FIXED POINT THEOREMS OF KANNAN TYPE FOR CYCLICAL CONTRACTIVE CONDITIONS

Mihaela Petric, Boyan Zlatanov

Abstract. *The main aim of this paper is to obtain fixed point theorems for Kannan and Zamfirescu operators in the presence of cyclical contractive condition. A method for approximation of the fixed points is also provided, for which both a priori and a posteriori error estimates are given. Our results generalize, unify and extend several important fixed points theorems in literature. In order to illustrate the efficiency of our generalizations five significant examples are also given.*

Keywords: fixed point, cyclical operator, contractive condition, Kannan operators, Zamfirescu operators, error estimates

2010 Mathematics Subject Classification: 47H10, 54H25

1. Introduction

One of the most important results used in nonlinear analysis is the well-known Banach's contraction principle which basically shows that any contraction on a complete metric space (X, d), that is any mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq ad(x, y), \text{for all } x, y \in X, \tag{1}$$

where $0 \leq a < 1$ is a constant, has a unique fixed point. Notice that any contraction is continuous on X. It is natural to ask if there exist contractive conditions which do not imply the continuity of T all over the whole space X. This was answered in the affirmative way by R. Kannan [4] in 1968, who proved a fixed point theorem, which extends Banach's contraction principle to mappings that don't need to be continuous, by considering instead of (1) the next condition: there exists $a \in [0, 0.5)$ such that

$$d(Tx, Ty) \leq a\big[d(x, Tx) + d(y, Ty)\big], \text{for all } x, y \in X. \tag{2}$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T. One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [3], is based on a condition similar to (2): there exist $c \in [0, 0.5)$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \text{for all } x, y \in X. \tag{3}$$

Rhoades proved in [11], that the contractive conditions (1), (2) and (3), as well as (1) and (2), respectively, are independent. In 1972, Zamfirescu obtained a very interesting fixed point theorem, which is a generalization of Banach's, Kannan's and Chatterjea's fixed point theorems. In [2] Berinde V.

completed the Kannan's and Zamfirescu's fixed point theorem with the error estimates and the rate of the convergence for the Picard iteration.

On the other hand, in [5] W.A. Kirk, P.S. Srinivasan and P. Veeramani obtained an extension of Banach's fixed point theorem by considering a cyclical contractive condition, as given by the next theorem.

Theorem 1.1. *([5])* Let A and B be two nonempty closed subsets of a complete metric space, and suppose $T : A \cup B \to A \cup B$ satisfies the following conditions:

$$T(A) \subseteq B \text{ and } T(B) \subseteq A; \tag{4}$$

and

$$d(Tx, Ty) \leq ad(x, y), \text{ for all } x \in A, y \in B. \tag{5}$$

where $a \in (0, 1)$. Then T has a unique fixed point in $A \cap B$.

Further in [5], this theorem was extended to the union of $p \geq 2$ nonempty sets, $A_1, A_2, ..., A_p, A_{p+1} = A_1$. A mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ satisfying

$$T(A_i) \subseteq A_{i+1} \text{ for all } i \in \{1, 2, \ldots, p\}, \tag{6}$$

is called a *cyclical operator*. Also, inspired by the results in [5], other fixed point theorems were obtained. In [6, 9, 14] it was defined the notion of *cyclical representation* of the space X with respect to the operator T and fixed points theorems were obtained for mappings defined on these cyclical representation. Also in [7] other important results from the fundamental metric fixed point theory were extended for cyclic assumptions, i.e., Chatterjea, Bianchini, Reich, Hardy-Rogers, Ćirić, and in [8] the same author considered Ćirić-Reich-Rus type operators.

Consequently, the main aim of this paper is to obtain the fixed point theorems for Kannan and Zamfirescu operators using cyclical conditions. For all fixed point theorems we will also provide error estimates.

It is possible in some of theorems with cyclic contractive conditions, for part of the proofs to use [10]. For the sake of simplicity we prefer to use the technique of the classical proof of Stefan Banach's fixed point Theorem.

