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New Rules For Assessing Mathematical Competencies

User guide

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USER GUIDE

Snezhana Gocheva-Ilieva and Araceli Queiruga-Dios

Editors

2020

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New Rules for Assessing Mathematical Competencies: USER GUIDE

Snezhana Gocheva-Ilieva and Araceli Queiruga-Dios

(Editors)

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PREFACE

I hear and I forget,

I see and I remember,

I do and I understand.

This is a core principle of learning in any human activity. But in mathematical learning this principle acquires a much more relevant position.

Mathematics is mainly to know-how-to-do. It is more to know a method than knowing contents. Even the content can and should be seen in the light of the main objective of the mathematical activity. This activity consists of addressing more or less complex real-world situations that admit a very peculiar rational treatment: mathematization, which involves an adequate symbolization, a rational rigorous manipulation to reach the effective domain of reality being explored.

In addition, learning (and mathematical learning has this property as it is no stranger to human activity) is, above all, a personal activity.

However, the task of teaching must be collective. The development of quality teaching materials to guide our students along the way pointed before, cannot and should not be an individual task. New teaching challenges demand new answers, far from the memorial exposures and the merely repetitive or more or less laborious problems of the mathematical operations involved. In addition, the teaching scenario has changed radically in recent years due to the rise of information technologies. At present, the traditional class has become another scenario as we are in the field of u-learning.

In the last decade different organizations have reflected on the changes in teaching practice. The basic questions about what to teach, how to teach and how to evaluate are now answered from theoretical positions very different from the traditional ones. This leads to a change in the language used to describe the new objectives in the teaching of mathematics.

Change in teaching system has not been limited to daily teaching practice. The deepest changes in recent years have come together with changes in evaluation and assessment systems. In this sense, the report prepared by the SEFI Mathematics Working Group ("A Framework for Mathematics Curricula in Engineering Education") places special emphasis on the acquisition of certain competencies (specific and transversal competencies) by students, and also in how to certify those competencies through the new evaluation and assessment systems.

With the conducting element of this SEFI report, some educational innovation projects at national and European level have been developed. One of those European projects is the RULES_MATH Erasmus+ project in which different European institutions of higher education have participated: Slovak University of Technology in Bratislava (Slovakia), Ankara Haci Bayram Veli University

(Turkey), The Czech Technical University in Prague (Czech Republic) , University of Plovdiv Paisii Hilendarski (Bulgaria), Spanish National Research Council (Spain), Higher Institute of Engenharia de Coimbra (Portugal), Technological University Dublin (Ireland), Technical University of Civil Engineering of Bucharest (Romania), coordinated by the University of Salamanca.

The teaching teams of the universities participating in this project have been concerned with developing new materials that can be used as class materials and as assessment materials, depending on the needs of each group of students. That work is the material basis of this book. In it you can see different types of materials that can be used as part of the competencies assessment of students who take different engineering courses.

The dissemination of these materials will undoubtedly stimulate the development of new materials that will help teachers in different countries to change the current evaluation systems to adapt to the evaluation in competences that is the new educational paradigm.

Gerardo Rodríguez Sánchez

CONTENTS

List of Abbreviations and Acronyms

Acronym	Meaning
AC	Analysis and Calculus
DM	Discrete Mathematics
GE	Geometry
LA	Linear Algebra
LO	Learning Outcomes
SP	Statistics and Probability

Introduction

This document is intendend as a guide for professors, lecturers or tutors who want to implement the competencies-based assessment with their students.

In order to use a competencies-based methodology, it is necessary to change the usual way of teaching and learning. Students must improve their knowledge taking an active role in classes.

What is shown in this guide is an example of an evaluation activity.

The report: “A Framework for Mathematics Curricula in Engineering Education” (Alpers, et al., 2013) provides a European framework for the teaching of Mathematics in engineering degrees. That document includes a part which contains rather specific, fine-grained specification of potential content-oriented learning outcomes (LO). Its intended usage is that lecturers and/or curriculum designers develop from it a concrete curriculum for a specific study course.

Once the LO list has been developed with specific activities, there is a development process in which the student acquires the competencies that are enumerated through activities marked by the teacher in different scenarios: the classroom, laboratory practices, small projects, search for information on the network, etc. Through these regulated activities or others, the student must acquire some knowledge and skills that we try to evaluate with the example that is proposed. That is to say, in short, this proposal tries to measure if the student has acquired the knowledge and competencies for his activity prior to the evaluation process.

In this document, all mathematical competencies are included. We have also included the objectives, contents, and learning outcomes from the Core Level 1, and a possible exam for some specified LO. This exam could be used as a small project (miniproject) that can be worked in groups or individually or through a combination of both options (for example, by starting the work individually and then sharing and discussing the results within a group).

Competencies

The notion of mathematical competency has been presented in all the PISA mathematics frameworks from the very beginning in the late 1990s. However, the actual role of mathematical competencies in the PISA outcomes has been subject to considerable evolution across the five PISA surveys completed so far; that is, until 2013.

Recently, it has been adopted as a framework for the study of the mathematical competency. Mathematical competence can be considered as the sum of various competencies. The eight competencies considered that our student must acquire are:

C1. Thinking mathematically

This competency comprises knowledge of the kind of questions that are dealt with in mathematics and the types of answers mathematics can and cannot provide, and the ability to pose such questions. It includes the recognition of mathematical concepts and an understanding of their scope and limitations as well as extending the scope by abstraction and generalisation of results. This also includes an understanding of the certainty mathematical considerations can provide.

C2. Reasoning mathematically

This competency includes on the one hand the ability to understand and assess an already existing mathematical argumentation (chain of logical arguments), in particular to understand the notion of proof and to recognise the central ideas in proofs. It also includes the knowledge and ability to distinguish between different kinds of mathematical statements (definition, if-then-statement, iff-statement etc.). On the other hand, it includes the construction of chains of logical arguments and hence of transforming heuristic reasoning into own proofs (reasoning logically).

C3. Posing and solving mathematical problems

This competency comprises on the one hand the ability to identify and specify mathematical problems (be they pure or applied, open-ended or closed) and on the other hand the ability to solve mathematical problems (including knowledge of the adequate algorithms). What really constitutes a problem is not well defined and it depends on personal capabilities whether or not a question is considered as a problem. This has to be borne in mind, for example when identifying problems for a certain group of students.

C4. Modelling mathematically

This competency also has essentially two components: the ability to analyse and work in existing models (find properties, investigate range and validity, relate to modelled reality) and the ability to 'perform active modelling' (structure the part of reality that is of interest, set up a mathematical model and transform the questions of interest into mathematical questions, answer the questions mathematically, interpret the results in reality and investigate the validity of the model, monitor and control the whole modelling process). This competency has been investigated in more detail by Blomhoj & Jensen (2003).

C5. Representing mathematical entities

This competency includes the ability to understand and use mathematical representations (be they symbolic, numeric, graphical and visual, verbal, material objects etc.) and to know their relations, advantages and limitations. It also includes the ability to choose and switch between representations based on this knowledge.

C6. Handling mathematical symbols and formalism

This competency includes the ability to understand symbolic and formal mathematical language and its relation to natural language as well as the translation between both. It also includes the rules of formal mathematical systems and the ability to use and manipulate symbolic statements and expressions according to the rules.

C7. Communicating in, with, and about mathematics

This competency includes on the one hand the ability to understand mathematical statements (oral, written or other) made by others and on the other hand the ability to express oneself mathematically in different ways.

C8. Making use of aids and tools

This competency includes knowledge about the aids and tools that are available as well as their potential and limitations. Additionally, it includes the ability to use them thoughtfully and efficiently.

These competencies are transversal to the different mathematical topics and can be mastered at different levels (Romero Albaladejo et al., 2015).

CHAPTER 1

Analysis and Calculus

Guide for a Problem

1.1. Analysis and Calculus

AC1, AC4, AC5, AC7, AC8

Ascensión Hernández Encinas, Araceli Queiruga-Dios and Víctor Gayoso Martínez

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 1, to be achieved with the development of this activity guide are the following:

1. To define and to sketch the functions sinh, cosh, tanh and their reciprocal functions.
2. To state the domain and range of the inverse hyperbolic functions.
3. To recognise and use basic hyperbolic identities.
4. To apply the functions to a practical problem (for example, a suspended cable).
5. To understand how the functions are used in simplifying certain standard integrals.
6. To define and to recognise an odd function and an even function.
7. To understand the properties 'concave' and 'convex' properties and to identify, from its graph, where a function is concave and where it is convex.
8. To define and locate points of inflection on the graph of a function.
9. To understand the concepts of continuity and smoothness.
10. To obtain definite and indefinite integrals of rational functions in partial fraction form.
11. To apply the method of integration by parts to indefinite and definite integrals.
12. To use the method of substitution on indefinite and definite integrals.
13. To solve practical problems which require the evaluation of integrals.
14. To recognise simple examples of improper integrals.
15. To use the formula for the maximum error in a Trapezoidal rule estimate and in a Simpson's rule estimate.
16. To find the length of part of a plane curve and the curved surface area of a solid of revolution.

Table 1 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 1: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Analysis and Calculus									
AC1	1. Hyperbolic functions								
AC11	Define and sketch the functions sinh, cosh, tanh	yellow	red	yellow	yellow	yellow	green	red	yellow

AC12	Sketch the reciprocal functions cosech, sech and coth								
AC13	State the domain and range of the inverse hyperbolic functions								
AC14	Recognise and use basic hyperbolic identities								
AC15	Apply the functions to a practical problem (for example, a suspended cable)								
AC16	Understand how the functions are used in simplifying certain standard integrals								
AC4	4. Functions								
AC41	Define and recognise an odd function and an even function								
AC42	Understand the properties 'concave' and 'convex' properties								
AC43	Identify, from its graph, where a function is concave and where it is convex								
AC44	Define and locate points of inflection on the graph of a function								
AC5	5. Differentiation								
AC51	Understand the concepts of continuity and smoothness								
AC7	7. Methods of integration								
AC71	Obtain definite and indefinite integrals of rational functions in partial fraction form								
AC72	Apply the method of integration by parts to indefinite and definite integrals								
AC73	Use the method of substitution on indefinite and definite integrals								
AC74	Solve practical problems which require the evaluation of an integral								
AC75	Recognise simple examples of improper integrals								
AC76	Use the formula for the maximum error in a trapezoidal rule estimate								
AC77	Use the formula for the maximun error in a Simpson's rule estimate								
AC8	8. Applications of integration								
AC81	Find the length of part of a plane curve								
AC82	Find the curved surface area of a solid of revolution								

2 CONTENTS

1. Hyperbolic functions.
2. Points of inflection, continuity, smoothness, concavity and convexity on the graph of a function.
3. Different integration methods for indefinite and definite integrals.
4. Simple examples of improper integrals.
5. Formula for the maximum error in a trapezoidal rule estimate and in a Simpson's rule estimate.
6. Length of part of a plane curve and the curved surface area of a solid of revolution.

3 TEST

The time to solve this exam will be 3 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 2.

Table 2: Learning outcomes (LO) involved in this assessment activity.

	Analysis and Calculus
AC1	1. Hyperbolic functions
AC11	Define and sketch the functions sinh, cosh, tanh
AC12	Sketch the reciprocal functions cosech, sech and coth
AC13	State the domain and range of the inverse hyperbolic functions
AC14	Recognise and use basic hyperbolic identities
AC15	Apply the functions to a practical problem (for example, a suspended cable)
AC16	Understand how the functions are used in simplifying certain standard integrals
AC4	4. Functions
AC41	Define and recognise an odd function and an even function
AC42	Understand the properties 'concave' and 'convex'
AC43	Identify, from its graph, where a function is concave and where it is convex
AC44	Define and locate points of inflection on the graph of a function
AC5	5. Differentiation
AC51	Understand the concepts of continuity and smoothness
AC7	7. Methods of integration
AC71	Obtain definite and indefinite integrals of rational functions in partial fraction form
AC72	Apply the method of integration by parts to indefinite and definite integrals
AC73	Use the method of substitution on indefinite and definite integrals
AC74	Solve practical problems which require the evaluation of an integral
AC75	Recognise simple examples of improper integrals
AC76	Use the formula for the maximum error in a trapezoidal rule estimate
AC77	Use the formula for the maximun error in a Simpson's rule estimate
AC8	8. Applications of integration
AC81	Find the length of part of a plane curve
AC82	Find the curved surface area of a solid of revolution

3.1 Multiple-Choice Questions

Solve the following test and determine the solution associated to the following statements:

1. The value of $\int_0^1 |x^2 + x - 6|\, dx$ is:
   i) -31/6
   ii) 31/6
   iii) None of the above
2. If $f(x) = \int_{-x}^{2x} 3t^2 \sin(t) dt$, then $f'(-\pi/2)$ is
   i) $\frac{3}{4}\pi^2$
   ii) $-\frac{3}{4}\pi^2$
   iii) None of the above
3. If $f(x)$ is a positive function such that $\lim_{x\to\infty} \frac{f(x)}{\sqrt[4]{x^3}} = 1$, then $\int_1^\infty f(x)dx$ is:
   i) Convergent
   ii) Divergent
   iii) The criterion cannot be determined
4. Applying Simpson's formula to the function $f(x) = \sin(x)$ in the interval $[-\pi, \pi]$ then $\int_{-\pi}^{\pi} \sin(x)dx$ is:
   i) -1
   ii) 4
   iii) 0

3.2 Question 1

1. Calculate the following integral:

$$\int \left(\sinh(x+a) - x\log(x) + \frac{x^3 + 3x}{x^2 + 4x + 10} \right) \mathrm{d}x$$

2. Indicate the path, domain, relative extremes and inflection points, the possible symmetries, the concavity and continuity of the function shown in Figure 1.

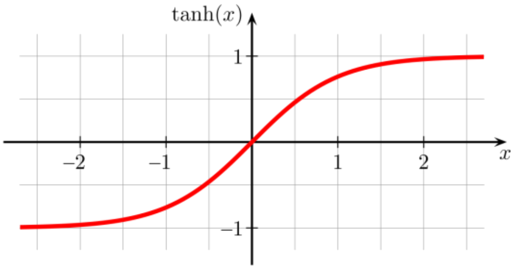

Figure 1. Graph of tanh (x)

3.3 QUESTIONS 2

The following images (Figure 2) collect news that appeared in different newspapers in recent months. Taking into account the information that appears in this news, answer the following questions:

Noticias de Gijón | Noticias de Oviedo

La rotura de la catenaria bloquea la línea férrea entre Oviedo y El Berrón

Otra incidencia, a media tarde, afectó al tramo entre Gijón y El Berrón, provocando retrasos

Oviedo, E. G. | 30.08.2017 | 21:36

La rotura de un cable en la catenaria en la línea férrea entre las estaciones de Colloto y Meres interrumpió esta tarde parte del tráfico ferroviario de Renfe ancho métrico (Feve) y provocó suspensiones y retrasos. Pasadas las ocho de la tarde la compañía se vio obligada a devolver los billetes a los usuarios que en la estación de Oviedo ya se encontraban en el interior del convoy que no pudo salir con destino a El Berrón. Muchos usuarios manifestaron su protesta por la escasa capacidad de la empresa para facilitar información. Se habilitaron transportes alternativos por carretera.

La rotura de la catenaria bloquea la línea férrea entre Oviedo y El Berrón

Levante-EMV » Valencia

La rotura de la catenaria afecta a cientos de viajeros de la línea 1 durante 10 horas

La avería suprime el paso de 30 trenes y corta el servicio entre la Canyada y Paterna

Mónica Ros Valencia | 04.10.2009 | 03:37

Dieciocho días después de que Ferrocarrils de la Generalitat Valenciana (FGV) anunciara el esperado incremento de horarios en la frecuencia de trenes de la línea 1 de Metrovalencia -de Llíria a Torrent y de Torrent a Llíria- se produjo la primera avería. Ocurrió ayer, a las 8.30 horas.

A la altura del apeadero de Fuente del Jarro -poco antes de llegar a la estación de Paterna- la catenaria del metro que iba de Llíria a Valencia se rompió "de manera fortuita", según fuentes de FGV. El convoy se paró en seco al dejar de recibir suministro eléctrico, por "motivos que se desconocen" ya que la avería "ni tuvo lugar en un paso a nivel ni se debe al paso pesado de vehículos, como en otras ocasiones".

Los operarios de FGV estuvieron trabajando durante diez horas para sustituir el cable y restituir el servicio. Los trabajos se prolongaron "más de lo esperado" -aseguraron las mismas fuentes- ya que, en principio, se esperaba restablecer el servicio a medio día.

Operarios de FGV arreglan la avería que cortó el tramo comprendido entre la Canyada y Paterna durante 10 horas. m.a. montesinos

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VALENCIA

La línea 1 de metro, cortada por la rotura de la catenaria

A. TALAVERA | CARLET

El servicio de la línea 1 de Metrovalencia entre las estaciones de Alginet y Carlet estuvo interrumpido durante la mayor parte del día de ayer debido a la rotura de la catenaria por parte de un camión en el paso a nivel de Ausiàs March.

Sobre las doce del mediodía, el vehículo se enredó con los cables de la línea, al sobrepasar la altura máxima, lo que provocó además de daños en la propia catenaria la rotura del paso a nivel.

Los trabajos para restaurar el servicio se prolongaron hasta las diez de la noche de ayer ya que la lluvia intermitente durante toda la tarde impidió que se realizaran con normalidad. Sin embargo, todos los trenes de esta línea 1 circulan desde esta mañana sin problemas por la zona de la Ribera.

Durante la interrupción, Ferrocarrils de la Generalitat estableció un servicio alternativo de dos autobuses entre las estaciones de Alginet y Carlet, mientras que el resto de la línea no se vio afectada por el suceso.

La rotura no afectó a los trenes que realizan el trayecto de Valencia hasta Alginet ni en el tramo que comunica Carlet y Villanueva de Castellón, de forma que únicamente quedó fuera de servicio durante la jornada de ayer el apeadero de Ausiàs March, muy poco frecuentado.

El servicio se restablecerá hoy. :: FRANCISCO GARCÍA

Figure 2. News in some Spanish newspapers

1. Do you know what a catenary is? The dictionary of the Royal Spanish Academy (RAE) defines the catenary with 3 meanings:

   i) Pertainig or relative to the chain.

   ii) Curve formed by a chain, rope or similar thing suspended between two points not located in the same vertical.

   iii) Suspension system in the air of an electric cable, maintained at a fixed height of the ground, from which by means of a trolley or pantograph, some vehicles take over, such as trams, trains, etc.

   Which of the meanings is the one applied in the previous news?

2. Public transport systems such as the metro or the train cover a very important function. Any service rupture causes great losses to some companies, besides the discomfort of the passengers that some public transports like the metro or the train (Figure 3).

Figure 3. Picture of a train and some catenaries

From the mathematical point of view, the catenary is the curve taken in the second meaning of the dictionary of the RAE. Their equation is $y = a \cosh \frac{x}{a}$, where $a = \left(\frac{\tau_h}{\lambda}\right)$ is a positive number, considering the weight per unit length (λ) constant and where (τ_h) is the horizontal tension that appears at the ends of the cable.

