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ON ANOTHER PROOF OF THE SCHUR PROPERTY IN MUSIELAK–ORLICZ SEQUENCE SPACES[1]

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Abstract. We show that if the dual of a Musielak–Orlicz sequence space ℓ_Φ is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis then ℓ_Φ has the Schur property.

Key words: Musielak–Orlicz Sequence Spaces
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1. Introduction

A Banach space X is said to have Schur property if every weakly null sequence is norm null. It is well known that ℓ_1 has Schur property and it's dual ℓ_∞ is stabalized asymptotic ℓ_∞ space with respect to the unit vector basis. The only ℓ_p space that has Schur property is ℓ_1 and thus if there is isomorphic copy of ℓ_p, $p > 1$ in X then X has not Schur property. If $\{X_\alpha\}_{\alpha\in I}$, I an index set, X_α are Banach spaces and and $X = (\oplus_{\alpha\in I} X_\alpha)_1$ is their ℓ_1–sum, then the space X has Schur property iff each factor X_α has it [10]. A Mushielak–Orlicz sequence space has Schur property, provided its dual is stabalized asymptotic ℓ_∞ space with respect to the unit vector basis [12]. The proof is based on a result of Kaminska and Masylo [4]. It turns out that there is a direct proof using the ideas going back to the time of Banach.

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2. Preliminaries

A standard Banach space terminology can be found in [5].

Let us recall that an Orlicz function M is even, continuous, nondecreasing, convex function defined for $t \geq 0$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We say that M is non–degenerate Orlicz function if $M(t) > 0$ for every $t > 0$. A sequence $\Phi = \{\Phi_i\}_{i=1}^{\infty}$ of Orlicz functions is called a Musielak–Orlicz function.

The Musielak–Orlicz sequence space ℓ_Φ, generated by a Musielak–Orlicz function Φ is the set of all real sequences $\{x_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \Phi_i(\lambda x_i) < \infty$ for some $\lambda > 0$. The Luxemburg's norm is defined by

$$\|x\|_\Phi = \inf \left\{ r > 0 : \sum_{i=1}^{\infty} \Phi_i(x_i/r) \leq 1 \right\}.$$

We denote by h_Φ the closed linear subspace of ℓ_Φ, generated by all $x \in \ell_\Phi$, such that $\sum_{i=1}^{\infty} \Phi_i(\lambda x_i) < \infty$ for every $\lambda > 0$.

If the Musielak–Orlicz function Φ consists of one and the same function M, then we obtain the Orlicz sequence spaces ℓ_M and h_M.

Let $1 \leq p_i$, $i \in \mathbb{N}$ be a sequence of reals. The Musielak–Orlicz sequence space ℓ_Φ, where $\Phi = \{t^{p_i}\}_{i=1}^{\infty}$ is called Nakano sequence space and is denoted by $\ell_{\{p_i\}}$.

An extensive study of Orlicz and Musielak–Orlicz spaces can be found in [5], [9].

Definition 2.1. *We say that a Musielak–Orlicz function Φ satisfies δ_2 condition at zero if there exist $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$ such that*

$$\Phi_n(2t) \leq K\Phi_n(t) + c_n$$

for every $t \geq \mathbb{R}$ and $n \in \mathbb{N}$ if $\Phi_n(t) \leq \beta$.

The spaces ℓ_Φ and h_Φ coincide iff Φ has δ_2 condition at zero.

Recall that given Musielak–Orlicz functions Φ and Ψ the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms iff Φ is equivalent to Ψ that is for some constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$ it holds

$$\Phi_n(Kt) \leq \Psi_n(t) + c_n \quad \text{and} \quad \Psi_n(Kt) \leq \Phi_n(t) + c_n$$

for every $t \geq \mathbb{R}$ and $n \in \mathbb{N}$ such that the first inequality is satisfied if $\Psi_n(t) \leq \beta$ and the second one holds if $\Phi_n(t) \leq \beta$ (e.g. [3]).