2. Fixed point theorem for cyclic Kannan operators

We extend the fixed point theorem of Kannan using cyclical assumptions.

Theorem 2.1. Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space X and suppose $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$, is a cyclical operator, i.e. satisfies the condition (6), such that

$$d(Tx, Ty) \leq a\big[d(x, Tx) + d(y, Ty)\big], \text{for all } x \in A_i, y \in A_{i+1}, \text{ for } 1 \leq i \leq p. \tag{7}$$

where $a \in \big[0, \frac{1}{2}\big)$ is a constant. Then

(i) T has a unique fixed point x^* in $\bigcap_{i=1}^{p} A_i$.

(ii) The Picard iteration $\{x_n\}$ given by

(8) $$x_{n+1} = Tx_n, n \geq 0,$$

converges to x^* for any starting point $x_0 \in \bigcup_{i=1}^{p} A_i$;

(iii) The following estimates hold

(9) $$d(x_n, x^*) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1), n \geq 0;$$

(10) $$d(x_{n+1}, x^*) \leq \frac{\lambda}{1-\lambda} d(x_n, x_{n+1}), n > 0;$$

where $\lambda = \frac{a}{1-a}$.

(iv) The rate of convergence of Picard iteration is given by

(11) $$d(x_n, x^*) \leq \frac{a}{1-a} d(x_{n-1}, x^*), n = 1, 2, \ldots$$

Proof. Let $x_0 \in \bigcup_{i=1}^{p} A_i$. So there exists $i \in \{1, 2, \ldots, p\}$ such that $x_0 \in A_i$, and from (6) we have that $x_1 = Tx_0 \in A_{i+1}$. Then by (7) we get $d(x_1, x_2) \leq \frac{a}{1-a} d(x_0, x_1)$. Therefore, denoting $\lambda := \frac{a}{1-a}$ we have $0 \leq \lambda < 1$, since $a \in \left[0, \frac{1}{2}\right)$, and the inequality $d(x_1, x_2) \leq \lambda d(x_0, x_1)$. By induction, we obtain $d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1), n = 0, 1, 2 \ldots$. Thus, for any numbers $n, m \in \mathbb{N}, m > 0$ we have

(12) $$d(x_n, x_{n+m}) \leq \sum_{k=n}^{n+m-1} d(x_k, x_{k+1}) \leq \lambda^n \frac{(1-\lambda^m)}{1-\lambda} d(x_1, x_0).$$

Since $\lambda \in [0, 1)$ it results that $\lambda^n \to 0$ which shows us that the sequence $\{x_n\}$ is a Cauchy sequence in $\bigcup_{i=1}^{p} A_i$, a subspace of a complete metric space. Consequently $\{x_n\}$ converges to some $x^* \in \bigcup_{i=1}^{p} A_i$. However in view of (6) the sequence $\{x_n\}$ has an infinite number of terms in each A_i, for all $i \in \{1, 2, \ldots, p\}$. Therefore $x^* \in \bigcap_{i=1}^{p} A_i$. So $\bigcap_{i=1}^{p} A_i \neq \emptyset$.

Now, we will prove that x^* is a fixed point of T. Let $i \in \{1, 2, \ldots, p\}$ such that $x^* \in A_i$ and $Tx^* \in A_{i+1}$. Then, by triangle inequality and (7), we get

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \leq d(x^*, x_{n+1}) + a[d(x_n, x_{n+1}) + d(x^*, Tx^*)].$$

Taking the limit when $n \to \infty$, we obtain $d(x^*, Tx^*) = 0$, i.e. x^* is a fixed point of T. We still have to prove that x^* is the unique fixed point of T. Arguing by contradiction, suppose there exists $y^* \in \bigcap_{i=1}^{p} A_i$ such that $x^* \neq y^*$ and $Ty^* = y^*$. From (7) we have

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq a[d(x^*, Tx^*) + d(y^*, Ty^*)].$$

It results that $d(x^*, y^*) = 0$, a contradiction.