You can see some catenaries (Figure 4) in the following representation for different values of the parameter a

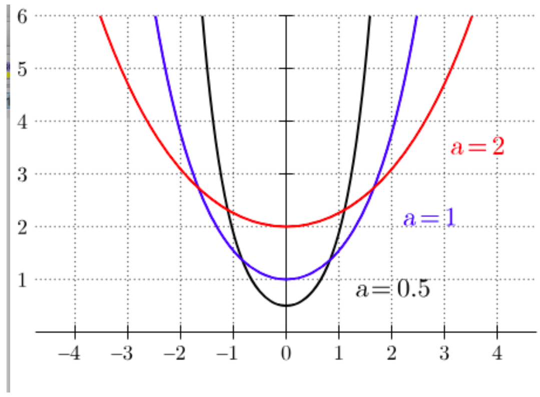

Figure 4. Catenaries graphs

Calculate the area bounded by the curve and the OX axis in the interval $[0, 1]$ for $a = 1$. Also calculate the arc length of the catenary between 0 and a, knowing that the length of a curve is calculated through the formula $L = \int_A^B \sqrt{1 + (f'(x))^2}\, dx$ for the parameter values that appear in the previous graphs.

Finally, calculate the lateral surface obtained by rotating the previous catenary around the axis OX of the previous catenary, knowing that the area is given by

$$A_L = 2\pi \int_A^B |f(x)| \sqrt{1 + (f'(x))^2}\, dx$$

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. The value of $\int_0^1 |x^2 + x - 6|\, dx$ is:
   i) -31/6
   ii) 31/6
   iii) None of the above

   Solution:

   If 0 < x <1 then $x^2 + x - 6 = (x-2)(x+3) \leq 0$ and $|x^2 + x - 6| = -x^2 - x + 6$. Thus:

$$\int_0^1 |x^2 + x - 6| dx = \int_0^1 (-x^2 - x + 6) dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x\right]_0^1 = \frac{31}{6}$$

   Option ii) is the correct one.

2. If $f(x) = \int_{-x}^{2x} 3t^2\, sin(t) dt$, then f ’(-π /2) is
   i) ¾ π^2
   ii) -¾ π^2
   iii) None of the above

   Solution:

   We can write:

$$\begin{aligned} f(x) &= \int_{-x}^{2x} 3t^2 \sin(t)\, dt \\ &= \int_{-x}^{1} 3t^2 \sin(t)\, dt + \int_{1}^{2x} 3t^2 \sin(t)\, dt \\ &= -\int_{1}^{-x} 3t^2 \sin(t)\, dt + \int_{1}^{2x} 3t^2 \sin(t)\, dt. \end{aligned}$$

   Applying the fundamental theorem of Calculus, the derivative is:

$$f'(x) = -3x^2 \sin(-x)\,(-1) + 3\; 4x^2 \sin(2x)\;\; 2 = 24x^2 \sin(2x) - 3x^2 \sin(x)$$

   As $\sin(-\mathrm{x}) = -\sin(\mathrm{x}), \sin\left(-\frac{\pi}{2}\right) = -1$ and $\sin(-\pi) = 0$, we obtain:

$$f'\left(-\frac{\pi}{2}\right) = \frac{3\pi^2}{4}$$

   Option i) is the correct one.

3. If $f(x)$ is a positive function such that $\lim_{x\to\infty} \frac{f(x)}{\sqrt[4]{x^3}} = 1$, then $\int_1^\infty f(x) dx$ is:
   i) Convergent
   ii) Divergent
   iii) The criterion cannt be determined

Solution:

According to comparison criterium at the limit, the character of $\int_1^\infty f(x)dx$ is the same as the character of $\int_1^\infty \sqrt[4]{x^3}dx$. Then $\int_1^\infty \sqrt[4]{x^3}dx = \int_1^\infty x^{\frac{4}{3}}dx = \lim_{b\to\infty}\left[\frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1}\right]_1^b = \lim_{b\to\infty}\frac{4b^{\frac{7}{4}}}{7} - \frac{4}{7} = \infty$. Thus, $\int_1^\infty f(x)dx$ is divergent.

Option ii) is the correct one.

4. By applying Simpson's formula to the function $f(x) = \sin(x)$ in the interval $[-\pi, \pi]$, then $\int_{-\pi}^{\pi} \sin(x)dx$ is:

   i) -1

   ii) 4

   iii) 0

   Solution:

   As sin(-π) = sin (0) = sin (π) = 0, then by using Simpson's formula: $\int_{-\pi}^{\pi} \sin(x)\,dx = 0$.

   Option iii) is the correct one.

LO: AC7.

4.2 QUESTIONS 1

1. Calculate the following integral:

$$\int (\sinh(x+a) - x\log(x) + \frac{x^3+3x}{x^2+4x+10})\mathrm{d}x$$

Solution:

$$-4x + \frac{3x^2}{4} + 11\sqrt{\frac{2}{3}}\, arc.tan\frac{2+x}{\sqrt{6}} + cosh(a)cosh(x) - \frac{1}{3}x^2 Log|x| + \frac{9}{2}Log|10+4x+x^2| + \sinh(a)\sinh(x)$$

2. Indicate the path, domain, relative extremes and inflection points, the possible symmetries, the concavity and continuity of the function shown in Figure 1.

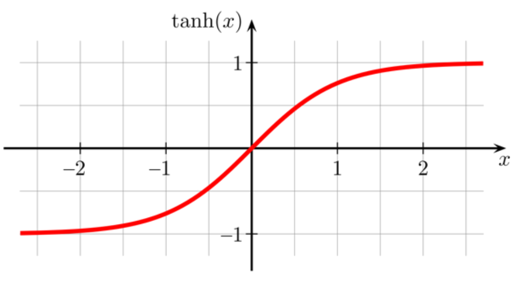

Figure 5. Graph of tanh (x)

Solution:

f (*x*) = tanh (*x*).

Dom (f) = R.

Im(*f*) = (-1,1).

The function is continuous in R.

It has not maximum or minimum.

It has an inflection point in (0, 0).

From (-∞, 0) the function is convex and from (0, ∞) it is concave.

It's symmetric at the origin of coordinates.

LO: AC1, AC4, AC5.

4.3 QUESTIONS 2

The following images (Figure 2) collect news that appeared in different newspapers in recent months. Taking into account the information that appears in this news, answer the following questions:

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?

Dieciocho días después de que Ferrocarrils de la Generalitat Valenciana (FGV) anunciara el esperado incremento de horarios en la frecuencia de trenes de la línea 1 de Metrovalencia -de Llíria a Torrent y de Torrent a Llíria- se produjo la primera avería. Ocurrió ayer, a las 8.30 horas.

A la altura del apeadero de Fuente del Jarro -poco antes de llegar a la estación de Paterna- la catenaria del metro que iba de Llíria a Valencia se rompió "de manera fortuita", según fuentes de FGV. El convoy se paró en seco al dejar de recibir suministro eléctrico, por "motivos que se desconocen" ya que la avería "ni tuvo lugar en un paso a nivel ni se debe al paso pesado de vehículos, como en otras ocasiones".

Los operarios de FGV estuvieron trabajando durante diez horas para sustituir el cable y restituir el servicio. Los trabajos se prolongaron "más de lo esperado" -aseguraron las mismas fuentes- ya que, en principio, se esperaba restablecer el servicio a medio día.

Operarios de FGV arreglan la avería que cortó el tramo comprendido entre la Canyada y Paterna durante 10 horas. m.a. montesinos

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VALENCIA

La línea 1 de metro, cortada por la rotura de la catenaria

05.04.12 - 00:11 - A. TALAVERA | CARLET

El servicio de la línea 1 de Metrovalencia entre las estaciones de Alginet y Carlet estuvo interrumpido durante la mayor parte del día de ayer debido a la rotura de la catenaria por parte de un camión en el paso a nivel de Ausiàs March.

Sobre las doce del mediodía, el vehículo se enredó con los cables de la línea, al sobrepasar la altura máxima, lo que provocó además de daños en la propia catenaria la rotura del paso a nivel.

Los trabajos para restaurar el servicio se prolongaron hasta las diez de la noche de ayer ya que la lluvia intermitente durante toda la tarde impidió que se realizaran con normalidad. Sin embargo, todos los trenes de esta línea 1 circulan desde esta mañana sin problemas por la zona de la Ribera.

Durante la interrupción, Ferrocarrils de la Generalitat estableció un servicio alternativo de dos autobuses entre las estaciones de Alginet y Carlet, mientras que el resto de la línea no se vio afectada por el suceso.

La rotura no afectó a los trenes que realizan el trayecto de Valencia hasta Alginet ni en el tramo que comunica Carlet y Villanueva de Castellón, de forma que únicamente quedó fuera de servicio durante la jornada de ayer el apeadero de Ausiàs March, muy poco frecuentado.

El servicio se restablecerá hoy. :: FRANCISCO GARCÍA

Figure 6. News in some Spanish newspapers

1. Do you know what is a catenary? The dictionary of the Royal Spanish Academy (RAE) defines the catenary with 3 meanings:
   i) Pertainig or relative to the chain.
   ii) Curve made by a chain, rope or similar thing suspended between two points not located in the same vertical.
   iii) Suspension system in the air of an electric cable, maintained at a fixed height of the ground, from which by means of a trolley or pantograph, some vehicles take over, such as trams, trains, etc.

   Which of the meanings is the one applied in the previous news?

2. Public transport systems such as the metro or the train cover a very important function. Any service rupture rupture causes great losses to some companies, besides the discomfort of the passengers (Figure 3).

Figure 7. Picture of a train and some catenaries

From the mathematical point of view, the catenary is the curve taken in the second meaning of the dictionary of the RAE. Their equation is $y = a\cosh\frac{x}{a}$, where $a = \left(\frac{\tau_h}{\lambda}\right)$ is a positive number, considering the weight per unit length (λ) constant and where (τ_h) is the horizontal tension that appears at the ends of the cable.

You can see some catenaries (Figure 4) in the following representation for different values of the parameter a

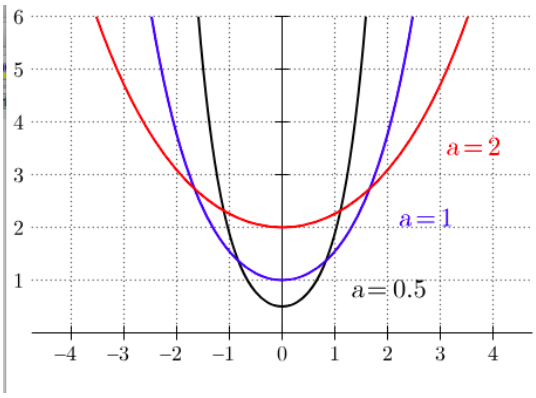

Figure 8. Graph of Catenaries

3. Calculate the area bounded by the curve and the OX axis in the interval [0,1] for a=1. In addition, calculate the arc length of the catenary between 0 and a, knowing that the length of a curve is calculated through the formula $L = \int_A^B \sqrt{1 + (f'(x))^2}\,dx$ for the parameter values that appear in the previous graphs.

   Finally calculate the lateral surface obtained by rotating the previous catenary around the axis OX, knowing that the said area is given by $A_L = 2\pi\int_A^B |f(x)|\sqrt{1 + (f'(x))^2}\,dx$

   i) $\int_0^1 \sinh(x)\,dx = \cosh(1) - cosh(0)$

   $$\cosh(x) = \frac{e^x + e^{-x}}{2} \to \cosh(1) = \frac{e^1 + e^{-1}}{2} = \frac{e^2 + 1}{2e}$$

$$\int_0^1 \sinh(x)\,dx = cosh(1) - 1 = \frac{e^2+1}{2e} - 1$$

ii) $L = \int_A^B \sqrt{1+(f'(x))^2}\,dx = \int_0^a \sqrt{1+\left(\sinh\left(\frac{x}{a}\right)\right)^2}\,dx$

As it is known $\cosh^2 x - \sinh^2 x = 1$ then:

$$\int_0^a \sqrt{1+\left(\sinh\left(\frac{x}{a}\right)\right)^2}\,dx = a\int_0^a \cosh\frac{x}{a}\,dx = a(\sinh(1) - \sinh(0)) = a\sinh(1)$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \to \sinh(1) = \frac{e^1 - e^{-1}}{2} = \frac{e^2-1}{2e}$$

$$L = a\frac{e^2-1}{2e}$$

If a= 0.5, then $L = \frac{e^2-1}{4e}$

If a= 1, then $L = \frac{e^2-1}{2e}$

If a= 2, then $L = \frac{e^2-1}{e}$

iii) $A_L = 2\pi\int_A^B |f(x)|\sqrt{1+(f'(x))^2}\,dx = 2\pi a\int_0^a cosh\frac{x}{a}\sqrt{1+\left(\sinh\left(\frac{x}{a}\right)\right)^2}\,dx =$

$2\pi a\int_0^a cosh^2\frac{x}{a}dx$

$$\cosh(x) = \frac{e^x+e^{-x}}{2} \to \cosh^2(x) = \frac{e^{2x}+e^{-2x}+2}{4}$$

$$\int cosh^2\frac{x}{a}\,dx = \int \frac{e^{2x}+e^{-2x}+2}{4}dx = \frac{a}{4}\left(\frac{e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}}{2}\right) + \frac{x}{2} = \frac{a}{4}\sinh\left(\frac{2x}{a}\right) + \frac{x}{2}$$

$$A_L = 2\pi a\int_0^a cosh^2\frac{x}{a}dx = 2\pi a\left[\frac{a}{4}\sinh\left(\frac{2x}{a}\right) + \frac{x}{2}\right]_0^a = 2\pi a\left(\frac{a}{4}\sinh(2) + \frac{a}{2}\right)$$

$$\sinh(2) = \frac{e^2 - e^{-2}}{2} = \frac{e^4-1}{2e^2}$$

$$A_L = 2\pi a\left(\frac{a}{4}\sinh(2) + \frac{a}{2}\right) = \pi a^2\left(\frac{e^4-1}{4e^2} + 1\right)$$

LO: AC1, AC7, AC8.

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Guide for a Problem

Analysis and Calculus

1.2. Hyperbolic functions

AC1

Michael Carr, Paul Robinson and

Ciarán O Sullivan

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 3, to be achieved with the development of this activity guide are the following:

1. Define and sketch the functions sinh, cosh, tanh.
2. Sketch the reciprocal functions cosech, sech and coth.
3. State the domain and range of the inverse hyperbolic functions.
4. Recognize and use basic hyperbolic identities.
5. Apply the functions to a practical problem (for example, a suspended cable).
6. Understand how the functions are used in simplifying certain standard integrals.

Table 3 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

= very important

= medium important

= less important

Table 3: Competencies that we measure with the global test model that is proposed

		C1	C2	C3	C4	C5	C6	C7	C8
AC1	Hyperbolic Functions								
AC11	Define and sketch the functions sinh, cosh, tanh								
AC12	Sketch the reciprocal functions cosech, sech and coth								
AC13	State the domain and range of the inverse hyperbolic functions								
AC14	Recognise and use basic hyperbolic identities								
AC15	Apply the functions to a practical problem (for example, a suspended cable)								
AC16	Understand how the functions are used in simplifying certain standard integrals								

2 CONTENTS

1. Define and sketch the functions sinh, cosh, tanh.
2. Sketch the reciprocal functions cosech, sech and coth.
3. State the domain and range of the inverse hyperbolic functions.
4. Recognize and use basic hyperbolic identities.
5. Apply the functions to a practical problem (for example, a suspended cable).
6. Understand how the functions are used in simplifying certain standard integrals.

3 TEST

The time to solve this exam will be 100 minutes (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 4.

Table 4: Learning outcomes (LO) involved in this assessment activity.

Analysis and Calculus	
AC1	Hyperbolic Functions
AC11	Define and sketch the functions sinh, cosh, tanh.
AC12	Sketch the reciprocal functions cosech, sech and coth.
AC13	State the domain and range of the inverse hyperbolic functions
AC14	Recognize and use basic hyperbolic identities.
AC15	Apply the functions to a practical problem (for example, a suspended cable).
AC16	Understand how the functions are used in simplifying certain standard integrals.

3.1 Multiple-Choice Questions

1. The function f shown is:

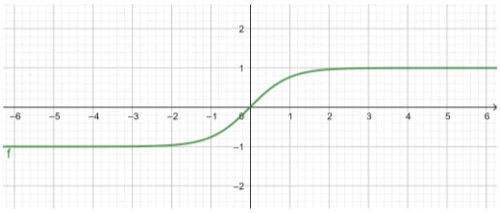


Figure 9

   A) Cosh(x)
   B) Sinh(x)
   C) Tanh(x)
   D) Cosh(9x)

2. The function f shown is

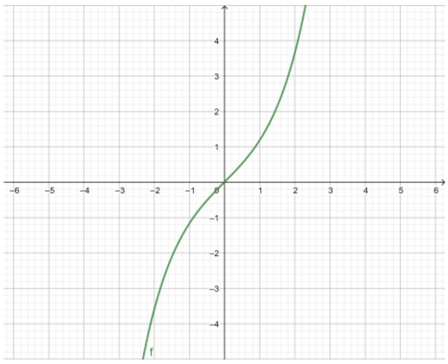


Figure 10

   A) Cosh(x)
   B) Sinh(x)
   C) Tanh(x)
   D) Cosh(9x)

3. The function f shown is

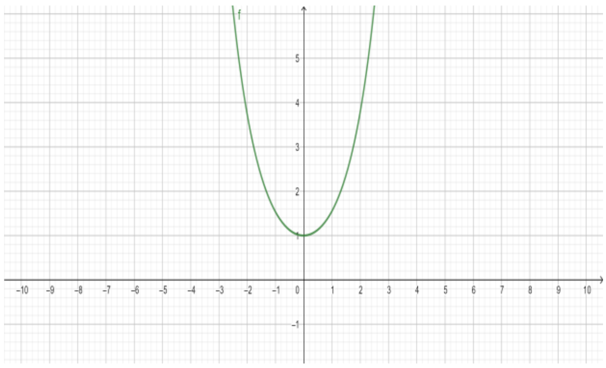


Figure 11

A) Cos(x)

B) Sinh(x)

C) Tanh(x)

D) Cosh(9x)

4. The function f shown is

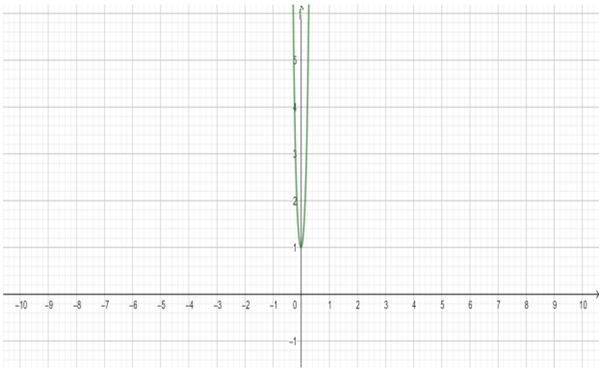


Figure12

A) Cosh(x)

B) Sinh(x)

C) Tanh(x)

D) Cosh(9x)

5. The function *f* shown is

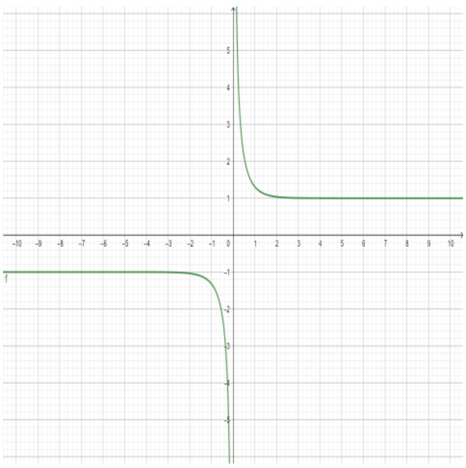


Figure 13

A) Cotanh(x)

B) Sech(x)

C) Cosh(x)

D) Sinh(x)