Throughout this paper Φ will always denote a Musielak–Orlicz function. As the Schur property is preserved by isomorphisms without loss of generality we may assume that Φ consists entirely of non–degenerate Orlicz functions, such that for every $i \in \mathbb{N}$ the Orlicz function Φ_i is differentiable, $\Phi_i^{'}(0) = 0$ and $\Phi_i(1) = 1$ [12].

Definition 2.2. *For a Musielak–Orlicz function $\Phi = \{\Phi_j\}_{j=1}^{\infty}$, such that $\lim_{t\to 0} \Phi_j(t)/t = 0$ for every $j \in \mathbb{N}$ define*

$$\Psi_j(x) = \sup\{t|x| - \Phi_j(t) : t \geq 0\}$$

and $\Psi = \{\Psi_j\}_{j=1}^{\infty}$ is called function complementary to Φ.

Let us note that the condition $\lim_{t\to 0} \Phi_j(t)/t = 0$ for every $j \in \mathbb{N}$ secures that the complementary function G_j is always non-degenerate. Observe that if the Musielak–Orlicz function Ψ is complementary to the Musielak–Orlicz function Φ, then Φ is function complementary to Ψ.

Throughout this paper the function complementary to the Musielak–Orlicz function Φ is denoted by Ψ.

It is well known that $h_\Phi^* \cong \ell_\Psi$. Well known equivalent norm in ℓ_Φ is the Orlicz norm $\|x\|_\Phi^O = \sup\left\{\sum_{j=1}^{\infty} x_j y_j : \sum_{j=1}^{\infty} \Psi_j(y_j) \leq 1\right\}$, which satisfies the inequalities (see e.g.[2])

$$\|\cdot\|_\Phi \leq \|\cdot\|_\Phi^O \leq 2\|\cdot\|_\Phi .$$

We will use the Hölder's inequality: $\sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_\Phi^O \|y\|_\Psi$, which holds for every $x = \{x_j\}_{j=1}^{\infty} \in \ell_\Phi$ and $y = \{y_j\}_{j=1}^{\infty} \in \ell_\Psi$, where Φ and Ψ are complementary Musielak–Orlicz functions.

By $\{e_j\}_{j=1}^{\infty}$ and $\{e_j^*\}_{j=1}^{\infty}$ we denote the unit vector basis in h_Φ and h_Ψ respectively. For a Banach space X with a basis $\{v_i\}_{i=1}^{\infty}$ and an element $x \in X$, $x = \sum_{i=1}^{\infty} x_i v_i$ we define supp $x = \{i \in \mathbb{N} : x_i \neq 0\}$. We write $n \leq x$ if $n \leq \min\{\text{supp } x\}$ and $x < y$ if $\max\{\text{supp } x\} < \min\{\text{supp } y\}$. We say that x is a block vector with respect to the basis $\{v_i\}_{i=1}^{\infty}$ if $x = \sum_{i=p}^{q} x_i v_i$ for some finite p and q and we say that x is a normalized block vector if it is a block vector and $\|x\| = 1$.

The notion of stabilized asymptotic ℓ_p spaces first appeared in [8] under the name of asymptotic ℓ_p spaces.

Definition 2.3. *A Banach space X is said to be stabilized asymptotic ℓ_∞ with respect to a basis $\{v_i\}_{i=1}^{\infty}$, if there exists a constant $C \geq 1$, such that for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$, so that whenever $N \leq x_1 < \cdots < x_n$ are successive normalized block vectors, then $\{x_i\}_{i=1}^{n}$ are C–equivalent to the unit vector basis of ℓ_∞^n, i.e.*

$$\frac{1}{C}\max_{1\le i\le n}|a_i| \le \left\|\sum_{i=1}^{n} a_i x_i\right\| \le C\max_{1\le i\le n}|a_i|.$$

The following characterization of the stabilized asymptotic ℓ_∞ Musielak–Orlicz sequence spaces is due to Dew:

Proposition 2.1. *(Proposition 4.5.1 [1]) Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a Musielak–Orlicz function. Then the following are equivalent:*
(i) h_Φ is stabilized asymptotic ℓ_∞ (with respect to its natural basis $\{e_j\}_{j=1}^\infty$);
(ii) there exists $\lambda > 1$ such that for all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $N \le p \le q$ and $\sum_{j=p}^{q}\Phi_j(a_j) \le 1$, then

$$\sum_{j=p}^{q}\Phi_j(a_j/\lambda) \le \frac{1}{n}.$$

3. Schur Property in Musielak–Orlicz Sequence Spaces

Theorem 1. *Let Φ be a Musielak–Orlicz function, which has δ_2 condition at zero and has a complementary function Ψ. Let h_Ψ is sabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ then ℓ_Φ has Schur property.*

Proof. Let h_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ but ℓ_Φ fail Schur property i.e. there exists a sequence $\{x^{(n)}\}_{n=1}^\infty$ which is weakly null in ℓ_Φ and such that $\|x^{(n)}\| \ge \varepsilon$ for some $\varepsilon > 0$ and every $n \in \mathbb{N}$. Thus by the equivalence of the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Phi^O$ it follows that $\|x^{(n)}\|_\Phi^O \ge 2\varepsilon$, i.e.

$$\sup\left\{\sum_{j=1}^{\infty} x_j^{(n)} y_j : \|y\|_\Psi \le 1\right\} \ge 2\varepsilon.$$

Therefore for every $n \in \mathbb{N}$ there exists $y^{(n)} \in h_\Psi$ such that $\|y^{(n)}\| = 1$ and $\sum_{j=1}^{\infty} x_j^{(n)} y_j^{(n)} \ge \varepsilon$. Because of the convergence of the last sum there exists $N_n \in \mathbb{N}$, such that

$$\sum_{j=N_n+1}^{\infty} x_j^{(n)} y_j^{(n)} \le \frac{\varepsilon}{5},\quad \sum_{j=1}^{N_n} x_j^{(n)} y_j^{(n)} \ge \frac{4\varepsilon}{5},\quad \left\|\sum_{j=N_n+1}^{\infty} x_j^{(n)} e_j\right\| \le \frac{\varepsilon}{10\lambda},$$

where $\lambda \geq 1$ is the constant from Proposition 2.1.

Now we will choose inductively a subsequence $\{x^{(n_i)}\}_{i=1}^{\infty}$ and a sequence $\{y^{(i)}\}_{i=1}^{\infty}$.

I) Let $n_1 = 1$. We choose a norm one vector $y^{(1)} = \{y_j^{(1)}\}_{j=1}^{\infty}$ in h_Ψ so that

$$\text{sign } y_j^{(1)} = \text{sign } x_j^{(1)} \quad \text{and} \quad \sum_{j=1}^{\infty} x_j^{(1)} y_j^{(1)} \geq \varepsilon.$$

Then we choose $N_1 \in \mathbb{N}$ so that

$$\sum_{j=N_1+1}^{\infty} x_j^{(1)} y_j^{(1)} \leq \frac{\varepsilon}{5}, \ \sum_{j=1}^{N_1} x_j^{(1)} y_j^{(1)} \geq \frac{4\varepsilon}{5}, \ \left\| \sum_{j=N_1+1}^{\infty} x_j^{(1)} e_j \right\| \leq \frac{\varepsilon}{10\lambda}.$$

II) Because $x^{(n)}$ is weakly null sequence we have that $\lim_{n\to\infty} x_j^{(n)} = 0$ for every $j \in \mathbb{N}$. Thus we can choose $n_2 \in \mathbb{N}$, $n_2 > n_1 = 1$ such that

$$\sum_{j=1}^{N_1} |x_j^{(n_2)}| \leq \frac{\varepsilon}{5}.$$

Now we choose a norm one vector $y^{(2)} = \{y_j^{(n_2)}\}_{j=1}^{\infty}$ in h_Ψ so that

$$\text{sign } y_j^{(2)} = \text{sign } x_j^{(n_2)} \quad \text{and} \quad \sum_{j=1}^{\infty} x_j^{(n_2)} y_j^{(2)} \geq \varepsilon.$$