Letting $m \to \infty$ in (12) we obtain the a priori estimate (9). Taking $x := x_{n-1}$ and $y := x_n$ in (7) we find: $d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$, and hence, by induction,

$$d(x_{n+k}, x_{n+k+1}) \leq \lambda^{k+1} d(x_{n-1}, x_n), k \geq 0,$$

which yields

$$d(x_n, x_{n+m}) \leq \sum_{k=0}^{m-1} d(x_{n+k}, x_{n+k+1}) \leq$$
$$\leq \sum_{k=0}^{m-1} \lambda^{k+1} d(x_{n-1}, x_n) \leq \frac{\lambda}{1-\lambda}(1 - \lambda^m) d(x_{n-1}, x_n).$$

Letting $m \to \infty$ we obtain the a posteriori estimate (10).

(iv) Let $i \in \{1, 2, \dots, p\}$ and $x \in A_i, y \in A_{i+1}$. By (7), and the triangle rule, we obtain:

$$d(Tx,Ty) \le a\big[d(x,Tx)+d(y,Ty)\big] \le a\big\{[d(x,y)+d(y,Ty)+d(Ty,Tx)]+d(y,Ty)\big\},$$

which yields $d(Tx,Ty) \le \frac{a}{1-a}d(x,y)+\frac{2a}{1-a}d(y,Ty)$, for all $x \in A_i, y \in A_{i+1}, 1 \le i \le p$. Now, taking $x := x_{n-1}$ and $y := x^*$ (since $x^* \in \bigcap_{i=1}^p A_i$), we obtain the relation (11). □

Notice that the assumption (i) in Theorem 2.1 was proved in [14] using fixed point structure arguments.

3. Fixed point theorem for cyclic Zamfirescu operators

Zamfirescu's theorem is a generalization of Banach's, Kannan's and Chatterjea's fixed point theorems. About the Zamfirescu's fixed point theorem we can assert the following result.

Theorem 3.1. Let $A_1, A_2, \dots, A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space X and suppose $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a cyclical operator, and there exist real numbers $a \in [0,1), b \in [0, \frac{1}{2})$ and $c \in [0, \frac{1}{2})$ such that for each pair $(x,y) \in A_i \times A_{i+1}$, for $1 \le i \le p$, at least one of the following is true:

(z1) $d(Tx,Ty) \le ad(x,y)$;
(z2) $d(Tx,Ty) \le b\big[d(x,Tx) + d(y,Ty)\big]$;
(z3) $d(Tx,Ty) \le c\big[d(x,Ty) + d(y,Tx)\big]$.

Then

(i) T has a unique fixed point x^* in $\bigcap_{i=1}^p A_i$.
(ii) The Picard iteration $\{x_n\}$ given by (8) converges to x^* for any starting point $x_0 \in \bigcup_{i=1}^p A_i$;
(iii) The following error estimates hold

$$d(x_n, x^*) \le \frac{\lambda^n}{1-\lambda}d(x_0, x_1), n \ge 0; \tag{13}$$

$$d(x_{n+1}, x^*) \le \frac{\lambda}{1-\lambda}d(x_n, x_{n+1}), n > 0; \tag{14}$$

(iv) The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \le \lambda d(x_{n-1}, x^*), n = 1, 2, \dots \tag{15}$$

where $\lambda = \max\big\{a, \frac{b}{1-b}, \frac{c}{1-c}\big\}$.

Proof. Let $i \in \{1, 2, \dots, p\}$ and two points $x \in A_i, y \in A_{i+1}$. Using the metric axiom's it is easy to prove that each one of the three relations (z1), (z2), (z3) can be written in the following equivalent manner (see [1]):

$$d(Tx,Ty) \le \lambda d(x,y) + 2\lambda d(x,Tx), \tag{16}$$

and

$$d(Tx, Ty) \le \lambda d(x, y) + 2\lambda d(x, Ty), \tag{17}$$

where $\lambda := \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$.