6. The function f shown is

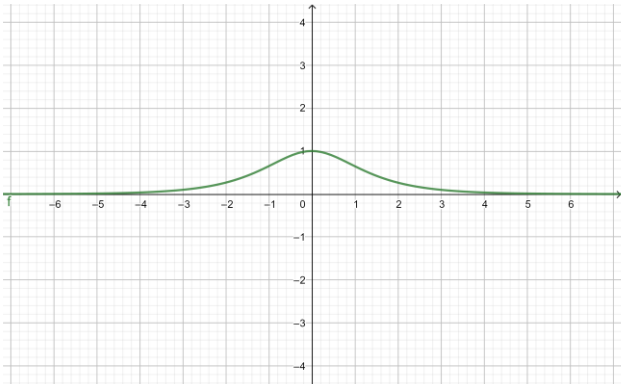


Figure 14

   A) Cotanh(x)
   B) Sechx
   C) Cosh x
   D) Sinh x

7. Cotanh(x) is equal to
   A) Tan^{-1} (x)
   B) $Tanh^{-1}$ (x)
   C) 1/Tan (x)
   D) 1/Tanh(x)

8. Sech(x) is equal to
   A) $Cosh^{-1}$ (x)
   B) $Sinh^{-1}$ (x)
   C) 1/Cosh(x)
   D) 1/Tanh(x)

3.2 Questions 1

1. Define Cosh(x).

2. Define Sinh (x).

3. Plot e^x and e^{-x} for x in the range -4, 4 in steps of 1.

4. Plot $(e^x + e^{-x})$ for x in the range -4, 4 in steps of 1.

5. Plot Cosh(x) for x in the range -4, 4 in steps of 1.

6. Plot Sinh(x) for x in the range -4, 4 in steps of 1.

7. Calculate sinh(x), cosh(x) and tanh(x) for x = 5.

8. Calculate cosh(2x) for x = 2.

9. Calculate cotanh(x) for x = 3.

10. Calculate sech(x) for x = 1.

11. Calculate cotanh(3x) for x = 2.

12. Verify that $Cosh^2(x)$- $Sinh^2(x)$=1.

13. Calculate the maximum value of the function f(x)=e^x for all x.

14. Calculate the minimum value of f(x)=e^x for all x.

15. Calculate the maximimum value of f(x)=e^{-x}.

16. What is the range of Cosh(x) ?

17. What is the range of Sinh(x) ?

18. Find $\int \frac{1}{x\sqrt{x^2+3^2}}dx$.

19. Find $\int \frac{1}{x\sqrt{16-x^2}}dx$.

3.3 Questions 2

For a 2 dimensional plate (shown below) of dimensions $\pi \times \pi$ the steady state heat distribution(U) is described by the Laplace equation

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 0$$

(1) U= 0 along edge x=0 for all 0 < y < π
(2) U= 0 along edge y=0 for all 0 < x < π
(3) U= 0 along edge x= π for all 0 < y < π
(4) U=10sin(7x) along edge y= π for all 0 < x < π

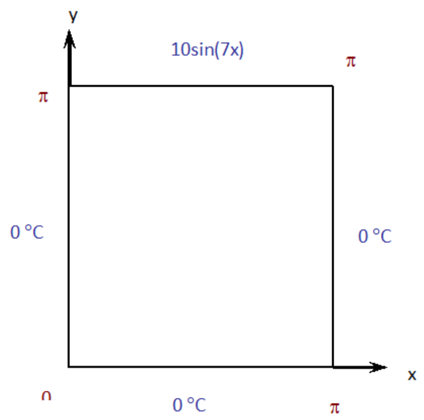


Figure 15

a) Find the temperature at the point a) x=1, y=1 and the point b) x=3, y=3

At i) 2.04×10^{-6} ii) 31.4

b) Using a package of your choice plot heat distribution over the entire plate.

4 SOLUTIONS

4.1 Multiple-Choice Question

1. C
2. B
3. A
4. D
5. A
6. B
7. D
8. C

4.2 Question 1

1. $Cosh\ (x) = \frac{e^x+e^{-x}}{2}$
2. $Sinh\ (x) = \frac{e^x-e^{-x}}{2}$

3 and 4. Plot of e^x and e^{-x} for x in the range -4, 4 in steps of 1and also plot of (e^x + e^{-x}) for x in the range -4, 4 in steps of 1:

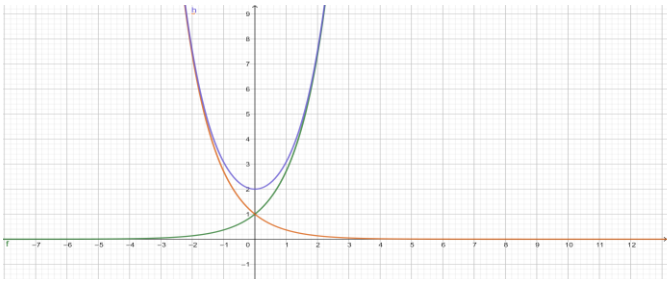
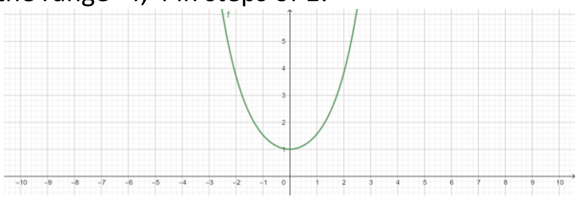


Figure 16

5. Plot of Cosh(x) for x in the range -4, 4 in steps of 1:



6. Plot of Sinh(x) for x in the range -4, 4 in steps of 1:

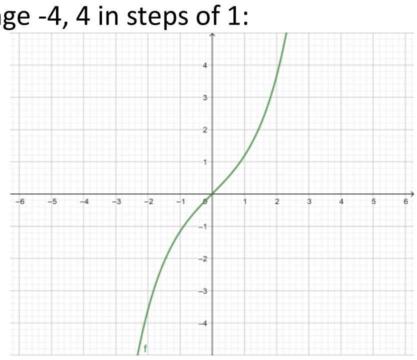


Figure 17

7. Calculate sinh(x), cosh(x) and tanh(x) for x = 5:

   Sinh(5)= 74.2032 Cosh(5) =74.2099 Tanh(5)=0.9999

8. Calculate cosh(2x) for x=2:

   Cosh(4)= 27.3

9. Calculate cotanh x) for x=3:

   Contanh(3)=1.004

10. Calculate sech(x) for x= 1:

    Sech(1)=0.648

11. Calculate cotanh(3x) for x= 2:

    Contanh(6)=1.0001

12. Given the definition of Cosh(x)=(e^x+e^{-x})/2 and Sinh(x)= (e^x-e^{-x})/2 verify that

    $Cosh^2x- Sinh^2x=1$:

    $(e^x+e^{-x})^2/4 + (e^x-e^{-x})^2/4 =1/4(e^{2x} +2+ e^{-2x}) -1/4(e^{2x} -2+ e^{-2x})=1/4(2+2)=1$

13. Calculate the maximum value of the function $f(x)=e^x$for all x.

    Max value -> ∞

14. Calculate the minimum value of $f(x)=e^x$ for all x:

    Min value -> 0

15. Calculate the maximimum value of $f(x)=e^{-x}$:

    Max value -> ∞

16. What is the range of Cosh(x):

    Range= 1 to ∞

17. What is the range of Sinh(x):

    Range = - ∞ to ∞

18. Find $\int \frac{1}{x\sqrt{x^2+3^2}}dx$= -1/3 $cosech^{-1}$ (x/3)

19. Find $\int \frac{1}{x\sqrt{16-x^2}}dx$=-1/4$sech^{-1}$x/4

4.3 QUESTION 2

U= (10/sinh(7 π)) (sin(7x)sinh(7y))

a) The temperature at the point i) x=1, y=1 is U= 0.000002°C and temperature at the point ii) x=3, y=3 is U=3.14°C .

b) Sample plot:

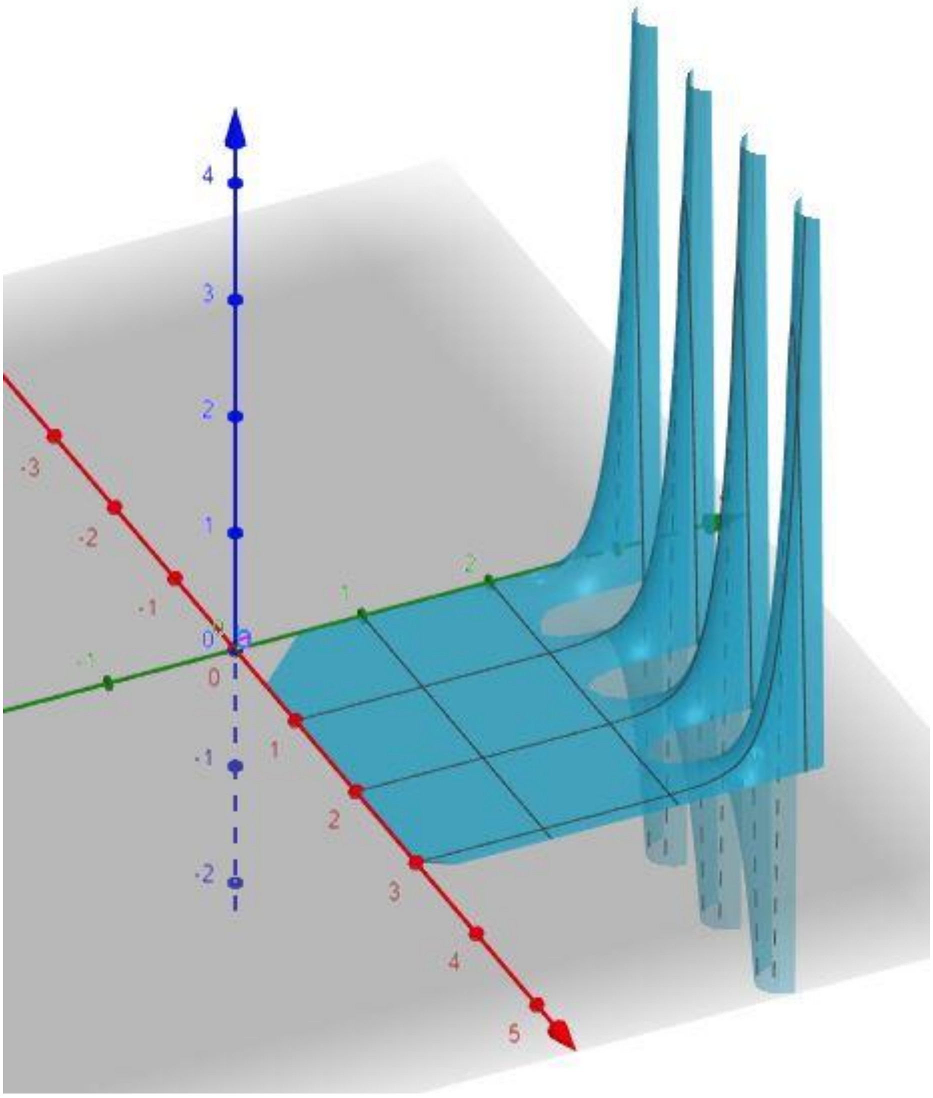


Figure 18

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Guide for a Problem

Analysis and Calculus

1.3. Rational functions

AC2

Marie Demlova and Petr Habala

1 OBJECTIVES

The aim of this activity guide is to achieve certain learning outcomes. As a result of learning this material you should be able to:

AC21. Sketch the graph of a ratio of two linear factors or a linear factor over the product of two linear factors.

AC22. Find the partial fractions decomposition for the given function f.

The Table 1 shows how these Learning Outcomes address individual competencies.

Table 5 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- = strong connection
- = medium connection
- = weak connection

Table 5: Learning outcomes and competencies.

		C1	C2	C3	C4	C5	C6	C7	C8
Analysis and Calculus									
AC2	2. Rational functions								
AC21	Sketch a rational function								
AC22	Decompose into partial fractions								

The question 1 can also incorporate some graphing tool to confirm the sketch, and then it also addresses the competency C8.

2 CONTENTS

1. Rational functions.
2. Partial fractions decomposition.

3 TEST

The time to solve this exam will be 120 minutes (approximately).
The learning outcomes to be achieved with this procedure are detailed in Table 6.

Table 6: Learning outcomes (LO) involved in this assessment activity.

Analysis and Calculus	
AC2	2. Rational functions
AC21	Sketch the graph of a ratio of two linear factors or a linear factor over the product of two linear factors.
AC22	Find the partial fractions decomposition for the given function f.

The statement of the exam is included below:

3.1 Multiple-Choice Questions

Solve the following test, reasoning if it's True or False.

1. Which of the following graphs is the graph of the rational function $f(x) = \frac{x}{x+1}$?

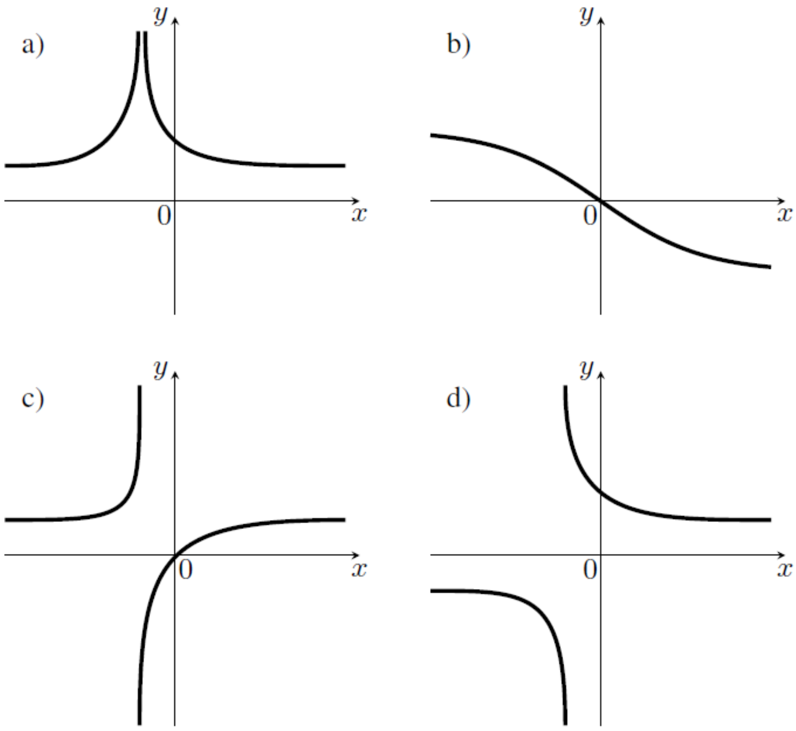

Figure 19

2. Which of the following functions is already a partial fraction (i.e. it does not need to be decomposed any further).

a) $\dfrac{x^2+1}{x^2+x+1}$.

b) $\dfrac{x}{x^3+x^2+x+1}$.

c) $\dfrac{2x+1}{x^2+4x+3}$.

d) $\dfrac{x+1}{2x^2+1}$.

e) $\dfrac{10x}{(2+x)^2}$.

Co-funded by the Erasmus+ Programme of the European Union

3. Which of the following general forms is correct for $\frac{13x^2-23}{(x-1)^2(x^2+1)}$?

a) $\frac{A}{(x-1)^2} + \frac{Bx+C}{x^2+1}$
b) $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$
c) $\frac{A}{x-1} + \frac{Bx}{x^2+1}$
d) $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x^2+1}$

3.2 Questions 1

1. Sketch the graph of the function:

$$f(x) = \frac{2x}{2x-1}.$$

2. Sketch the graph of the function:

$$f(x) = \frac{2x+2}{(x-1)(x+3)}.$$

3. Find the partial fraction decomposition for the function:

$$f(x) = \frac{4-x}{(x-1)^2(2x+1)}.$$

4. Find the partial fraction decomposition for the function:

$$f(x) = \frac{9x+2}{(x^2+2x+2)(x-2)}.$$

3.3 Questions 2

Finding a partial fractions decomposition can be long and boring. Fortunately there are tricks that can save work.

Consider a rational function $\frac{p(x)}{q(x)}$ whose denominator involves a linear factor $(x-a)^n$, where n is the proper power, that is, $q(x) = (x-a)^n Q(x)$ for some function $Q(x)$ such that $Q(a) \neq 0$.

The partial fraction decomposition then looks like this:

$$\frac{p(x)}{(x-a)^n Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots \frac{A_n}{(x-a)^n} + \frac{P(x)}{Q(x)}.$$

The “cover-up” trick, or a “hiding trick”, or Heaviside’s method works as follows: To get A_n, cover (hide, scratch out) the $(x-a)^n$ part in the denominator of the given rational function $\frac{p(x)}{(x-a)^n Q(x)}$ and substitute the corresponding zero point a for x into the expression that is left:

$$A_n = \left.\frac{p(x)}{(////)^n Q(x)}\right|_{x=a} = \frac{P(a)}{Q(a)}.$$

For instance, in the decomposition

$$\frac{x+1}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

we easily get

$$A = \left.\frac{(x+1)}{(////)(x-3)}\right|_{x=1} = \frac{2}{-2} = -1, \quad B = \left.\frac{(x+1)}{(x-1)(////)}\right|_{x=3} = \frac{2}{2} = 1$$

1. Find the partial fraction decomposition for the function

$$f(x) = \frac{x+4}{(x-1)(x+1)(x-2)}.$$

The main disadvantage of this approach is that it only works for linear factors, and in fact only for their highest powers. Consider this example:

$$\frac{x+9}{(x-1)^2(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-3}.$$

Here we easily get

$$B = \left.\frac{(x+9)}{(////)(x-3)}\right|_{x=1} = -5, \quad C = \left.\frac{(x+9)}{(x-1)^2(////)}\right|_{x=3} = 3.$$

However, in order to get A we would have to hide just the factor $(x-1)$ in the denominator and one would be left there:

$$A = \left.\frac{x+9}{(////)(x-1)(x-3)}\right|_{x=1} = \frac{10}{0} = ?$$

Still, usually this trick provides us with at least some coefficients, which simplifies further work.

Another interesting method works as follows: Write the general fraction decomposition, then multiply both sides by a power of x so that none of the resulting fractions tends to infinity, but not all of them go to zero. This essentially means that we multiply the equation by x. Then send x to infinity and check on what comes out of it.

Consider this example:

$$\frac{x^2+13}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

The hiding trick finds one constant:

$$A = \left.\frac{x^2+13}{(////)(x^2+1)}\right|_{x=1} = 7.$$

We need two more. We substitute for A into the decomposition equality and multiply it by x:

$$\frac{x^3+13x}{(x-1)(x^2+1)} = \frac{7x}{x-1} + \frac{Bx^2+Cx}{x^2+1}$$

We pass to infinity with x and obtain

$$1 = 7 + B \Longrightarrow B = -6.$$

It remains to determine C, and there we have to fall back on the default method: Multiply the decomposition equality by the denominator $(x-1)(x^2+1)$, cancel in fractions, then multiply out brackets and compare powers. However, since now there is only one constant left, our work is much easier:

$$x^2+13 = 7(x^2+1) + (C-6x)(x-1)$$
$$\Longrightarrow x^2+13 = x^2 + (C+6)x + (7-C)$$
$$\Longrightarrow C = -6.$$

2. Find the partial fraction decomposition for the function

$$f(x) = \frac{x+4}{(x-1)(x+1)(x-2)}.$$

Use the hiding trick for as many coefficients as possible, then find the rest using the trick with limit.

