) - F_2(\bar{u})| \\ &\leq 0.18|x - u| + 0.02|y - v| + 0.1|x - u| + 0.2|y - v| \\ &= 0.28|x - u| + 0.22|y - v| \\ &\leq 0.28(|x - u| + |y - v|) = 0.28\rho(\bar{x}, \bar{u}). \end{aligned}$$

Thus the two response functions satisfy Corollary 3.6 with contractive constant $\alpha = 0.28$.

The coupled fixed point $\xi \approx 7.757$ and $\eta \approx 7.687$ is the same as the market equilibrium point obtained by the classical Stackelberg's maximization in Section 2.3.

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ПОДХОД ЧРЕЗ НЕПОДВИЖНИ ТОЧКИ КЪМ МОДЕЛА НА СТАКЕЛБЕРГ В ОЛИГОПОЛНИ ПАЗАРИ

Мартин Павлов, Васил Желински, Боян Златанов

Резюме. В статията се изследва връзката между равновесието на Стакелберг в дуополни пазари и теорията на двойки неподвижни точки. Вместо да се разчита на диференцируемост и максимизация на функциите на печалба, взаимодействието между лидера и последователя се моделира чрез функции на реакция, удовлетворяващи смесено свиващо условие. Чрез обобщаване на принципа на Банах за свиващи изображения върху наредени двойки от изображения се доказва обща теорема за съществуване и единственост на двойки неподвижна точка, заедно с априорни и апостериорни оценки на грешката и скоростта на сходимост. Получените резултати осигуряват единна рамка в контекста на неподвижните точки за анализ на равновесието в олигополни пазари. Представен е пример с линеен дуополен модел на Стакелберг, който показва, че формулировка с помощта на неподвижни точки възпроизвежда класическото равновесие и потвърждава информационното предимство на лидера.