(i) Let $x_0 \in \bigcup_{i=1}^{p} A_i$ and let $x_n = T^n x_0, n = 1, 2, \ldots$, be the Picard sequence. It follows that there exist $i \in \{1, 2, \ldots, p\}$ such that $x_0 \in A_i$ and $x_1 = Tx_0 \in A_{i+1}$, due to (6). In addition, from (17) we get $d(x_1, x_2) \le \lambda d(x_0, x_1)$, which can be generalized by induction to $d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1), n \ge 0$. Thus, for any numbers $n, m \in \mathbb{N}, m > 0$ we have

$$d(x_n, x_{n+m}) \le \sum_{k=n}^{n+m-1} d(x_k, x_{k+1}) \le \frac{\lambda^n(1-\lambda^m)}{1-\lambda} d(x_1, x_0). \tag{18}$$

Now, in a similar manner to that in the proof of Theorem 2.1 we deduce that $\{x_n\}$ is a Cauchy sequence for each $x_0 \in \bigcup_{i=1}^{p} A_i$ and hence a convergent sequence, too. Let x^* be it's limit. In view of (6) an infinite number of terms of this sequence lie in each A_i, for all $i = 1, 2, \ldots, p$. Therefore $x^* \in \bigcap_{i=1}^{p} A_i \ne \emptyset$. To prove that x^* is a fixed point of T we will use (16):

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_n, Tx^*) \le \lim_{n\to\infty} [\lambda d(x_{n-1}, x^*) + 2\lambda d(x_{n-1}, x_n)] = 0.$$

Therefore $d(x^*, Tx^*) = 0$. Now, suppose that T has another fixed point $y^* \in \bigcap_{i=1}^{p} A_i$, $x^* \ne y^*$. Again, by using (16), we obtain

$$d(x^*, y^*) = d(Tx^*, Ty^*) \le \lambda d(x^*, y^*) + 2\lambda d(x^*, Tx^*),$$

which implies $d(x^*, y^*) = 0$, since $\lambda < 1$, i.e., x^* is the unique fixed point of T in $\bigcap_{i=1}^{p} A_i$.

(iii) Letting $m \to \infty$ in (18) we obtain the a priori estimate (13). Taking $x := x_n$ and $y := x_{n-1}$ in (17) we find:

$$d(x_n, x_{n+1}) \le \lambda d(x_{n-1}, x_n), \tag{19}$$

and hence, by induction:

$$d(x_{n+k}, x_{n+k+1}) \le \lambda^{k+1} d(x_{n-1}, x_n), k \ge 0,$$

which yields

$$d(x_n, x_{n+m}) \le \sum_{k=0}^{m-1} d(x_{n+k}, x_{n+k+1}) \le$$

$$\le \sum_{k=0}^{m-1} \lambda^{k+1} d(x_{n-1}, x_n) \le \frac{\lambda(1-\lambda^m)}{1-\lambda} d(x_{n-1}, x_n).$$

Letting $m \to \infty$ in the last inequality we obtain the a posteriori estimate (14).

(iv) Let $x \in \bigcup_{i=1}^{p} A_i$. Then for any $n > 0$ there exists $i_n \in \{1, 2, \ldots, p\}$ such that $x_n \in A_{i_n}$. Since $x^* \in \bigcap_{i=1}^{p} A_i$ we can consider $x^* \in A_{i_n+1}$. Then by (16), with $x := x^*$ and $y := x_n$, we get the desired inequality. $\square$

4. Remarks and Examples

In a similar manner, it is easy to prove the following fixed point result which extends Banach's fixed point theorem, when the map T is defined on the union of $p > 1$ nonempty sets. This result completes Theorem 1.1 with error estimates and the rate of the convergence.