4 SOLUTION

4.1 Multiple-Choice Questions

1. Which of the following graphs is the graph of the rational function

$$f(x) = \frac{x}{x+1}?$$

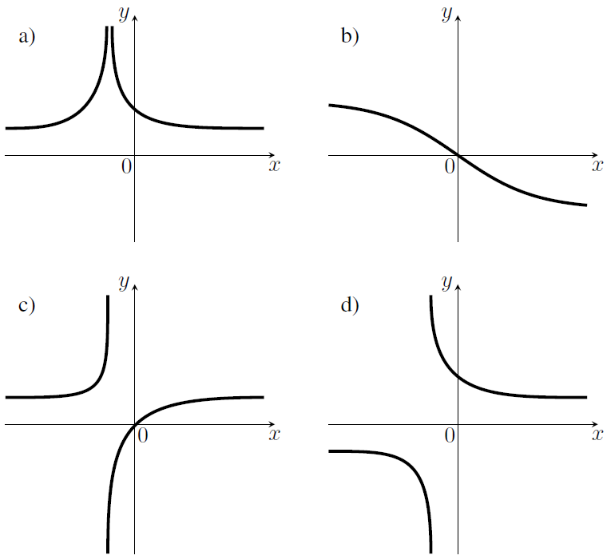

Figure 20

Solution:

The correct answer is c).

LO: AC21.

2. Which of the following functions is already a partial fraction (i.e. it does not need to be decomposed any further).

a) $\dfrac{x^2+1}{x^2+x+1}$.

b) $\dfrac{x}{x^3+x^2+x+1}$.

c) $\dfrac{2x+1}{x^2+4x+3}$.

d) $\dfrac{x+1}{2x^2+1}$.

e) $\dfrac{10x}{(2+x)^2}$.

Solution:

The correct answer is d).

LO: AC22.

3. Which of the following general forms is correct for

$$f(x) = \frac{13x^3-23}{(x+1)^2(x^2+1)}?$$

a) $\dfrac{A}{(x-1)^2} + \dfrac{Bx+C}{x^2+1}$

b) $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1}$

c) $\dfrac{A}{x-1} + \dfrac{Bx}{x^2+1}$

d) $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x^2+1}$

Solution:

The correct answer is b)

LO: AC22.

4.2 QUESTIONS 1

1. Sketch the graph of the function

$$f(x) = \frac{2x}{2x-1}.$$

Solution:

We see that the domain is interrupted at $x = \frac{1}{2}$, so the graph will consist of two parts. To see how they go we find limits at infinity and negative infinity, and also one-sided limits at all points of discontinuity, which here means at $x = \frac{1}{2}$.

$$\lim_{x\to\infty}\left(\frac{2x}{2x-1}\right) = 1, \qquad \lim_{x\to-\infty}\left(\frac{2x}{2x-1}\right) = 1,$$
$$\lim_{x\to\frac{1}{2}^+}\left(\frac{2x}{2x-1}\right) = \infty, \qquad \lim_{x\to\frac{1}{2}^-}\left(\frac{2x}{2x-1}\right) = -\infty.$$

These limits tell us where the parts of the graph should "end".

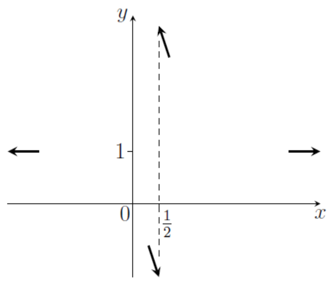

Figure 21

We know that a rational function with linear factors cannot make any "waves", the endpoints should be connected with simple curves of suitable concavity. This essentially anchors the parts of the graph in place. To get the picture more precise we determine the intersections: There is just one given by $f(0) = 0$. Thus we guess the following graph.

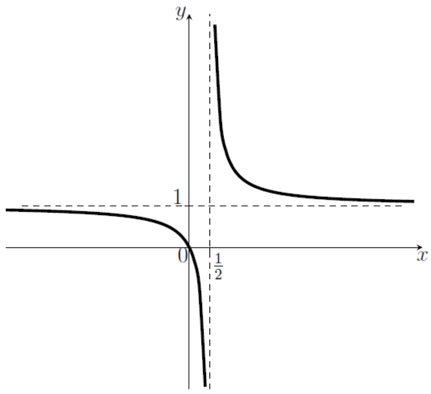

Figure 22

Incidentally, we can notice that the function can be written as $f(x) = 1 + \frac{1}{2x-1}$. Hence its graph will be a hyperbola, indeed.

We can confirm our guessed shape using the standard graph sketching procedure:

The function $f(x)$ is defined for every $x \in \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ (in other words for every real number except $\frac{1}{2}$); and it is continous and differentiable on its domain.

Since the first derivative $f'(x) = -\frac{2}{(2x-1)^2}$ which is always negative. Hence, the function $f(x)$ s decreasing on all intervals of its domain.

Moreover, its second derivative $f''(x) = \frac{8}{(2x-1)^3}$ is negative for $x < \frac{1}{2}$, hence $f(x)$ is concave down on $\left(-\infty, \frac{1}{2}\right)$, and $f''(x)$ is positive for $x > \frac{1}{2}$, hence $f(x)$ is concave up on the $\left(\frac{1}{2}, \infty\right)$.

LO: AC21.

2. Sketch the graph of the function

$$f(x) = \frac{2x+2}{(x-1)(x+3)}.$$

Solution:

We see that the domain is interrupted at $x = -3$ and $x = 1$, so the graph will consist of three parts. To see how they go we find limits at infinity and negative infinity, and also one-sided limits at all points of discontinuity, which here means at $x = -3$ and $x = 1$.

$$\lim_{x\to\infty}\left(\frac{2x+2}{(x-1)(x+3)}\right) = 0\,, \qquad \lim_{x\to-\infty}\left(\frac{2x+2}{(x-1)(x+3)}\right) = 0\,,$$
$$\lim_{x\to(-3)^+}\left(\frac{2x+2}{(x-1)(x+3)}\right) = \infty\,, \qquad \lim_{x\to(-3)^-}\left(\frac{2x+2}{(x-1)(x+3)}\right) = -\infty$$
$$\lim_{x\to1^+}\left(\frac{2x+2}{(x-1)(x+3)}\right) = \infty\,, \qquad \lim_{x\to1^-}\left(\frac{2x+2}{(x-1)(x+3)}\right) = -\infty$$

These limits tell us where the patrs of the graph should "end".

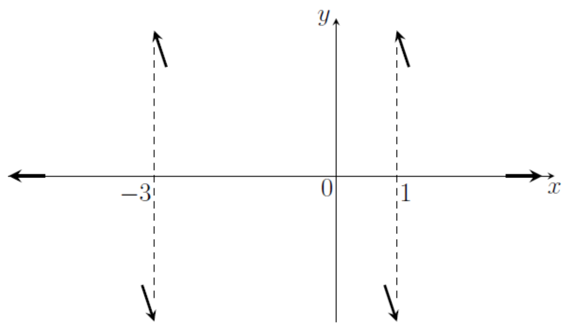

Figure 23

We know that a rational function with linear factors cannot make any "waves", the endpoints should be connected with simple curves of suitable concavity. This essentially anchors the parts of the graph in place. To get the picture more precise we determine the intersections: $f(0) = -\frac{2}{3}$ is the y-intercept, the x intercept is given by $f(x) = 0$, it is $x = -1$. Thus we guess the following graph.

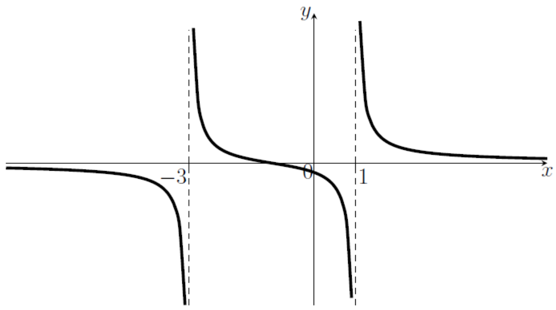

Figure 24

Since the middle part has two vertical asymptotes, it cannot be concave up nor concave down on the whole interval $(-3, 1)$. Thus there must be an inflection point there. We do not know where, in the picture we used the point $(-1, 0)$ as it makes for a nice picture, but it may be wrong.

We can confirm this shape and find out about that inflection point using the standard graph sketching procedure.

The domain of $f(x)$ is $(-\infty, -3) \cup (-3,1) \cup (1, \infty)$ and the function is continuous and differentiable on its domain.

The first derivative is

$$f'(x) = -2\frac{x^2 + 2x + 5}{(x-1)^2(x+3)^2}$$

which is always negative. Hence the function is decreasing on the open intervals $(-\infty, -3), (-3,1)$ and $(1, \infty)$. We can also determine concavity.

The second derivative is

$$f''(x) = 4\frac{x^3 + 3x^2 + 15x + 13}{(x-1)^3(x+3)^3}.$$

The numerator does not invite further analysis. However, we can try whether $x = -1$ is really a point of inflection: $f''(x) = 0$. Yes, we got lucky.

LO: AC21.

3. Find the partial fraction decomposition for the function

$$f(x) = \frac{4-x}{(x-1)^2(2x+1)}.$$

Solution:

Partial fractions will look like

$$\frac{4-x}{(x-1)^2(2x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{2x+1}$$

for suitable reals A, B, C.

This means

$$4 - x = A(x-1)(2x+1) + C(x-1)^2,$$

that is,

$$4 - x = (2B + C)x^2 + (2A - B - 2C)x + (A - B + C).$$

Two polynomials are equal if they have the same coefficients with corresponding powers of x.

Hence we can get a system od three linear equations for unknown A, B, C.

$$\begin{aligned} 0 &= \quad\;\; 2B + C \\ -1 &= 2A - B - 2C \\ 4 &= \;\; A - B + C \end{aligned}$$

Solving the system we obtain $A = 1, B = -1, C = 2$.

Hence

$$\frac{4-x}{(x-1)^2(2x+1)} = \frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{2}{2x+1}$$

We can often obtain at least some constants in an easier way, which will simplify remaining calculations. We start with the equation

$$4 - x = A(2x+1) + B(x-1)(2x+1) + C(x-1)^2$$

that should be true for all values x, hence it must be true also if we substitute some value for x.

If the denominator in our fraction has linear components, we can substitute their zero points.

Substituting $x = 1$ we get

$$3 = 3A + 0 + 0, \quad \text{hence } A = 1.$$

Substituting $x = -\frac{1}{2}$ we get

$$\frac{9}{2} = 0 + 0 + \frac{9}{4}C, \quad \text{hence } C = 2.$$

We need one more equation to derive B. For instance, we may substitute $x = 0$ to obtain

$$4 = A \cdot 1 + B \cdot (-1) + C \cdot 1.$$

Since we already know A and C, we easily get $B = -1$.

LO: AC22.

4. Find the partial fraction decomposition for the function

$$f(x) = \frac{9x+2}{(x^2+2x+2)(x-2)}$$

Solution:

Partial fractions will look like

$$\frac{9x+2}{(x^2+2x+2)(x-2)} = \frac{Ax+B}{x^2+2x+2} + \frac{C}{x-2}$$

for suitable reals A, B, C. (Notice that the polynomial $x^2 + 2x + 2$ cannot be factorized into product of two linear real polynomials hence the numerator of the first fraction will be a polynomial of degree 1.)

This means

$$9x + 2 = (Ax+B)(x-2) + C(x^2+2x+2)$$

that is,

$$9x + 2 = (A + B)x^2 + (-2A + B + 2C)x + (-2B + 2C).$$

Two polynomials are equal if they have the same coefficients with corresponding powers of x.

Hence we can get a system od three linear equations for unknown A, B, C.

$$\begin{aligned} 0 &= A \qquad\quad + C \\ 9 &= -2A + B + 2C \\ 2 &= \qquad -2B + 2C \end{aligned}$$

There is a linear factor in the denominator, so we can calculate C using substitution.
Substituting $x = 2$ into

$$9x + 2 = (Ax + B)(x - 2) + C(x^2 + 2x + 2)$$

we get

$$20 = (2A + B) \cdot 0 + C \cdot 10,$$

from which we get $C = 2$.

When we put this into the original system, we get $A = -2$ from the first equation and $B = 1$ from the third.

Hence

$$\frac{9x + 2}{(x^2 + 2x + 2)(x - 2)} = \frac{1 - 2x}{x^2 + 2x + 2} + \frac{2}{x - 2}.$$

LO: AC22.

4.3 Questions 2

1. Find the partial fraction decomposition for the function

$$f(x) = \frac{x + 7}{(x - 1)(x + 1)(x - 2)}.$$

Solution:

$$\frac{x + 7}{(x - 1)(x + 1)(x - 2)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x - 2}$$

Our special trick:

$$A = \left.\frac{x + 7}{(////)(x + 1)(x - 2)}\right|_{x=1} = -4, \qquad B = \left.\frac{x + 7}{(x - 1)(////)(x - 2)}\right|_{x=-1} = 1.$$

$$C = \left.\frac{x + 7}{(x - 1)(x + 1)(////)}\right|_{x=2} = 3$$

2. Find the partial fraction decomposition for the function

$$f(x) = \frac{x+5}{(x-1)^2(x+1)}.$$

Use the hiding trick for as many coefficients as possible, then find the rest using the trick with limit.

Solution:

The general form is

$$\frac{x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Hiding trick:

$$B = \left.\frac{x+5}{(////)(x+1)}\right|_{x=1} = 3, \qquad C = \left.\frac{x+5}{(x-1)^2(////)}\right|_{x=-1} = 1.$$

We substitute these into the decomposition and multiply it by x:

$$\frac{x^2+5}{(x-1)^2(x+1)} = \frac{Ax}{x-1} + \frac{3x}{(x-1)^2} + \frac{x}{x+1}$$

Passing to infinity with x we obtain

$$0 = A + 0 + 1 \Longrightarrow A = -1.$$

LO: AC22.

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Guide for a Problem

Analysis and Calculus

1.4. Complex numbers

AC3

Daniela Richtáriková

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 7, to be achieved with the development of this activity guide are the following:

1. To explain complex plane.
2. To explain and use polar and cartesian coordinates.
3. To explain relationship and transition between polar and cartesian coordinates.
4. To describe regions of complex plane by polar coordinates.
5. To state and make operations in all representations of complex numbers: algebraic, trigonometric and exponential.
6. To compare the difficulties in calculations with respect to operations.
7. To explain and use de Moivre theorem for rational exponent.
8. To calculate powers and roots of complex numbers.
9. To solve simple power equations in complex numbers.

Table 7 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 7: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
ANALYSIS AND CALCULUS									
AC3	3. Complex numbers								
AC31	State and use Euler's formula	yellow	yellow	yellow	yellow	green	green	yellow	red
AC32	State and understand De Moivre's theorem for a rational index	green	green	yellow	yellow	green	green	green	red
AC33	Find the roots of a complex number	green	green	yellow	yellow	green	green	green	red
AC34	Link trigonometric and hyperbolic functions	green	green	yellow	yellow	green	green	green	red
AC35	Describe regions in the plane by restricting the modulus and / or the argument of a complex	green	green	yellow	yellow	red	red	green	yellow

2 CONTENTS

1. Complex Plane, Cartesian and Polar coordinates.
2. Algebraic form of a complex number, operations.
3. Trigonometric form of a complex number, operations, De Moivre's theorem.
4. Euler form of a complex number, operations.
5. Relationship between forms.
6. Roots of complex numbers.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 8.

Table 8: Learning outcomes (LO) involved in this assessment activity.

Analysis and Calculus	
AC3	3. Complex numbers
AC31	State and use Euler's formula
AC32	State and understand De Moivre's theorem for a rational index
AC33	Find the roots of a complex number
AC34	Link trigonometric and hyperbolic functions
AC35	Describe regions in the plane by restricting the modulus and / or the argument of a complex

3.1 Multiple choice Questions

Instruction: The number of correct answers varies between 0 and 4. The sum of all correct answers is 10.

1. Imaginary part in trigonometric form of a complex number is expressed through
    a) Cosine of the argument
    b) Sine of the argument
    c) Tangent of the argument
    d) Euler number
2. Real part in trigonometric form of a complex number is expressed through
    a) Cosine of the argument
    b) Sine of the argument
    c) Tangent of the argument
    d) Euler number
3. Which of the pairs are considered to be complex conjugates?
    a) $ci + d, -ci + d$
    b) $a + bi, -(a + bi)$
    c) $cos\,\varphi - i\,sin\,\varphi, cos\,\varphi + i\,sin\,\varphi$
    d) $e^{-i\varphi}, e^{i\varphi}$
4. Images of complex conjugates in Gauss plane are
    a) Symmetric about horizontal axis
    b) Symmetric about imaginary axis
    c) Symmetric about origin of axes
    d) Not symmetric
5. The solution of equation $x^4 - 16 = 0$ is
    a) {2}
    b) {2, -2}
    c) {2, 2i, $\sqrt{2}$, -2}
    d) {2, 2i, -2i, -2}

6. Which of the triads $\{x_1, x_2, x_3\}$are possible solutions of the cubic equation$x^3 = z$? Give the reason. Sketch the numbers.

   a) $\{2, 2(cos\frac{2}{3}\pi + i\,sin\frac{2}{3}\pi), 2(cos\left(\frac{2}{3}\pi\right) - i\,sin\left(\frac{2}{3}\pi\right))\}$

   b) $\{2, 2(cos\frac{2}{3}\pi + i\,sin\frac{2}{3}\pi), 2(cos\left(-\frac{2}{3}\pi\right) + i\,sin\left(-\frac{2}{3}\pi\right))\}$

   c) $\{e^{i3}, e^{i(3+\frac{2}{3}\pi)}, e^{i(3+\frac{4}{3}\pi)}\}$

3.2 QUESTION 1

1. Given two complex numbers z_1 and z_2: $Re(z_1) = 1, Im(z_1) = \sqrt{3}$, the argument of z_2 is $-\frac{\pi}{6}$ and its absolute value is 2.

   a) Express z_1 and z_2 in all forms (algebraic, trigonometric, exponential). Determine their cartesian and polar coordinates.

   b) First, think over a strategy of calculation, and calculate

      i) The division $z = z_1/z_2$ (product, sum or subtraction)

      ii) Give the reason for your strategy. Was it the most efficient way?

      iii) Interpret calculation of the argument in graphical way.

      iv) Which kind of complex numbers is the result z from? Give the reason.

      v) Display the result *z* in complex plane.

2. Given the complex number z_1 : $Re(z_1) = 1, Im(z_1) = \sqrt{3}$

   a) Calculate z_1^4 in trigonometric form. Interpret calculation of the argument in graphical way. Write the result in exponential form.

   b) Calculate z_1^{70} in exponential form.

   c) Calculate $\sqrt[3]{z}$. Interpret calculation of the argument in graphical way. What shape is created by the roots of z?

   d) How $z^{p/q}$ will be calculated? Express both in trigonometric and exponential form.

3. Given z_2, the argument of z_2 is $-\frac{\pi}{6}$ and its absolute value is 2. Calculate in trigonometric form

   a) The sum of complex conjucates z_2 and $\bar{z}_2$.

   b) The subtraction of complex conjucates z_2 and $\bar{z}_2$.

   c) Express Cos(2) using an operation of complex conjugate numbers suggested by you. Write the expression via the exponential form of complex numbers.

   d) Express Sin(2) using an operation of complex conjugate numbers suggested by you. Write the expression via the exponential form of complex numbers.

3.3 Question 2

A wall clock is designed in the shape of square with the horizontal edge of the length 32 cm.

a) Determine the vertices of the square in terms of complex numbers in all forms: algebraic, goniometrical and exponentional. The middle of the clock lies in the origin of complex plane.