Then we choose $N_2 \in \mathbb{N}$, $N_2 > N_1$, such that

$$\sum_{j=N_2+1}^{\infty} x_j^{(n_2)} y_j^{(2)} \leq \frac{\varepsilon}{5}, \ \sum_{j=1}^{N_2} x_j^{(n_2)} y_j^{(2)} \geq \frac{4\varepsilon}{5}, \ \left\| \sum_{j=N_2+1}^{\infty} x_j^{(n_2)} e_j \right\| \leq \frac{\varepsilon}{10\lambda}.$$

Thus

$$\sum_{j=N_1+1}^{N_2} x_j^{(n_2)} y_j^{(2)} = \sum_{j=1}^{N_2} x_j^{(n_2)} y_j^{(2)} + \sum_{j=1}^{N_1} x_j^{(n_2)} y_j^{(2)} \geq \frac{4\varepsilon}{5} - \sum_{j=1}^{N_1} |x_j^{(n_2)}| \geq \frac{4\varepsilon}{5} - \frac{\varepsilon}{5} = \frac{3\varepsilon}{5}.$$

III) If we have chosen $\{n_i\}_{i=1}^{k-1}$, $\{x^{(n_i)}\}_{i=1}^{k-1}$, $\{y^{(i)}\}_{i=1}^{k-1}$, $\{N_i\}_{i=1}^{k-1}$, then we choose $n_k \in \mathbb{N}$, $n_k > n_{k-1}$ such that

$$\sum_{j=1}^{N_{k-1}} |x_j^{(n_k)}| \leq \frac{\varepsilon}{5}.$$

Now we choose a norm one vector $y^{(k)} = \{y_j^{(k)}\}_{j=1}^{\infty}$ in h_Ψ so that

$$\text{sign } y_j^{(k)} = \text{sign } x_j^{(n_k)} \quad \text{and} \quad \sum_{j=1}^{\infty} x_j^{(n_k)} y_j^{(k)} \geq \varepsilon.$$

Then we choose $N_k \in \mathbb{N}$, $N_k > N_{k-1}$ such that

$$\sum_{j=N_k+1}^{\infty} x_j^{(n_k)} y_j^{(k)} \leq \frac{\varepsilon}{5}, \quad \sum_{j=1}^{N_k} x_j^{(n_k)} y_j^{(k)} \geq \frac{4\varepsilon}{5}, \quad \left\| \sum_{j=N_k+1}^{\infty} x_j^{(n_k)} e_j \right\| \leq \frac{\varepsilon}{10\lambda}.$$

Thus

$$\sum_{j=N_{k-1}+1}^{N_k} x_j^{(n_k)} y_j^{(k)} = \sum_{j=1}^{N_k} x_j^{(n_k)} y_j^{(k)} + \sum_{j=1}^{N_{k-1}} x_j^{(n_k)} y_j^{(k)} \geq \frac{4\varepsilon}{5} - \sum_{j=1}^{N_{k-1}} |x_j^{(n_k)}| \geq \frac{4\varepsilon}{5} - \frac{\varepsilon}{5} = \frac{3\varepsilon}{5}.$$

Let consider the block basic sequence $y^{(k)} = \displaystyle\sum_{j=N_{k-1}+1}^{N_k} y_j^{(k)} e_j^*$. We claim that

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \Psi_j \left(\frac{y_j^{(n)}}{\lambda} \right) = \lim_{n\to\infty} \sum_{j=p_n}^{q_n} \Psi_j \left(\frac{y_j^{(n)}}{\lambda} \right) = 0, \tag{1}$$

where $\lambda \geq 1$ is the constant from Proposition 2.1.