Corollary 4.1. Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space, and suppose $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a cyclical mapping, i.e.

$$T(A_i) \subseteq A_{i+1} \text{ for all } i \in \{1, 2, \ldots, p\},$$

where $A_{p+1} = A_1$, and

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x \in A_i$, $y \in A_{i+1}$, $1 \leq i \leq p$, where $a \in [0, 1)$ is a constant. Then

(i) T has a unique fixed point x^* in $\bigcap_{i=1}^p A_i$;

(ii) The Picard iteration $\{x_n\}$ defined by (8) converges to x^*, for any $x_0 \in \bigcup_{i=1}^p A_i$;

(iii) The following a priori and a posteriori error estimates

$$d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1), n \geq 0; \tag{20}$$

$$d(x_n, x^*) \leq \frac{a}{1-a} d(x_{n-1}, x_n), n > 0; \tag{21}$$

hold;

(iv) The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \leq ad(x_{n-1}, x^*), n = 1, 2, \ldots \tag{22}$$

We will illustrate the obtained results by some examples.

Example 4.1: Consider the space $C_{[0,1]}$ endowed with the metric $d(f, g) = \max\limits_{x\in[0,1]} |f(x) - g(x)|$.

Define the subsets $A = \overline{\left\{\sum_{k=0}^n \alpha_k x^{2k}, \alpha_k \geq 0, \sum_{k=1}^n \alpha_k \leq 1, n \in \mathbb{N}\right\}}$ and $B = \overline{\left\{\sum_{k=1}^n \beta_k x^{2k-1}, \beta_k \geq 0, \sum_{k=1}^n \beta_k \leq 1, n \in \mathbb{N}\right\}}$ of $C_{[0,1]}$: By definition the sets A and B are closed and bounded subsets in $C_{[0,1]}$. Define the map $T : C_{[0,1]} \to C_{[0,1]}$ by $Tf = 1/2 \int_0^x f(t)dt$. We will show that the map T satisfies the conditions of Theorem 1.1. Let $f \in A$. We need the show that Tf is in B.

Case I) Let $f(x) = \sum_{k=1}^n \alpha_k x^{2k}$, $\alpha_k \geq 0$, $\sum_{k=1}^n \alpha_k \leq 1$. Then

$$Tf = 1/2 \int_0^x f(t)dt = 1/2 \int_0^x \sum_{k=1}^n \alpha_k t^{2k} = \sum_{k=1}^n \frac{\alpha_k}{2(2k+1)} x^{2k+1} \in B,$$

because $\sum_{k=1}^n \frac{\alpha_k}{2(2k+1)} \leq 1$.

Case II) There exists a sequence $f_n(x) = \sum_{k=1}^{p_n} \alpha_k^{(n)} x^{2k}$, $\alpha_k^{(n)} \geq 0$, $\sum_{k=1}^n \alpha_k^{(n)} \leq 1$, which is uniformly convergent to f. By the uniform convergence of $\{f_n\}_{n=1}^\infty$ we have that $\lim_{n\to\infty} \frac{1}{2} \int_0^x f_n(t)dt = \frac{1}{2} \int_0^x f(t)dt$.

By Case I) we have $\frac{1}{2} \int_0^x f_n(t)dt = \sum_{k=1}^{p_n} \beta_k^{(n)} x^{2k+1} = g_n \in B$, where $\beta_k^{(n)} = \frac{\alpha_k^{(n)}}{2(2k+1)}$. By the uniform convergence of the sequence $\{f_n\}_{n=1}^\infty$ we have that for every

$\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for every $n, m \geq N$ holds

$$\max_{x\in[0,1]} |f_n(x) - f_m(x)| < \varepsilon.$$

Then it is easy to see that
$\max_{x\in[0,1]} |g_n(x) - g_m(x)| \leq \frac{1}{2} \max_{x\in[0,1]} \left|\sum_{k=1}^{p_n} \alpha_k^{(n)} x^{2k} - \sum_{k=1}^{p_m} \alpha_k^{(m)} x^{2k}\right|$
$= \frac{1}{2} \max_{x\in[0,1]} |f_n(x) - f_m(x)| < \varepsilon$, which ensures that $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in B. Therefore there exists $g \in B$, such that $\lim_{n\to\infty} g_n = g$ and thus $\frac{1}{2}\int_0^x f(t)dt = g \in B$.

The proof that $T(B) \subseteq A$ is similar. It is well known that T is a contraction. The constant zero is a fixed point of the map T and $0 \in A \cap B$.