An engraving machine engraves the clock numerals images on the clock face with radius 14 cm. The middle of the clock lies in the origin of complex plane, the places of images are determined as complex numbers and they are engraved gradually one by one by turning the tool on reference position of the image.

b) Propose the computation formula for the algorithm to engrave

   i) All 12 numerals images.

   ii) The images of 1, 9, and 4.

c) Propose the region description in means of ranges for modulus and arguments to colour the annullus part between 1 and 9, in the distance from 3 cm to 10 cm from the middle of the clock (see the figure).

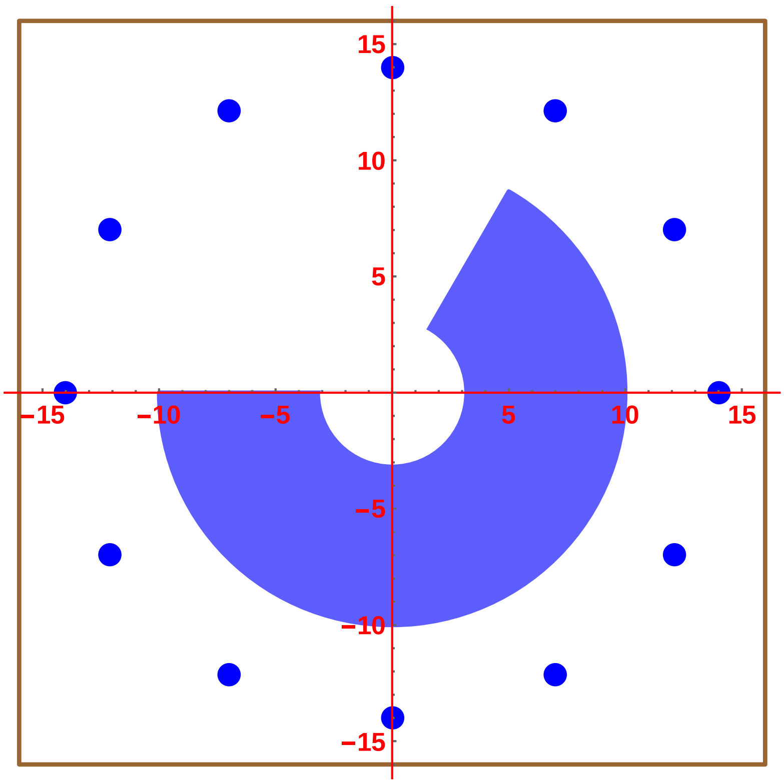


Figure 25

4 SOLUTION

4.1 MULTIPLE CHOICE QUESTIONS

Instruction: The number of correct answers varies between 0 and 4. The sum of all correct answers is 10.

1. Imaginary part in trigonometric form of a complex number is expressed through

   a) Cosine of the argument

   b) Sine of the argument

   c) Tangent of the argument

   d) Euler number

   Correct answer: b)

   LO: AC34.

   Competencies: C1, C5, C6.

2. Real part in trigonometric form of a complex number is expressed through

   a) Cosine of the argument

   b) Sine of the argument

   c) Tangent of the argument

   d) Euler number

   Correct answer: a)

   LO: AC34.

3. Which of the pairs are considered to be complex conjugates?

   a) $ci + d, -ci + d$

   b) $a + bi, -(a + bi)$

   c) $cos\,\varphi - i\,sin\,\varphi, cos\,\varphi + i\,sin\,\varphi$

   d) $e^{-i\varphi}$,$e^{i\varphi}$

   Other possibilities: $a + bi$, $a - bi$, $ci + d$, $ci - d$, $ci + d$, $-(ci + d)$, $cos\,\varphi + i\,sin\,\varphi$, $-\,cos\,\varphi + i\,sin\,\varphi$, $sin\,\varphi + i\,cos\,\varphi$, $sin\,\varphi - i\,cos\,\varphi$, $e^{i\varphi}$,$\frac{1}{e^{i\varphi}}$, $-e^{i\varphi}$,$e^{i\varphi}$)

   Correct answers: a), c), d)

   LO: AC31, AC34.

4. Images of complex conjugates in complex plane are

   a) Symmetric about horizontal axis

   b) Symmetric about imaginary axis

c) Symmetric about origin of axes

d) Not symmetric

Correct answer: a)

LO: AC33, AC34.

5. The solution of equation $x^4 - 16 = 0$ is

a) {2}

b) {2, -2}

c) {2, 2i, $\sqrt{2}$, -2}

d) {2, 2i, -2i, -2}

Correct answer: d)

LO: AC32, AC33.

6. Which of the triads $\{x_1, x_2, x_3\}$are possible solutions of the cubic equation$x^3 = z$? Give the reason. Sketch the numbers.

a) 2, $2(cos\frac{2}{3}\pi + i\,sin\frac{2}{3}\pi)$, $2(cos\left(\frac{2}{3}\pi\right) - i\,sin\left(\frac{2}{3}\pi\right))$ }

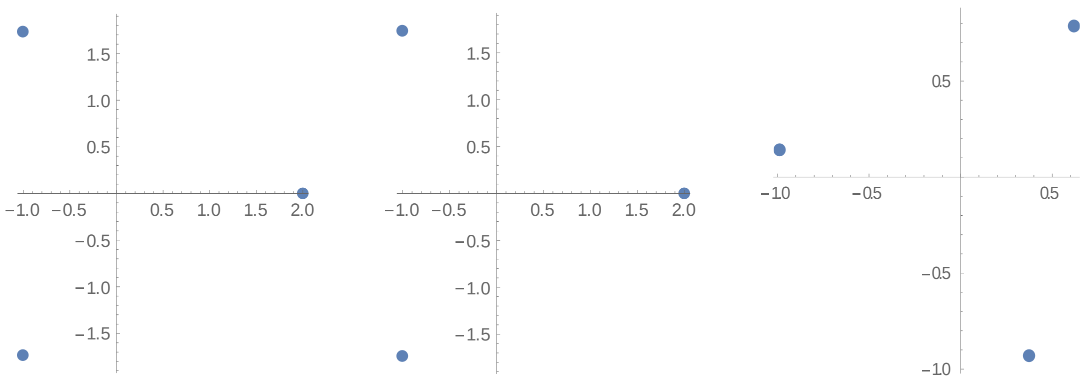
b) {2, $2(cos\frac{2}{3}\pi + i\,sin\frac{2}{3}\pi)$, $2(cos\left(-\frac{2}{3}\pi\right) + i\,sin\left(-\frac{2}{3}\pi\right))$ }

c) $\{e^{i3}, e^{i(3+\frac{2}{3}\pi)}, e^{i(3+\frac{4}{3}\pi)}\}$

Correct answers: a), b), c)

Solution:

All triads are possible roots of a cubic equation, since they form equilateral triangles (at each case the modulus is 2 and the arguments differ by 2/3π) and in addition, a) and b) are the same triads since $cos\left(\frac{2}{3}\pi\right) = cos\left(-\frac{2}{3}\pi\right)$ and $-sin\left(\frac{2}{3}\pi\right) = sin\left(-\frac{2}{3}\pi\right)$



a b c

Figure 26

LO: AC31, AC32, AC33, AC34.

4.2 QUESTION 1

1. Given two complex numbers z_1 and z_2: $Re(z_1) = 1, Im(z_1) = \sqrt{3}$, the argument of z_2 is $-\frac{\pi}{6}$ and it's absolute value is 2:

a) Expres z_1 and $z_2$in all forms (algebraic, trigonometric, exponential). Determine their cartesian and polar coordinates.

b) First, think over a strategy of calculation, and calculate

   i) The division $z = z_1/z_2$ (product, sum or subtraction).

   ii) Give the reason for your strategy. Was it the most efficient?

   iii) Interpret calculation of the argument in graphical way.

   iv) Which kind of complex numbers is the result *z* from? Give the reason.

   v) Display the result *z* in complex plane.

Solution:

a) $z_1 = 1 + \sqrt{3}i = 2\left(cos\frac{\pi}{3} + i\,sin\frac{\pi}{3}\right) = 2e^{\frac{\pi}{3}i}$

Coordinates cartesian $Re(z_1) = 1, Im(z_1) = \sqrt{3}, z_1 = [x, y] = [1, \sqrt{3}]$

Coordinates polar $|z_1| = r = 2, \varphi = \frac{\pi}{3}, z_1 = [r, \varphi] = [2, \frac{\pi}{3}]$

$z_2 = \sqrt{3} - i = 2\left(cos\left(-\frac{\pi}{6}\right) + i\,sin\left(-\frac{\pi}{6}\right)\right) = 2\left(cos\frac{\pi}{6} - i\,sin\frac{\pi}{6}\right) = 2e^{-\frac{\pi}{6}i}$

Coordinates cartesian $Re(z_2) = \sqrt{3}, Im(z_2) = -1, z_2 = [x, y] = [\sqrt{3}, -1]$

Coordinates polar $|z_2| = r = 2, \varphi = -\frac{\pi}{6}, z_1 = [r, \varphi] = [2, -\frac{\pi}{6}]$

b)

   i) $z = z_1/z_2 = i$

   ii) pure imaginary

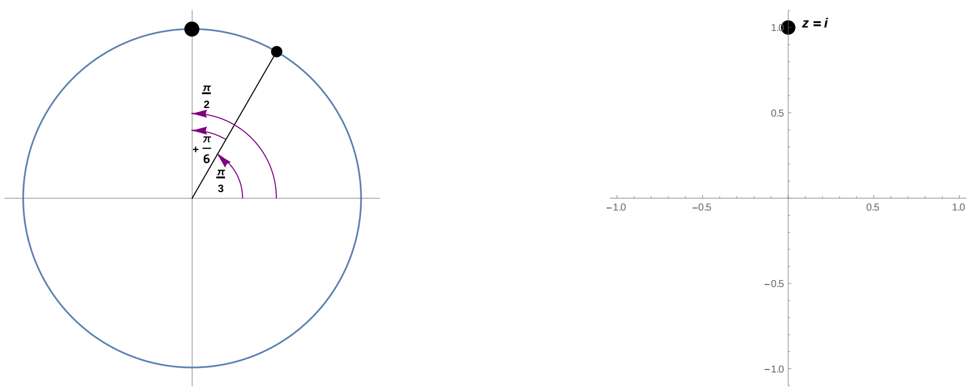


Figure 27

LO: AC31, AC34.

2. Given the complex numbers z_1: $Re(z_1) = 1, Im(z_1) = \sqrt{3}$:
   a) Calculate z_1^4 in trigonometric form. Interpret calculation of the argument in graphical way. Write the result in exponential form.
   b) Calculate z_1^{70} in exponential form.
   c) Calculate $\sqrt[3]{z}$. Interpret calculation of the argument in graphical way. What shape is created by the roots of z?
   d) How $z^{p/q}$ will be calculated? Express both in trigonometric and exponential form.

Solution:

a) $z_1^4 = 2^4(cos\frac{4}{3}\pi + i\, sin\frac{4}{3}\pi)$ =$-8 - 8\sqrt{3}i$ = $2^4 e^{\frac{4}{3}\pi}$

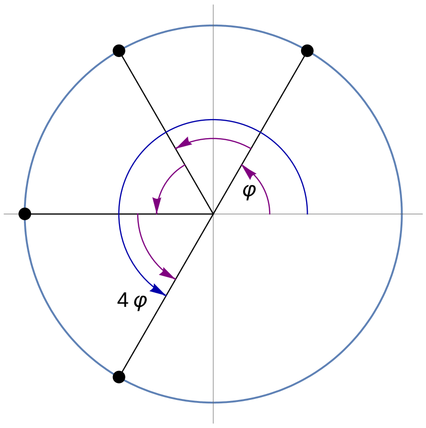


Figure 28

b) $z_1^{70} = 2^{70}(cos\left(70\frac{\pi}{3}\right) + i\, sin\left(70\frac{\pi}{3}\right)) = 2^{70}(cos\frac{4}{3}\pi + i\, sin\frac{4}{3}\pi)$ = $2^{70}e^{\frac{4}{3}\pi}$, $(70\frac{\pi}{3} = 33 \times 2\pi + \frac{4}{3}\pi)$

c) $\sqrt[3]{z} = \sqrt[3]{i} = \sqrt[3]{e^{\frac{\pi}{2}i}}: i = x^3 \Rightarrow x_1 = e^{\frac{\pi}{6}i}, x_2 = e^{\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)i} = e^{\frac{5\pi}{6}i}, x_3 = e^{\left(\frac{\pi}{6}+\frac{2\times2\pi}{3}\right)i} = e^{\left(\frac{3\pi}{2}\right)i}$

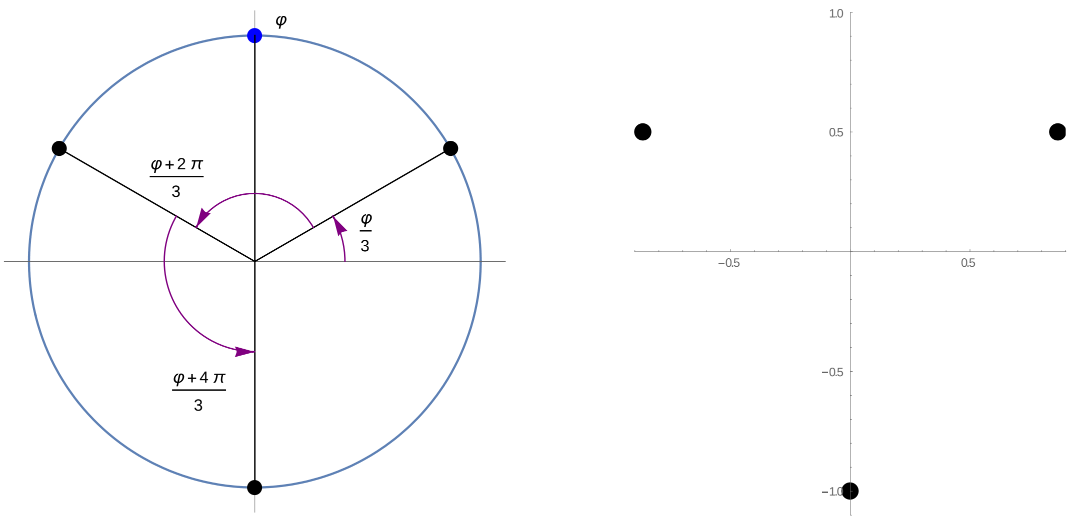


Figure 29

d) $z^{p/q} = |z^{p/q}|(cos\frac{p\varphi+k2\pi}{q} + i\,sin\frac{p\varphi+k2\pi}{q}) = |z^{p/q}|e^{\frac{p\varphi+k2\pi}{q}i}, k \in Z$

LO: AC31, AC32, AC33, AC34.

3. Given z_2, the argument of z_2 is $-\frac{\pi}{6}$ and it's absolute value is 2. Calculate in trigonometric form
   a) The sum of complex conjucates z_2 and $\bar{z}_2$.
   b) The subtraction of complex conjucates z_2 and $\bar{z}_2$.
   c) Express Cos(2) using an operation of complex conjugate numbers suggested by you. Write the expression via the exponential form of complex numbers.
   d) Express Sin(2) using an operation of complex conjugate numbers suggested by you. Write the expression via the exponential form of complex numbers.

Solution:

a) $z_2 + \bar{z}_2 = 2\left(cos\frac{\pi}{6} - i\,sin\frac{\pi}{6}\right) + 2\left(cos\frac{\pi}{6} + i\,sin\frac{\pi}{6}\right) = 4\,cos\frac{\pi}{6}$

b) $z_2 - \bar{z}_2 = 2\left(cos\frac{\pi}{6} - i\,sin\frac{\pi}{6}\right) - 2\left(cos\frac{\pi}{6} + i\,sin\frac{\pi}{6}\right) = -4i\,sin\frac{\pi}{6}$

c) $cos(\varphi) = \frac{z_2+\bar{z}_2}{2|z_2|} \Rightarrow cos\,2 = \frac{(cos\,2+i\,sin\,2)+(cos\,2-i\,sin\,2)}{2} = \frac{e^{2i}+e^{-2i}}{2}$

d) $sin(\varphi) = \frac{z_2-\bar{z}_2}{2i|z_2|} \Rightarrow sin\,2 = \frac{(cos\,2+i\,sin\,2)-(cos\,2-i\,sin\,2)}{2i} = \frac{e^{2i}-e^{-2i}}{2i}$

4. Sketch the set of complex numbers z in complex plane for which $0 \le |z| \le 3$ and $\varphi \in \left\langle -\frac{\pi}{4}, \frac{\pi}{2} \right\rangle$

Solution:

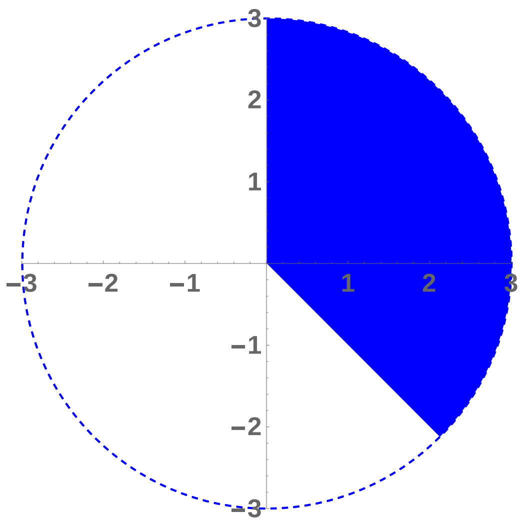


Figure 30

LO: AC31, AC32, AC33, AC34.

4.3 QUESTION 2

A wall clock is designed in the shape of square with the horizontal edge of the length 32 cm.

a) Determine the vertices of the square in terms of complex numbers in all forms: algebraic, goniometrical and exponentional. The middle of the clock lies in the origin of complex plane.

An engraving machine engraves the clock numerals images on the clock face with radius 14 cm. The middle of the clock lies in the origin of complex plane, the places of images are determined as complex numbers and they are engraved gradually one by one by turning the tool on reference position of the image.

b) Propose the computation formula for the algorithm to engrave:

   i) All 12 numerals images.

   ii) The images of 1, 9, and 4.

c) Propose the region description in means of ranges for modulus and arguments to colour the annullus part between 1 and 9, in the distance from 3 cm to 10 cm from the middle of the clock (see the figure).