Indeed by the assumption that h_Ψ is stabilized asymptotic ℓ_∞ space, there exists $\lambda > 1$ such that for every $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ so that whenever $N < N_{k-1} < N_k$ and $\displaystyle\sum_{j=N_{k-1}}^{N_k} \Psi_j(y_j^{(n)}) \leq 1$, then holds $\displaystyle\sum_{j=N_{k-1}}^{N_k} \Psi_j \left(\frac{y_j^{(n)}}{\lambda} \right) \leq 1/m$ for every $N < N_{k-1} < N_k$. Thus $\displaystyle\lim_{n\to\infty} \sum_{j=N_{k-1}}^{N_k} \Psi_j \left(\frac{y_j^{(n)}}{\lambda} \right) = 0$.

Now by (1) it follows that there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of naturals, such that

$$\sum_{i=1}^{\infty} \sum_{j=N_{k_i-1}+1}^{N_{k_i}} \Psi_j \left(\frac{y_j^{(k_i)}}{\lambda} \right) \leq 1. \tag{2}$$

Let $y = \sum_{i=1}^{\infty} y^{(k_i)}$. By (2) $y \in \ell_\Psi$ and $\|y\| \leq \lambda$. So we can write the chain of inequalities:

$$\begin{aligned}
\|y(x^{(n_{k_s})})\| &= \left|\sum_{i=1}^{\infty}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} x_j^{(n_{k_s})} y_j^{(k_i)}\right| \geq \left|\sum_{j=N_{k_{s-1}}+1}^{N_{k_s}} x_j^{(n_{k_s})} y_j^{(k_s)}\right| \\
&- \left|\sum_{i=1}^{s-1}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} x_j^{(n_{k_s})} y_j^{(k_i)}\right| - \left|\sum_{i=s+1}^{\infty}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} x_j^{(n_{k_s})} y_j^{(k_i)}\right| \\
&\geq \frac{3\varepsilon}{5} - \sum_{i=1}^{s-1}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} \left|x_j^{(n_{k_s})}\right| - \sum_{i=s+1}^{\infty}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} \left|x_j^{(n_{k_s})} y_j^{(k_i)}\right| \\
&\geq \frac{3\varepsilon}{5} - \sum_{i=1}^{N_{k_{s-1}}} |x_j^{(n_{k_s})}| \\
&\quad - \left\|\sum_{i=s+1}^{\infty}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} x_j^{(n_{k_s})} e_j\right\|_{\Phi}^{O} \left\|\sum_{i=s+1}^{\infty}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} y_j^{(k_i)} e_j^*\right\| \\
&\geq \frac{3\varepsilon}{5} - \frac{\varepsilon}{5} - 2\left\|\sum_{i=s+1}^{\infty}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} x_j^{(n_{k_s})} e_j\right\|_{\Phi} \left\|\sum_{i=1}^{\infty} y^{(k_i)}\right\| \\
&\geq \frac{2\varepsilon}{5} - 2\lambda\left\|\sum_{i=s+1}^{\infty}\sum_{j=N_{k_{i-1}}+1}^{N_{k_i}} x_j^{(n_{k_s})} e_j\right\|_{\Phi} \geq \frac{2\varepsilon}{5} - \frac{2\lambda\varepsilon}{10\lambda} = \frac{\varepsilon}{5},
\end{aligned}$$

a contradiction with the weak convergence of $\{x^{(n_{k_s})}\}_{s=1}^{\infty}$ to zero. □

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ВЪРХУ ДРУГО ДОКАЗАТЕЛСТВО НА СВОЙСТВОТО НА ШЕР В ПРОСТРАНСТВА НА МУШИЕЛАК–ОРЛИЧ

Б. Златанов

Резюме. Показали сме, че ако спрегнатото пространство на редично пространство на Мушиелак–Орлич е стабилизирано асимптотично ℓ_∞ пространство спрямо каноничния базис, то пространството ℓ_Φ притежава свойството на Шур.