Let us mention that the sets A and B consists not only of polynomial functions. For example $\frac{1}{4}e^{x^2} \in A$. Indeed $\frac{1}{4}e^{x^2} = \lim_{n\to\infty}\left(\frac{1}{4} + \sum_{k=1}^{n} \frac{x^{2k}}{4.k!}\right)$. Thus the function $\frac{1}{8}\int_0^x e^{x^2} dx$ is in B.

Example 4.2: Consider the function $f(x) = -\frac{\mathrm{sign}(x)}{3}|x\sin(1/x)|$ if $x \neq 0$ and $f(0) = 0$. Obviously $f : [0, \pi] \to [-\pi, 0]$ and $f : [-\pi, 0] \to [0, \pi]$. It is easy to see that for $x \in [0, \pi]$ and $y \in [-\pi, 0]$ holds $|f(x) - f(y)| \leq \frac{1}{3}(|f(x) - x| + |f(y) - y|)$ and for $y \in [0, \pi]$ and $x \in [-\pi, 0]$ holds $|f(x) - f(y)| \leq \frac{1}{3}(|f(x) - x| + |f(y) - y|)$. So f satisfies all the conditions of Theorem 2.1 and thus it has a fixed point which is the intersection of the sets $[0, \pi]$ and $[-\pi, 0]$.

It is interesting in this example that there is no a constant $a > 0$ such that $|f(x) - f(y)| \leq a|x - y|$. Indeed if we take $x_n = \frac{1}{2n\pi + \frac{1}{n}}$ and $y_n = \frac{1}{2n\pi}$ then we have $\lim_{n\to\infty} \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \lim_{n\to\infty} \frac{2n\pi\sin(1/n)}{3/n} = +\infty$ and therefore there is no $a > 0$ so that $|f(x_n) - f(y_n)| \leq a|x_n - y_n|$.

Example 4.3: Consider the map $T(\{x_k\}) = \{f(x_k)\}_{k=1}^{\infty}$, where f is the function defined in Example 4.2, $T : \ell_2 \to \ell_2$.

Consider the sets $A_1 = \{\{x_k\}_{k=1} : \sum_{k=1}^{\infty} x_k^2 \leq 1, x_k \geq 0\}$ and $A_2 = \{\{x_k\}_{k=1} : \sum_{k=1}^{\infty} x_k^2 \leq 1, x_k \leq 0\}$. Obviously $T : A_1 \to A_2$ and $T : A_2 \to A_1$. It is easy to show, that for $x = \{x_k\}_{k=1}^{\infty}$ and $y = \{y_k\}_{k=1}^{\infty} \in A_2$ holds $\|Tx - Ty\| \leq \dfrac{1}{3}(\|Tx - x\| + \|Ty - y\|)$. So T satisfies all the conditions of Theorem 2.1 and thus it has a fixed point which is the intersection of the sets A_1 and A_2.

Example 4.4: Consider the sets: $A_1 = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \cup \left\{\frac{-1}{2n}\right\}_{n=1}^{\infty}$ and $A_2 = \{0\} \cup \left\{\frac{-1}{n}\right\}_{n=1}^{\infty} \cup \left\{\frac{1}{2n-1}\right\}_{n=1}^{\infty}$. Define the map

$$Tx = \begin{cases} -\dfrac{x}{x+4}, & x \in A_1 \\ -\dfrac{x}{4}, & x \in A_2 \end{cases}$$

It is easily to be checked that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$.

For any $x \in A_1$ and any $y \in A_2$ we have the chain of inequalities $|Tx - Ty| = \left|\frac{x}{x+4} - \frac{y}{4}\right| \leq \frac{1}{3}(|x| + |y|) \leq \frac{1}{3}\left(\left|\frac{-x}{x+4} - x\right| + \left|\frac{-y}{4} - y\right|\right) = \frac{1}{3}(|Tx - x| + |Ty - y|)$. So T satisfies all the conditions of Theorem 2.1 and thus it has a fixed point $x^* = 0 \in A_1 \cap A_2$. It is interesting in this example that the intersection of A_1 and A_2 is with empty interior.