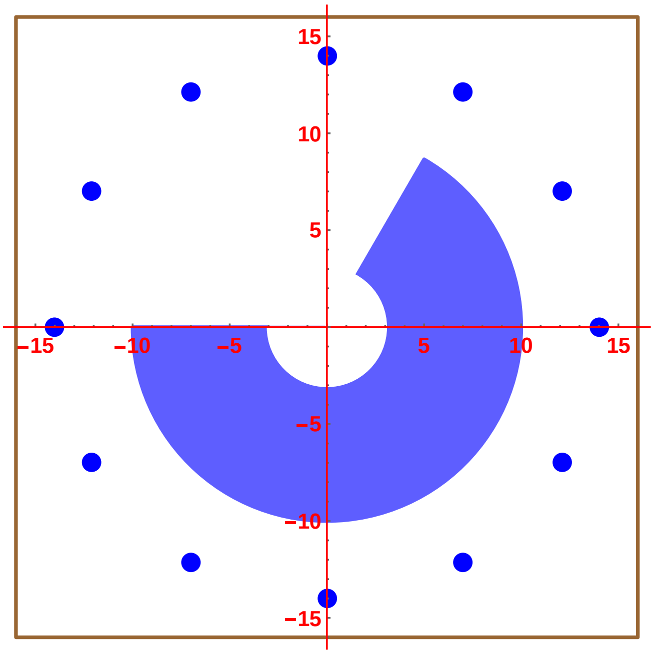


Figure 31

Solution:

a) The square vertices:

$$r = \sqrt{16^2 + 16^2} = 16\sqrt{2}, \varphi = \frac{\pi + k2\pi}{4}, k = 0, 1, 2, 3$$

$$\left\{16\sqrt{2}e^{\frac{\pi}{4}i}, 16\sqrt{2}e^{\frac{3\pi}{4}i}, 16\sqrt{2}e^{\frac{5\pi}{4}i}, 16\sqrt{2}e^{\frac{7\pi}{4}i}\right\}$$

$$\left\{16\sqrt{2}\left(\cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right), 16\sqrt{2}\left(\cos\frac{3\pi}{4} + i\,sin\frac{3\pi}{4}\right), 16\sqrt{2}\left(\cos\frac{5\pi}{4} + i\,sin\frac{5\pi}{4}\right) 16\sqrt{2}\left(\cos\frac{7\pi}{4} + i\,sin\frac{7\pi}{4}\right)\right\}$$

$$\{16 + 16i, -16 + 16i, -16 - 16i, 16 - 16i\}$$

b) Formulas

i) $z = 14\sqrt{2}e^{\frac{\pi+k2\pi}{12}}$, k= 0, 1, .., 11

ii) $z = 14\sqrt{2}e^{\frac{\pi+k2\pi}{3}}$, k= 0, 1, 2

c) Color region:

$$\left\{z = |z|e^{i\varphi}; 3 \leq |z| \leq 10, \varphi \in \left\langle -\pi, \frac{\pi}{3} \right\rangle \right\}$$

LO: AC3.

5 BIBLIOGRAPHY

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Guide for a Problem

Analysis and Calculus

1.5. Differentiation

AC5

Peter Letavaj

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 9, to be achieved with the development of this activity guide are the following:

1. To understand the concepts of continuity and smoothness.
2. To differentiate inverse functions.
3. To differentiate functions defined implicitly.
4. To differentiate functions defined parametrically.
5. To locate any points of inflection of a function.
6. To find greatest and least values of physical quantities.

Table 9 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 9: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Analysis and Calculus									
AC5	5. Differentiation								
AC51	Understand the concepts of continuity and smoothness	yellow	green	yellow	yellow	yellow	yellow	green	red
AC52	Differentiate inverse functions	yellow	green	green	yellow	yellow	yellow	green	red
AC53	Differentiate functions defined implicitly	yellow	yellow	green	yellow	yellow	red	yellow	yellow
AC54	Differentiate functions defined parametrically	yellow	red	red	yellow	yellow	yellow	yellow	yellow
AC55	Locate any points of inflection of a function	green	green	green	yellow	red	red	yellow	red
AC56	Find greatest and least values of physical quantities	green	green	green	yellow	red	red	yellow	red

2 CONTENTS

1. Definition and relationship between continuity and differentiability.
2. Points of inflection graphical localization.
3. Formal derivative of composite function and product of functions.
4. Derivative of inverse function.
5. Usage of derivative definition in approximative way.
6. Finding local extreme and global extreme on interval.
7. Derivative of function given parametrically.
8. Derivative of function given implicitly.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 10.

Table 10: Learning outcomes (LO) involved in this assessment activity.

Analysis and Calculus	
AC5	5. Differentiation
AC51	Understand the concepts of continuity and smoothness
AC52	Differentiate inverse functions
AC53	Differentiate functions defined implicitly
AC54	Differentiate functions defined parametrically
AC55	Locate any points of inflection of a function
AC56	Find greatest and least values of physical quantities

3.1 Multiple-choice questions

Solve the following test.

1. Function $f(x) = \frac{x^2-5x+6}{x-3}$ is not defined at $x = 3$. How should it be defined there in order to obtain continuous function?
   i) $f(3) = 5$
   ii) $f(3) = -5$
   iii) $f(3) = 1$
   iv) $f(3) = -1$
2. Which assertion is correct?
   i) Function which is continuous at a point x_0 is always differentiable at this point.
   ii) Function which is continuous at a point x_0 can be differentiable at this point.
   iii) Function which is continuous at a point x_0 is never differentiable at this point.
   iv) There is no relationship between continuity and differentiability.
3. In which situation the function $f(x)$ has got a point of inflection at x_0? Suppose that $f(x)$ is smooth function, i.e. all its derivatives are continuous.

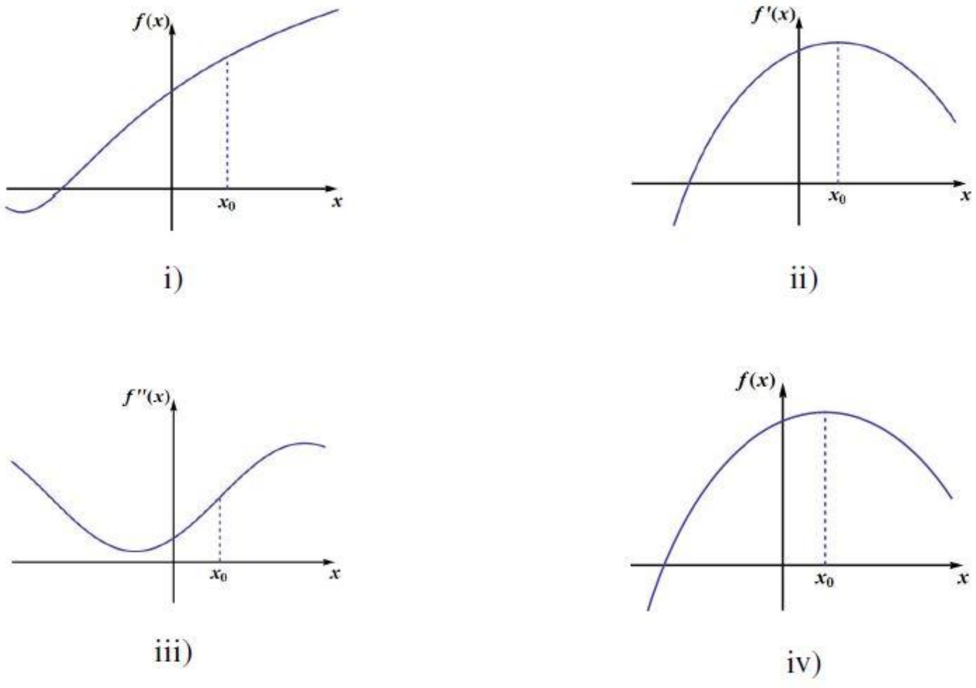


Figure 32

4. What is derivative of $f(g(x).h(x))$?
   i) $[f(g(x).h(x))]' = f'(g(x).h(x)).g'(x).h'(x)$
   ii) $[f(g(x).h(x))]' = f(g'(x).h'(x)).f'(x)$
   iii) $[f(g(x).h(x))]' = f(g'(x).h(x) + g(x).h'(x)).g(x).h(x)$
   iv) $[f(g(x).h(x))]' = f'(g(x).h(x)).[g'(x).h(x) + g(x).h'(x)]$

3.2 QUESTION 1

1. What is derivative of the function f^{-1} evaluated at 1, f^{-1} is inverse to the function $f: y = \tan(x)$?
2. What is derivative of the function f^{-1} evaluated at 10, f^{-1} is inverse to the function $f: y = x.\sqrt{x-1}$? (Existence of f^{-1} at 10 is guaranteed.)

3.3 QUESTION 2

Two friends Peter and Lucy downloaded mobile Application each. This App gives the location based on positioning system and works in different modes.

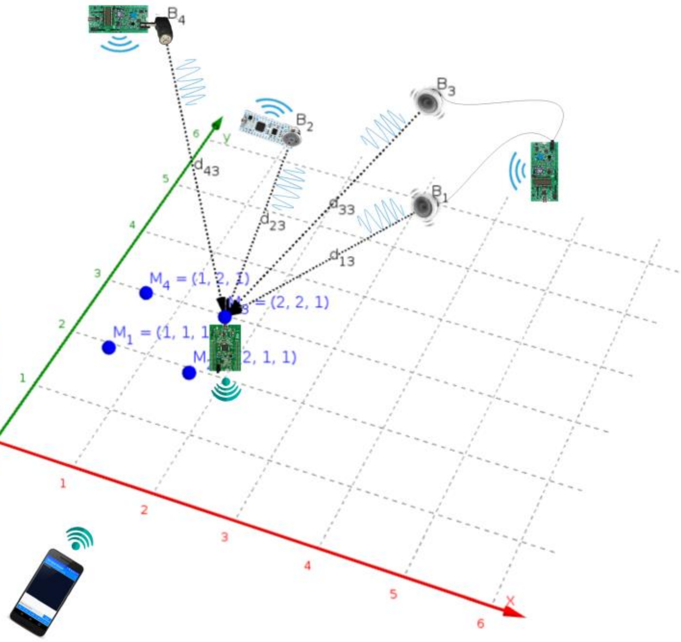


Figure 33

Today they travel to another town to amusement park to enjoy the day. And they are going to test the Application in all possible situations.

1. They got into the train and turned on the Application in mode DISCRETE at the moment of departure of the train. Unfortunately, they pushed the buttons with 5 seconds difference. The time on the clock in mobile devices was synchronized. The railway is a straight line. After a while, they finished active phase of collecting positions and asked for evaluation at the specific time $t = 100s$, which was in real equivalent to the time 12h 1min 40s. They obtained these graphs of position in meters depending on time in seconds.

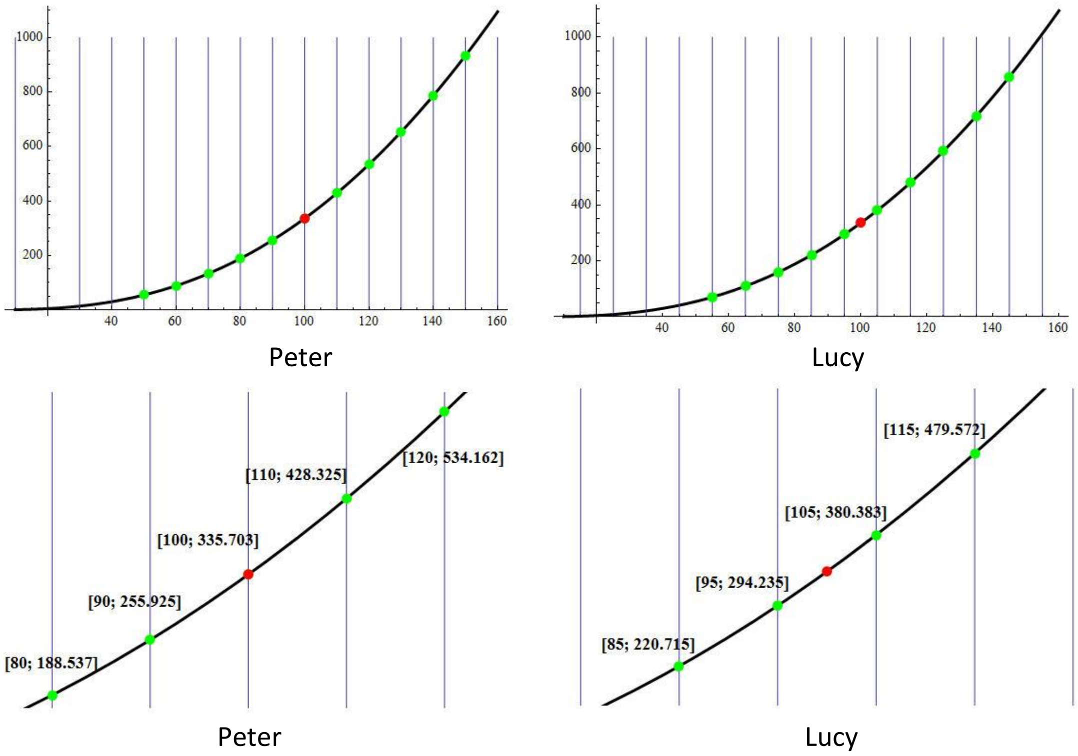


Figure 34

In next tasks i), ii) and iii) use only discrete values of coordinates!

i) How could Peter determine the velocity of the train at time 100s? What is the most precious value determined by him? (Hint: Velocity is derivative of the position. Find the derivative of the sketched function at the time 100*s*!).

ii) How could Lucy determine the most precious value of velocity at the time 100*s*?

iii) Give an approximate value of acceleration of the train in the time 100s. (Hint: Find the second derivative of the function sketched in the figure!).

2. Then they changed the mode to FITTING FUNCTION. The result they obtained (using collected data) was $f: y = -\left(\frac{x}{19}\right)^3 . log\left(\frac{x}{1000}\right)$, where *y* is distance and *x* is time.

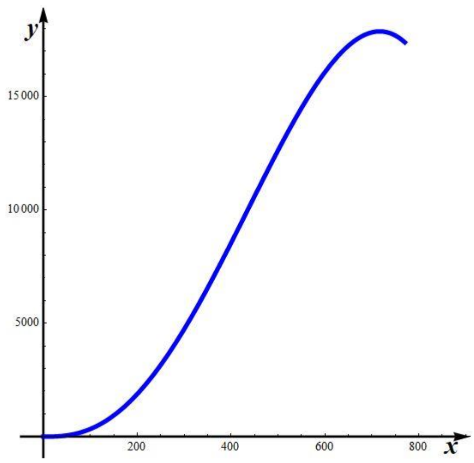


Figure 35

i) Give analytically velocity $v(100) = f'(100)$ and acceleration $a(100) = f''(100)$ of the train at time $t = 100s$!

ii) If the motion of the train would follow the function f, when will it stop (zero velocity) and how far from the beginning?

iii) What was the highest and the lowest velocity of the train in between *400s* and *600s*?

3. After arriving into park, Lucy got in the multi-rotational and shift device. She turned on mode PARAMETRIC and started the ride. After getting off, the Application gave her these functions:

$$x(t) = \frac{1}{3}t.(t-20).cos\left(\frac{\pi}{10}t\right)$$

$$y(t) = 15sin\left(\frac{\pi}{10}t\right)$$

$$z(t) = 15cos\left(\frac{\pi}{10}t\right)$$

i) Peter watched her and stopped the stopwatch at the time $t_1 = 6.32s$. It seemed to him, that she is moving very fast. What was her velocity $v(t_1)$ at that time? (Hint: $v(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$)

ii) On the other hand, she stopped the stopwatch at the time $t_2 = 10s$, when she almost fainted away. What was her acceleration $a(t_2)$ at that time? (Hint: $a(t) = \sqrt{x''(t)^2 + y''(t)^2 + z''(t)^2}$)

4. Peter visited another attraction, where his motion followed this graph:

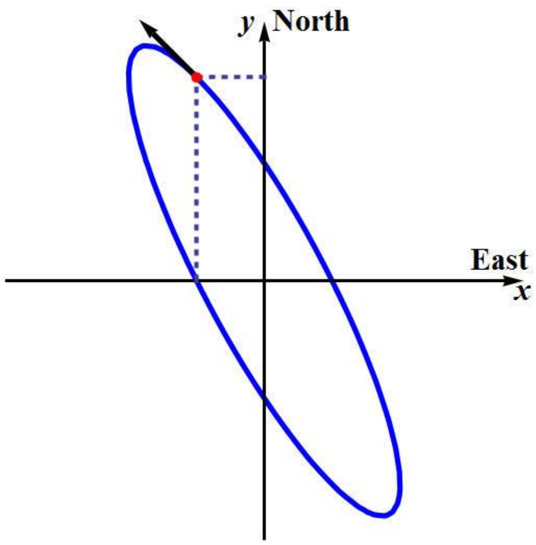


Figure 36

Beforehand he switched Application to the mode IMPLICIT. Application turned back this equation of rotated ellipse:

$$\frac{x^2}{50} + \frac{y^2}{150} + \frac{x.y}{50} = 1$$

In which point (coordinates) of trajectory did he moved northwestward, if he moved on the ellipse counterclockwise?

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test

1. Function $f(x) = \frac{x^2-5x+6}{x-3}$ is not defined at $x = 3$. How should it be defined there in order to obtain continuous function?

   i) $f(3) = 5$

   ii) $f(3) = -5$

   iii) $f(3) = 1$

   iv) $f(3) = -1$

   Solution:

   Function f is continuous at $x = 3$ if and only if $f(3) = \lim_{x\to 3} f(x)$.

$$\lim_{x\to 3}\frac{x^2-5x+6}{x-3} = \left|\frac{0}{0}\right| = \lim_{x\to 3}\frac{(x-3)(x-2)}{x-3} = \lim_{x\to 3}(x-2) = 3-2 = 1$$

   The solution is iii).

2. Which assertion is correct?

   i) Function which is continuous at a point x_0 is always differentiable at this point.

   ii) Function which is continuous at a point x_0 can be differentiable at this point.

   iii) Function which is continuous at a point x_0 is never differentiable at this point.

   iv) There is no relationship between continuity and differentiability.

   Solution:

   i) is not correct. For example $y = |x|$ is continuous at $x = 0$, but not differentiable there.

   iii) is not correct. For example $y = x$ is continuous and differentiable at $x = 0$.

   iv) is not correct. There is a relationship: If the function is differentiable at $x = 0$, then it is continuous there.

   ii) is correct. Functions from i) and iii) prove it.

   The solution is ii).

3. In which situation the function $f(x)$ has got a point of inflection at x_0? Suppose that $f(x)$ is smooth function, i.e. all its derivatives are continuous.

   Solution:

   iii) is not correct. Necessary condition for existence of point of inflection at x_0 is $f''(x_0) = 0$.

   i) and iv) are not correct. Graphically: tangent line to the graph of the function $f(x)$ at the point of inflection is a secant. (In vicinity of x_0, it is not above the graph only, like here, or it is not below the graph only.)

   ii) is correct. Clearly on the left from x_0 the function $f'(x)$ is increasing, i.e. $f''(x)$ is positive.

On the right from x_0 the function $f'(x)$ is decreasing, i.e. $f''(x)$ is negative. At x_0 there is local extreme of $f'(x)$, i.e. $f''(x) = 0$ (regarding existence of all continuous derivatives, particularly $f''(x)$).

The solution is ii).

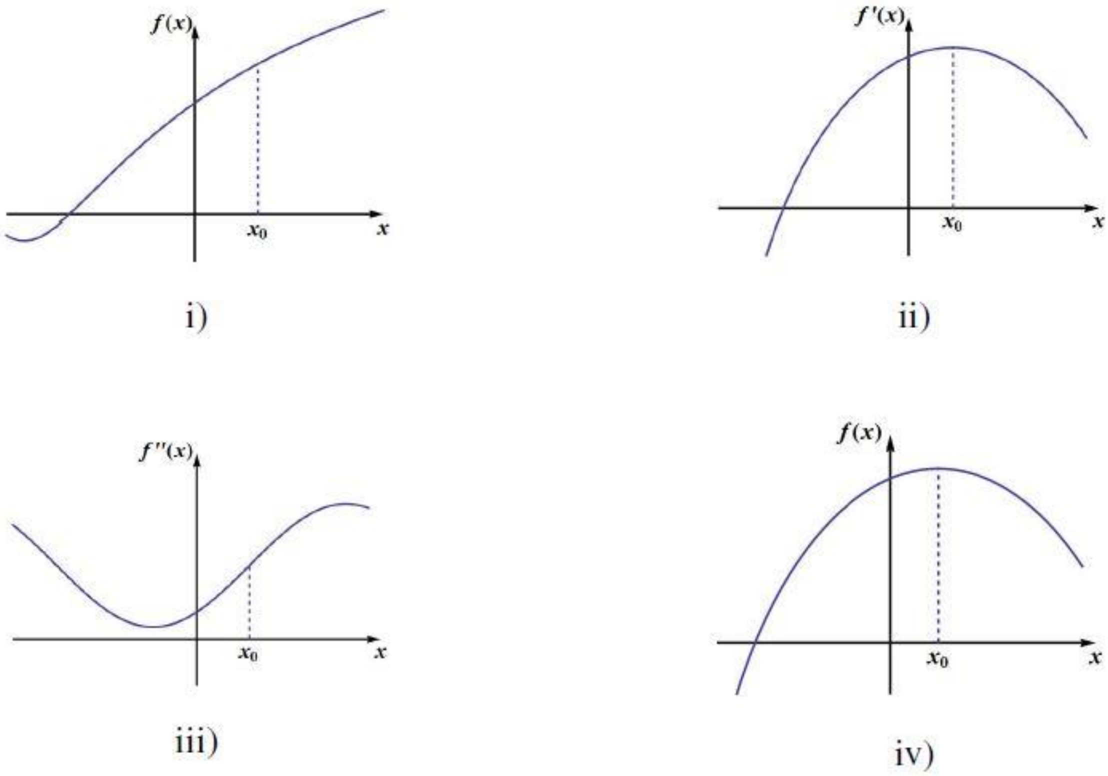


Figure 37

4. What is derivative of $f(g(x).h(x))$?

i) $[f(g(x).h(x))]' = f'(g(x).h(x)).g'(x).h'(x)$

ii) $[f(g(x).h(x))]' = f(g'(x).h'(x)).f'(x)$

iii) $[f(g(x).h(x))]' = f(g'(x).h(x) + g(x).h'(x)).g(x).h(x)$

iv) $[f(g(x).h(x))]' = f'(g(x).h(x)).[g'(x).h(x) + g(x).h'(x)]$

Solution:

We need to use two facts:

1. Derivative of composite function: $[f(u(x))]' = f'(u(x)).u'(x)$.

2. Derivative of product of two functions: $(g(x).h(x))' = g'(x).h(x) + g(x).h'(x)$.

If in 1. we substitute $u(x)$ by $g(x).h(x)$ and $u'(x)$ by $g'(x).h(x) + g(x).h'(x)$, we obtain iv).

The solution is iv).

LO: AC5: AC51, AC55, AC56.

4.2 QUESTION 1

1. What is derivative of the function f^{-1} evaluated at $x = 1$, which is inverse to the function $f: y = \tan(x)$?

   Solution:

   Inverse function of $f(x) = \tan(x)$ is $f^{-1}(x) = \arctan(x)$. Derivative of inverse function is $[f^{-1}(x)]' = \left(arctan(x)\right)' = \frac{1}{1+x^2}$. Thus $[f^{-1}(x)]'|_{x=1} = \frac{1}{1+1^2} = \frac{1}{2}$.

   Or using formula for derivative of inverse function: $[f^{-1}(y)]' = \frac{1}{f'(x)}$, where $y = f(x)$ or $x = f^{-1}(y)$. In our case y= 1 and $x = \arctan(1) = \frac{\pi}{4}$. Derivative of function $f(x)$ is $f'(x) = \left(tan(x)\right)' = \frac{1}{cos^2(x)}$. Then $[f^{-1}(y)]'|_{y=1} = \left.\frac{1}{\frac{1}{cos^2 x}}\right|_{x=\frac{\pi}{4}} = cos^2\frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

2. What is derivative of the function f^{-1} evaluated at 10, which is inverse to the function $f: y = x.\sqrt{x-1}$? (Existence of f^{-1} is guaranteed. Hint: $x = 5$.)

   Solution:

   Derivative of inverse function: $[f^{-1}(y)]' = \frac{1}{f'(x)}$, where $y = f(x)$ or $x = f^{-1}(y)$. In our case y $= 10$ and $x = 5$. Derivative of function $f(x)$ is

   $f'(x) = \left(x.\sqrt{x-1}\right)' = 1.\sqrt{x-1} + x.\frac{1}{2.\sqrt{x-1}}$.

   Then $[f^{-1}(y)]'|_{y=10} = \left.\frac{1}{\sqrt{x-1}+x.\frac{1}{2.\sqrt{x-1}}}\right|_{x=5} = \frac{1}{2+\frac{5}{4}} = \frac{4}{13}$.

LO: AC52.

4.3 QUESTIONS 2

Two friends Peter and Lucy downloaded mobile Application each. This App gives the location based on positioning system and works in different modes.

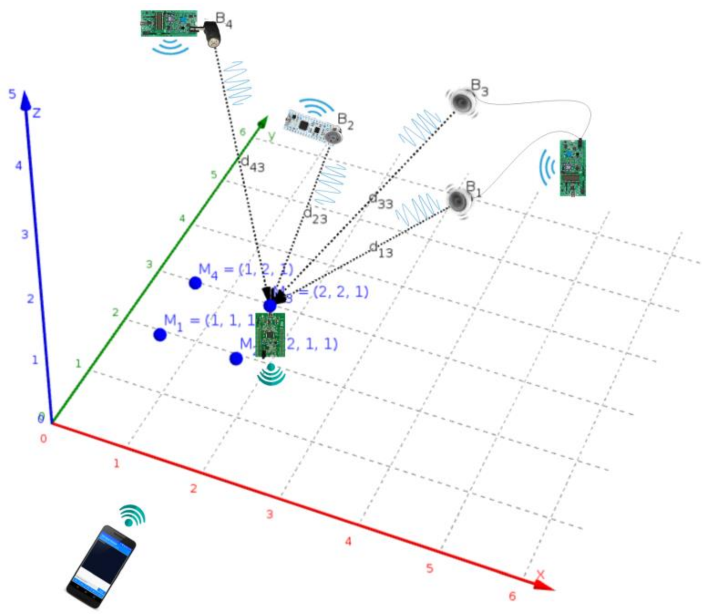
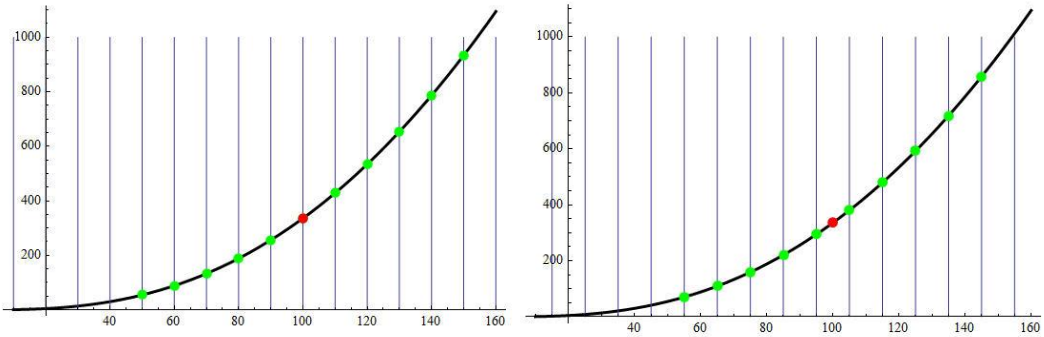


Figure 38

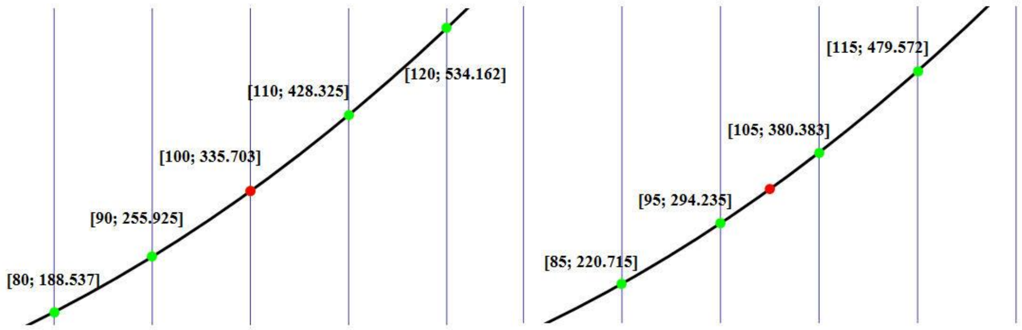
Today they travel to another town to amusement park to enjoy the day. And they are going to test the Application in all possible situations.

1. They got into the train and turned on the Application in mode DISCRETE at the moment of departure of the train. Unfortunately, they pushed the buttons with 5 seconds difference. The time on the clock in mobile devices was synchronized. The railway is a straight line. After a while, they finished active phase of collecting of positions and asked for evaluation at the specific time $t = 100s$, which was in real equivalent to the time 12h 1min 40s. They obtained these graphs of position in meters depending on time in seconds.



Peter Lucy

Figure 39



Peter Lucy

Figure 40

In next tasks i), ii) and iii) use only discrete values of coordinates!

i) How could Peter determine the velocity of the train at time 100s as accurately as possible? (Hint: Velocity is derivative of the position. Find the derivative of the sketched function at the time 100*s.*)

ii) How could Lucy determine, as accurately as possible, the value of velocity at the time 100*s*?

iii) Give an approximate value of acceleration of the train in the time 100s. (Hint: Find the second derivative of the function sketched in the figure.)

Solution:

i) Peter can use quotient $\frac{f(x)-f(x_0)}{x-x_0}$ from definition of derivative $f'(x) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ of function $f(x)$.

For example: $\frac{f(x)-f(x_0)}{x-x_0} = \frac{428.325-335.703}{110-100} = 9.26$ or $\frac{f(x)-f(x_0)}{x-x_0} = \frac{255.295-335.703}{90-100} = 8.00$.

Or he can use points $[110, 428.325]$ and $[90, 255.925]$ to determine the slope: $\frac{428.325-255.925}{110-90} = 8.62$, which is arithmetical mean of numbers 9.26 and 8.00 .

ii) Lucy can also determine the slope. She can use points $[105, 380.383]$ and $[95, 294.235]$. Slope of the line given by these two point is $\frac{380.383-294.235}{105-95} = 8.61$.

iii) We can use approximative values of first derivatives from i) 1st and 2nd cases. We can use points $[100, 9.26]$ and $[90, 8.00]$. First coordinates are time values, and second coordinates are velocity values. Similarly we use differential quotient from definition of second derivative $f''(x) = \lim_{x \to x_0} \frac{f'(x)-f'(x_0)}{x-x_0}$. We get $\frac{f'(x)-f'(x_0)}{x-x_0} = \frac{9.26-8.00}{100-90} = 0.126$.

2. Then they changed the mode to FITTING FUNCTION. The result they obtained (using collected data) was $f: y = -\left(\frac{x}{19}\right)^3 . log\left(\frac{x}{1000}\right)$, where *y* is distance and *x* is time.

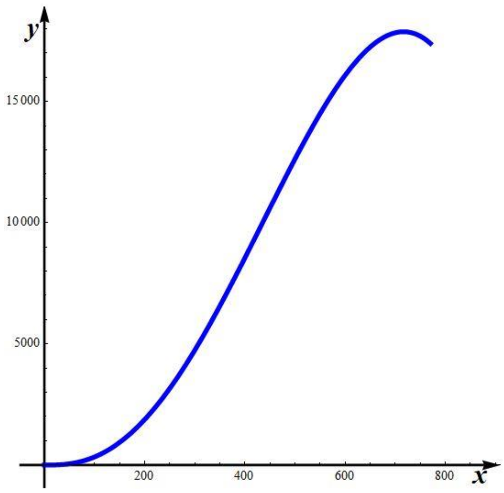


Figure 41

i) Give analytically velocity $v(100) = f'(100)$ and acceleration $a(100) = f''(100)$ of the train at time $t = 100s$!

ii) If the motion of the train would follow the function *f*, when will it stop (zero velocity) and how far from the beginning?

iii) What was the highest and the lowest velocity of the train in between *400s* and *600s*?

Solution:

i) $f'(x) = \left[-\left(\frac{x}{19}\right)^3 . log\left(\frac{x}{1000}\right)\right]' = -\left[\frac{3x^2}{19^3} . log\left(\frac{x}{1000}\right) + \frac{x^2}{19^3}\right]$ After replacing *x* by 100 we get $v(100) = f'(100) = 8.61$.

$f''(x) = -\left[\frac{3x^2}{19^3}.log\left(\frac{x}{1000}\right) + \frac{x^2}{19^3}\right]' = -\left[\frac{6x}{19^3}.log\left(\frac{x}{1000}\right) + \frac{5x}{19^3}\right]$ After replacing x by 100 we get $a(100) = f''(100) = 0.129$.

ii) We need to solve equation $v(x) = 0$ *(or* $f'(x) = 0$*), for the unknown x.*

$$-\left[\frac{3x^2}{19^3}.log\left(\frac{x}{1000}\right) + \frac{x^2}{19^3}\right] = 0$$

$$-\frac{x^2}{19^3}\left[3.log\left(\frac{x}{1000}\right) + 1\right]$$

$x = 0$ *or* $log\left(\frac{x}{1000}\right) = -\frac{1}{3}$, *which gives* $x = 716.5$.

$x = 716.5$ is physically correct answer.

iii) We need to find stationary points of $f'(x)$, i.e. points where $f''(x) = 0$ and to take in account end points of interval $\langle 400, 600\rangle$. Let us solve equation

$$-\left[\frac{6x}{19^3}.\log\left(\frac{x}{1000}\right) + \frac{5x}{19^3}\right] = 0$$

$$-\frac{x}{19^3}\left[6.\log\left(\frac{x}{1000}\right) + 5\right] = 0$$

$x = 0$ or $\log\left(\frac{x}{1000}\right) = -\frac{5}{6}$, which gives $x = 434.6$.

Since $x = 434.6$ belongs to interval $\langle 400,600\rangle$, we disregard $x = 0$.

$v(400) = f'(400) = 40.8$

$v(434.6) = f'(434.6) = 41.3$

$v(600) = f'(600) = 27.9$

The highest velocity of the train in between 400s and 600s was 41.3 and the lowest 27.9.

3. After ariving into park, Lucy got in the multi-rotational and shift device. She turned on mode PARAMETRIC and started the ride. After getting off, the Aplication gave her these functions:

$$x(t) = \frac{1}{3}t.(t-20).cos\left(\frac{\pi}{10}t\right)$$

$$y(t) = 15sin\left(\frac{\pi}{10}t\right)$$

$$z(t) = 15cos\left(\frac{\pi}{10}t\right)$$

i) Peter watched her and stopped the stopwatch at the time $t_1 = 6.32s$. It seemed to him, that she is moving very fast. What was her velocity $v(t_1)$ at that time? (Hint: $v(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$)

ii) On the other hand, she stopped the stopwatch at the time $t_2 = 10s$, when she almost fainted away. What was her acceleration $a(t_2)$ at that time? (Hint: $a(t) = \sqrt{x''(t)^2 + y''(t)^2 + z''(t)^2})$

Solution:

i) Let us make first derivatives of $x(t),\ y(t),\ z(t)$

$x'(t) = \frac{1}{3}.(t-20).cos\left(\frac{\pi}{10}t\right) + \frac{1}{3}t.cos\left(\frac{\pi}{10}t\right) - \frac{1}{3}t.(t-20).sin\left(\frac{\pi}{10}t\right).\frac{\pi}{10}$

$y'(t) = 15cos\left(\frac{\pi}{10}t\right).\frac{\pi}{10}$

$z'(t) = -15sin\left(\frac{\pi}{10}t\right).\frac{\pi}{10}$.

Let us evaluate them in $t = 6.32s$.

$x'(6.32) = 9.27$ and $y'(6.32) = -1.90$ and $z'(6.32) = -4.31$. It means that $v(6.32) = \sqrt{9.27^2 + (-1.90)^2 + (-4.31)^2} = 10.4$

ii) Let us make second derivatives of $x(t)$, $y(t)$, $z(t)$

$x''(t) = \frac{1}{3}.cos\left(\frac{\pi}{10}t\right) - \frac{1}{3}.(t-20).sin\left(\frac{\pi}{10}t\right).\frac{\pi}{10} + \frac{1}{3}.cos\left(\frac{\pi}{10}t\right) - \frac{1}{3}t.sin\left(\frac{\pi}{10}t\right).\frac{\pi}{10}$

$-\frac{1}{3}.(t-20).sin\left(\frac{\pi}{10}t\right).\frac{\pi}{10} - \frac{1}{3}t.sin\left(\frac{\pi}{10}t\right).\frac{\pi}{10} - \frac{1}{3}t.(t-20).cos\left(\frac{\pi}{10}t\right).\left(\frac{\pi}{10}\right)^2$

$y''(t) = -15sin\left(\frac{\pi}{10}t\right).\left(\frac{\pi}{10}\right)^2$

$z''(t) = -15cos\left(\frac{\pi}{10}t\right).\left(\frac{\pi}{10}\right)^2$.

Let us evaluate them in $t = 10s$.

$x''(10) = -3.96$ and $y''(10) = 0$ and $z''(10) = 1.48$. It means that $v(10) = \sqrt{(-3.96)^2 + 0^2 + 1.48^2} = 4.22$

4. Peter visited another attraction, where his motion followed this graph:

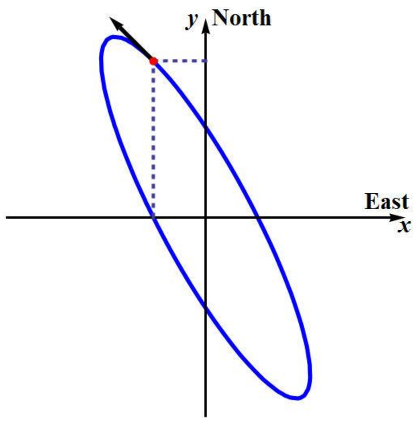


Figure 42

Beforehand he switched Application to the mode IMPLICIT. Application turned back this equation of rotated ellipse:

$$\frac{x^2}{50} + \frac{y^2}{150} + \frac{x.y}{50} = 1$$

In which point (coordinates) of trajectory did he moved northwestward, if he moved on the ellipse counterclockwise?

Solution:

Function $y(x)$ is given implicitly. We are looking for point $[x, y]$, such that $y'(x) = -1$. It represents northwest direction (and also south east).

Let us make derivative of both sides of given equation $\frac{x^2}{50} + \frac{y^2}{150} + \frac{x.y}{50} = 1$.

We get $\frac{2x}{50} + \frac{2y.y\prime}{150} + \frac{y}{50} + \frac{x.y\prime}{50} = 0$.

Let us express y'. We get $y' = -3\frac{2x+y}{2y+3x}$. It has to be equal -1.

So we get equation $-3\frac{2x+y}{2y+3x} = -1$, which after simplification is $y = -3x$. Let us use it together with equation of rotated ellipse. (System of two equations, two unknowns.)

$\frac{x^2}{50} + \frac{y^2}{150} + \frac{x.y}{50} = 1$ and is $y = -3x$

Solutions of the system are easily to find. They are $\left[\sqrt{50}, -3\sqrt{50}\right]$ and $\left[-\sqrt{50}, 3\sqrt{50}\right]$.

Taking in account, that motion is counterclockwise, only $\left[-\sqrt{50}, 3\sqrt{50}\right]$ is answer to the problem (see the image).

LO: AC5.

5 BIBLIOGRAPHY

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Guide for a Problem
Analysis and Calculus
1.6. Sequences and series
AC6

Marie Demlová and Petr Habala

1 OBJECTIVES

The aim of this activity guide is to achieve certain learning outcomes. As a result of learning this material you should be able to:

AC61. Determine limit of a given sequence.

AC62. Determine monotonicity of a given sequence (with proof).

AC63. Determine boundedness of a given sequence (with proof).