Example 4.5: Consider the sets: $A = [0,1] \bigcup \left(\bigcup_{n=1}^{\infty} \left[\frac{-1}{2^{2n-1}}, \frac{-1}{2^{2n}} \right] \right)$, $B = [-1,0] \bigcup \left(\bigcup_{n=1}^{\infty} \left[\frac{1}{2^{2n}}, \frac{1}{2^{2n-1}} \right] \right)$. Define the map $Tx = \frac{-x}{4}$. Then T satisfies the conditions of Theorem 1.1.

It is easy to see that $T\left(\bigcup_{n=1}^{\infty} \left[\frac{-1}{2^{2n-1}}, \frac{-1}{2^{2n}} \right] \right) \subseteq \left(\bigcup_{n=1}^{\infty} \left[\frac{1}{2^{2n}}, \frac{1}{2^{2n-1}} \right] \right)$ and $T\left(\bigcup_{n=1}^{\infty} \left[\frac{1}{2^{2n}}, \frac{1}{2^{2n-1}} \right] \right) \subseteq \left(\bigcup_{n=1}^{\infty} \left[\frac{-1}{2^{2n-1}}, \frac{-1}{2^{2n}} \right] \right)$ and 0 is the fixed point of T. The interior of the intersection $A \cap B$ is not an empty set, but the fixed point of T is not in the interior of $A \cap B$.

For any $x_0 \in A \cap B$, $x_n = Tx_{n-1} \in A \cap B$ and for any $x_0 \notin A \cap B$, $x_n = Tx_{n-1} \notin A \cap B$.

References

[1] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, Berlin, 2007 Rev. Roumaine Math. Pures Appl., 50 (2005), nos 5-6, 443-453

[2] V. Berinde, *Error estimates for approximating fixed points of quasi contractions*, General Mathematics Vol. 13, No. 2 (2005), 23-34

[3] S.K. Chatterjea, *Fixed-point theorems*, C.R. Acad. Bulgare Sci. 25(1972) 727-730

[4] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. 10(1968) 71-76

[5] W.A. Kirk, P.S. Srinivasan , P. Veeramani, *Fixed Points For Mappings Satisfying Cyclical Contractive Conditions*, Fixed Point Theory, Volume 4 No. 1(2003), 79-89

[6] M. Păcurar,I.A. Rus, *Fixed point theorems for cyclic φ-contractions*, Nonlinear Analysis: Theory, Methods and Applications, Vol 72, Issues 3-4, 1 February 2010, 1181-1187

[7] M.A. Petric, *Some results concerning cyclical contractive conditions*, General Mathematics (accepted)

[8] M.A. Petric, *Some remarks concerning Círíc-Reich-Rus operators*, Creative Math. and In., Vol 18(2009), no. 2, 188-193

[9] G. Petruşel, *Cyclic representation and periodic points*, Studia Univ. "Babeş-Bolyai" , Mathematica, Volume L, Number 3, September 2005, 107-112.

[10] P. D. Proinov, *A generalization of the Banach contraction principle with high order of convergence of successive approximations*, Nonlinear Anal. 67 (2007) 2361-2369.

[11] B.E. Rhoades, *A comparison of various definitions of contractive mappings*,Trans. Amer. Math. Soc. 226 (1977) 257-290

[12] I.A. Rus, *Principles and Applications of the Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979

[13] I.A. Rus,*Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001

[14] I.A. Rus, *Cyclic representations and fixed points*, Ann. T. Popoviciu Seminar Funct. Eq. Approx. Convexity 3 (2005) 171-178.

[15] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. (Basel), 23 (1972), 292-298

Mihaela Petric
Dep. of Mathematics and Computer Science
North University of Baia Mare,
e-mail: PetricMihaela@yahoo.com

Boyan Zlatanov
Fac. of Mathematics and Informatics
Plovdiv University,
e-mail: bobbyz@uni-plovdiv.bg