AC64. Sum up a given series based on geometric series.

AC65. Determine convergence/divergence of a given series of real numbers.

AC66. Determine the power expansion for a given function using known power expansions.

AC67. Find the linear and quadratic approximations of a given function with a given center.

AC68. Estimate the error of a Taylor approximation of given degree of a given function.

AC69. Find radius of convergence for a given power series.

Table 11 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- Green = very important
- Yellow = medium important
- Red = less important

Table 11. Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Analysis and Calculus									
AC6	6. Sequences and series								
AC61	Evaluate limit of a sequence	Yellow	Yellow	Green	Red	Red	Yellow	Red	Red
AC62	Determine monotonicity of a sequence:	Green	Green	Green	Red	Green	Green	Green	Red
AC63	Determine boundedness of a sequence	Green	Green	Green	Red	Green	Green	Green	Red
AC64	Find sum of geometric series	Red	Red	Yellow	Red	Red	Red	Red	Red
AC65	Determine convergence of a series	Green	Yellow	Green	Red	Yellow	Yellow	Yellow	Red
AC66	Expand function into power series	Yellow	Red	Green	Red	Yellow	Yellow	Red	Red
AC67	Find linear / quadratic approximation of a function	Green	Red	Green	Red	Green	Green	Yellow	Red (green box)
AC68	Estimate error of Taylor approximation	Green	Yellow	Green	Red	Green	Green	Red	Red (green box)
AC69	Determine radius of convergence for a series	Green	Yellow	Green	Red	Green	Green	Yellow	Red

The learning outcomes 7 and 8 can also incorporate Maple or another CAP to confirm that the expansion, error estimate, error etc. is correct. This addresses C8.

2 CONTENTS

1. Sequences of real numbers.
2. Limit evaluation.
3. Properties of sequences (monotonicity, boundedness).
4. Geometric series.
5. Testing convergence of series.
6. Taylor expansion and approximation of a given function, error estimate.
7. Radius of convergence of a power series.

3 EVALUATION

The time to solve this exam will be 180 minutes (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 12.

Table 12. Learning outcomes (LO) involved in this assessment activity.

Analysis and Calculus	
AC6	6. Sequences and series
AC61	Determine limit of a given sequence.
AC62	Determine monotonicity of a given sequence (with proof).
AC63	Determine boundedness of a given sequence (with proof).
AC64	Sum up a given series based on geometric series.
AC65	Determine convergence/divergence of a given series of real numbers.
AC66	Determine the power expansion for a given function using known power expansions.
AC67	Find the linear and quadratic approximations of a given function with a given centre.
AC68	Estimate the error of a Taylor approximation of given degree of a given function.
AC69	Find radius of convergence for a given power series.

3.1 Multiple-choice Questions

1. What are the properties of the sequence $a_n = \dfrac{n^2}{n+1}, n \geq 0$?
   a) a_n is bounded and monotone.
   b) a_n is bounded but not monotone.
   c) a_n is monotone but not bounded.
   d) a_n is not bounded nor monotone.

2. Which of the following sequences tend to zero?
   a) $a_n = \dfrac{2^n}{100n^2+1}$.
   b) $a_n = \dfrac{\ln(n)}{n^2+1}$.
   c) $a_n = \dfrac{13 \cdot 2^n}{n!-1}$.
   d) $a_n = \dfrac{n^3-1}{n^2+1}$.

3. Which of the following series are convergent?

a) $\sum \frac{1}{k^2+1}$.

b) $\sum \frac{1}{k+1}$.

c) $\sum \frac{1}{2^k}$.

d) $\sum (-1)^k$.

3.2 QUESTIONS

1. Determine the following limit:

$$\lim_{n\to\infty} \left(\frac{n^2+1}{2^n+1}\right).$$

2. Determine the monotonicity of the following sequence and prove your assertion:

$$a_n = \frac{1}{2n+1},\ n \geq 1.$$

3. Determine boundedness of the following sequence and prove your assertion:

$$a_n = \frac{n}{n^2+1},\ n \geq 0.$$

4. Find the sum of the following series:

$$\sum_{k=1}^{\infty} 3^{1-2k}.$$

5. Determine convergence/divergence of the following series:

$$\sum \frac{k}{2^k+1}.$$

6. Find the power expansion for the function $f(x) = e^{2x}$ with center $a = 0$ using known power expansions.
7. Find the power expansion for the function $f(x) = \frac{1}{1+2x}$ with center $a = 0$ using known power expansions.
8. Find linear and quadratic approximations of $f(x) = \sqrt{1+x}$ at $a = 0$.
9. The function $f(x) = cos(x)$ was approximated using the Taylor polynomial $T_{2n}(x)$ of degree 2n centered at $a = 0$. Estimate the error of this approximation for x from the interval [−1, 1].
10. Find the radius of convergence for the following power series:

$$\sum \frac{x^k}{3^k}.$$

3.3 PROJECT

Every practical application of mathematics ends with some calculations. Real numbers are expected as answers, because people who actually build things like cars or dams need to know what to manufacture. Unfortunately, we do not have computable formulas for all those popular mathematical symbols like $\sqrt{x}$, $ln(2)$, $sin(1)$, in fact majority of symbols that we routinely use in calculations cannot be evaluated precisely. We therefore have to do with approximate answers. One possible approach is via Taylor polynomials.

1. Derive Taylor polynomial of order 3 for the function $f(x) = arctan(x)$ with center $a = 0$. Use it to approximate $arctan(1)$.

Every application has specific requirements on how precise those approximate answers should be. Thus error control is an important part of real-life calculations.

2. Use the Lagrange remainder to estimate the error of your approximation of $arctan(1)$. Is it good enough to guarantee that two significant digits are correct?

The Taylor polynomial allows us to approximate various quantities using only the basic four algebraic operations. However, for real numbers with finite expansions (that is, floating point numbers used in computers and calculators) we have nice finite algorithms only for addition, subtraction and multiplication — with division it is a different story. Even a simple division like $\frac{1}{3.0}$ leads to infinite calculation, so we need to settle for approximate answers. Another problem is with speed, as the algorithm that we normally use can be too slow for some purposes. However, there are alternatives, and we will look at one here.

Consider some positive number d. Given an initial value x_0, we construct a recursive sequence given by the formula $x_{k+1} = x_k(2 - d.x_k)$..Note that only multiplication and subtraction are used.

3. a) Take $d = 5, x_0 = 0.05$, and calculate x_1, x_2, x_3, x_4, x_5, and x_6 according to the recursive formula above. Can you make a guess to which number this sequence converges?

   b) Take $d = 3, x_0 = 0.1$, and calculate x_1, x_2, x_3, x_4, x_5 according to the recursive formula above. Can you make a guess to which number this sequence converges?

   c) Take $d = 0.4, x_0 = 0.7$, and calculate x_1, x_2, x_3, x_4, x_5 according to the recursive formula above. Can you make a guess to which number this sequence converges?

This procedure is convergent if we start with suitably small x_0, and yields $\frac{1}{d}$. It is called the Newton-Raphson division and is actually used in real-life calculations.

4 SOLUTION

4.1 Multiple-choice Questions

1. What are the properties of the sequence $a_n = \frac{n^2}{n+1}, n \geq 0$?

   a) a_n is bounded and monotone.

   b) a_n is bounded but not monotone.

   c) a_n is monotone but not bounded.

   d) a_n is not bounded nor monotone.

   Solution:

   The solution is c).

   LO: AC62, AC63.

2. Which of the following sequences tend to zero?

   a) $a_n = \frac{2^n}{100n^2+1}$.
   b) $a_n = \frac{\ln(n)}{n^2+1}$.
   c) $a_n = \frac{13 \cdot 2^n}{n!-1}$.
   d) $a_n = \frac{n^3-1}{n^2+1}$.

   Solution:

   The solution is b) and c).

   LO: AC61.

3. Which of the following series are convergent?

   a) $\sum \frac{1}{k^2+1}$.
   b) $\sum \frac{1}{k+1}$.
   c) $\sum \frac{1}{2^k}$.
   d) $\sum (-1)^k$.

   Solution:

   The solution is a) and c).

   LO: AC65.

4.2 Questions

1. Determine the following limit:

$$\lim_{n\to\infty} \left(\frac{n^2+1}{2^n+1}\right).$$

Solution:

Direct solution using the scale of powers:

$$\lim_{n\to\infty}\left(\frac{n^2+1}{2^n+1}\right)=\lim_{n\to\infty}\left(\frac{n^2}{2^n}\frac{1+\frac{1}{n^2}}{1+\frac{1}{2^n}}\right)=0\cdot\frac{1+0}{1+0}=0$$

Solution by passing to limit of a function and using l'Hospital's rule:

$$\lim_{x\to\infty}\left(\frac{x^2+1}{2^x+1}\right)=\lim_{x\to\infty}\left(\frac{2x}{\ln(2)2^x}\right)=\lim_{x\to\infty}\left(\frac{2}{\ln(2)^2 2^x}\right)=0$$

LO: AC61.

2. Determine the monotonicity of the following sequence and prove your assertion:

$$a_n=\frac{1}{2n+1},\ n\geq 1$$

Solution:

The sequence is (strictly) decreasing.

Proof by definition: Let $n \epsilon$ **N**. We obviously have $n+1>n$, then $2(n+1)>2n$, so $2(n+1)+1>2n+1$, both are positive, so $\frac{1}{2(n+1)+1}<\frac{1}{2n+1}$, that is, $a_{n+1}<a_n$.

Proof by passing to a function: Consider $f(x)=\frac{1}{2x+1}$. Then $f'(x)=\frac{-2}{(2x+1)^2}<0$ on $[1,\infty)$, hence f is strictly decreasing on $[1,\infty)$.

LO: AC62.

3. Determine boundedness of the following sequence and prove your assertion:

$$a_n=\frac{n}{n^2+1},\ n\geq 0$$

Solution:

The sequence is bounded.

Proof of bounded below: When $n\geq 1$, we have $n\geq 0$ and $n^2+1>0$, hence $\frac{n}{n^2+1}>0$. Proof of bounded from above: We claim that $a_n\leq 1$. Equivalently, $n\leq n^2+1$. This can be written as $0\leq n^2-2n+1+n$, that is, $0\leq (n-1)^2+n$, which is true for $n\geq 1$.

LO: AC63.

4. Find the sum of the following series:

$$\sum_{k=1}^{\infty}3^{1-2k}$$

Solution:

$$\sum_{k=1}^{\infty} 3^{1-2k} = \sum_{k=1}^{\infty} 3\cdot(3^{-2})^k = 3\cdot\sum_{k=1}^{\infty}\left(\tfrac{1}{9}\right)^k = 3\cdot\frac{\frac{1}{9}}{1-\frac{1}{9}} = \frac{3}{8}$$

The summation formula is valid since $\left|\frac{1}{9}\right| < 1$.

LO: AC64.

5. Determine convergence/divergence of the following series:

$$\sum \frac{k}{2^k+1}$$

Solution:

Since $\frac{k}{2^k+1} \geq 0$ for $k \in \mathbf{N}$ we can apply the root and the ratio test.

The ratio test:

$$\lambda = \frac{a_{k+1}}{a_k} = \frac{\frac{k+1}{2^{k+1}+1}}{\frac{k}{2^k+1}} = \frac{k+1}{k}\,\frac{2^k+1}{2^{k+1}+1} = \frac{1+\frac{1}{k}}{1}\,\frac{1+\frac{1}{2^k}}{2+\frac{1}{2^k}} \to 1\cdot\frac{1+0}{2+0} = \frac{1}{2}.$$

Since $\lambda < 1$, the series is convergent.

For the root test it is better to simplify the denominator first. We use comparison for that:

$$0 \leq \frac{k}{2^k+1} \leq \frac{k}{2^k}.$$

Now we apply the root test to the series given by the expression on the right:

$$\varrho = \sqrt[k]{\frac{k}{2^k}} = \frac{k^{1/k}}{2} \to \frac{1}{2}.$$

Since $\varrho < 1$, the series $\sum \frac{k}{2^k}$ is convergent, hence also the given series converges.

LO: AC65.

6. Find the power expansion for the function $f(x) = e^{2x}$ with center $a = 0$ using known power expansions.

Solution:

We can use substitution $e^{2x} = e^y$ with $y = 2x$ and expand e^y:

$$e^{2x} = \sum_{k=0}^{\infty} \frac{1}{k!}(2x)^k = \sum_{k=0}^{\infty} \frac{2^k}{k!}x^k.$$

This expansion is valid for $x \in \mathbf{R}$ since it was true also for the original expansion of e^y.

LO: AC66.

7. Find the power expansion for the function $f(x) = \frac{1}{1+2x}$ with center $a = 0$ using known power expansions.

Solution:

$f(x) = \frac{1}{1-(-2x)}$. We can use the substitution $-2x = y$ and expand $\frac{1}{1-y}$:

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_{k=0}^{\infty}(-2x)^k = \sum_{k=0}^{\infty}(-2)^k x^k.$$

The expansion of $\frac{1}{1-y}$ is valid for $|y| < 1$, that is, for $|-2x| < 1$. Thus our expansion for f is valid for $|x| < \frac{1}{2}$.

LO: AC66.

8. Find linear and quadratic approximations of $f(x) = \sqrt{1+x}$ at $a = 0$.

Solution:

We have $f(0) = 1$. Also $f'(x) = \frac{1}{2\sqrt{1+x}}$, $f'(0) = \frac{1}{2}$, so

$$\sqrt{1+x} \approx 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x.$$

We also have $f''(x) = \frac{-1}{4\sqrt{1+x}^3}$, $f''(0) = \frac{-1}{4}$, so

$$\sqrt{1+x} \approx 1 + \frac{1}{2}(x-0) + \frac{1}{2}\cdot\frac{-1}{4}(x-0)^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2.$$

LO: AC67.

9. The function $f(x) = cos(x)$ was approximated using the Taylor polynomial $T_{2n}(x))$ of degree $2n$ centered at $a = 0$. Estimate the error of this approximation for x from the interval [−1,1].

Solution:

The Lagrange form of the remainder tells us that the error has the following bound:

$$|R_{2n}(x)| \leq \frac{1}{(2n+1)!}\max|f^{(2n+1)}(t)|\cdot|x-0|^{2n+1},$$

where the maximum is taken over the interval between $a = 0$ and x.

Since $f(x) = cos(x)$, the derivative in absolute value is either sine or cosine, so the maximum is at most equal to 1. We also have a bound on x, so we can estimate

$$|R_{2n}| \leq \frac{1}{(2n+1)!}\cdot 1\cdot 1^{2n+1} = \frac{1}{(2n+1)!}$$

LO: AC68.

10. Find the radius of convergence for the following power series:

$$\sum \frac{x^k}{3^k}.$$

Solution:

Root test applied to $\sum\left|\frac{x^k}{3^k}\right|$:

$$\varrho = \sqrt[k]{\frac{|x|^k}{3^k}} = \frac{|x|}{3}.$$

The series converges absolutely if $\frac{|x|}{3} < 1$, that is, if $|x| < 3$. Thus $r = 3$. (Note that it is a geometric series with base $q = \frac{x}{3}$).

LO: AC69.

4.3 Project

1. Derive Taylor polynomial of order 3 for the function $f(x) = arctan(x)$ with center $a = 0$. Use it to approximate $arctan(1)$.

   Solution:

   We prepare derivatives: $f(x) = arctan(x), f'(x) = \frac{1}{1+x^2}, f''(x) = \frac{-2x}{(1+x^2)^2}, f'''(x) = \frac{6x^2-2}{(1+x^2)^3}$.

   Now we prepare coefficients $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -2$, and weare ready to write the polynomial:

   $$T_3(x) = 0 + 1\cdot(x-0) + \frac{1}{2}\cdot 0\cdot(x-0)^2 + \frac{1}{3!}\cdot(-2)\cdot(x-0)^3 = x - \frac{1}{3}x^3.$$

   Now we can approximate:

   $$\arctan(1) \approx T_3(1) = \frac{2}{3} \approx 0.66666666\ldots$$

   LO: AC67

2. Use the Lagrange remainder to estimate the error of your approximation of $arctan(1)$. Is it good enough to guarantee that two significant digits are correct?

   Solution:

   The Lagrange form of reminder is $R_3(x) = \frac{1}{4!}f^{IV}(\xi)(x-0)^4$, where ξ is between 0 and x.

   In our case $x = 1, f^{IV}(x) = \frac{24x-24x^3}{(x^2+1)^4}$, so for $\xi \in [0,1]$ we can estimate $|f^{IV}(\xi)| \le \frac{24}{1} = 24$. Thus the error is at most

   $$|R_3(1)| \le \frac{1}{4!}\cdot 24\cdot 1^4 = 1.$$

   This is not very good, according to this estimate we cannot trust even the first significant digit in our approximation.

   Can we do better? We used very trivial estimate for the derivative, can we get something better with more care? The polynomial $24x - 24x^3$attains its maximum on $[0,1]$ at $x = \frac{1}{\sqrt{3}}$, and it is equal to $\frac{16}{\sqrt{3}} \le 10$, which leads to the estimate $|R_3(1)| \le \frac{10}{4!} \approx 0.41666$. Still not very good. In fact, while the actual error is somewhat smaller, it is still more than 0.1 and therefore the first digit is also wrong, indeed.

The Taylor polynomial approximation that we are using here does not improve in quality fast enough, we would have to take a very long polynomial just to get two significant digits right. Computers and calculators definitely use something else to approximate arctangent for us.

LO: AC68.

3. a) Take $d = 5, x_0 = 0.05,$ and calculate $x_1, x_2, x_3, x_4, x_5,$ and x_6 according to the recursive formula above. Can you make a guess to which number this sequence converges?

   b) Take $d = 3, x_0 = 0.1,$ and calculate x_1, x_2, x_3, x_4, x_5 according to the recursive formula above. Can you make a guess to which number this sequence converges?

   c) Take $d = 0.4, x_0 = 0.7,$ and calculate x_1, x_2, x_3, x_4, x_5 according to the recursive formula above. Can you make a guess to which number this sequence converges?

Solution

a) $x_0 = 0.05, x_1 = 0.0875, x_2 = 0.1367\ldots, x_3 = 0.17998\ldots, x_4 = 0.197995\ldots, x_5 = 0.19998\ldots, x_6 = 0.199999998\ldots$

Guess: Limit is $0.2 = \frac{1}{5}$.

b) $x_0 = 0.1, x_1 = 0.17, x_2 = 0.2533, x_3 = 0.3141\ldots, x_4 = 3322\ldots, x_5 = 0.333320\ldots$

Guess: Limit is $0.3333332\ldots = \frac{1}{3}$.

c) $x_0 = 0.7, x_1 = 1.204, x_2 = 1.828\ldots, x_3 = 2.3194\ldots, x_4 = 2.48696\ldots, x_5 = 2.49993\ldots$

Guess: Limit is $2.5 = \frac{1}{0.4}$.

LO: AC61, AC65.

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Guide for a Problem

Analysis and Calculus

1.7. Methods of integration

AC7

Ion Mierlus–Mazilu

1 OBJECTIVES

The objectives related to the required learning outcomes are:

1. Obtain definite and indefinite integrals of rational functions in partial fraction form.
2. Apply the method of integration by parts to indefinite and definite integrals.
3. Use the method of substitution on indefinite and definite integrals.
4. Solve practical problems which require the evaluation of an integral.
5. Recognize simple examples of improper integrals.
6. Use the formula for the maximum error in a trapezoidal rule estimate.
7. Use the formula for the maximum error in a Simpson's rule estimate.

As we have established the "technical part" of this chapter, in which we present *WHAT* students should accomplish, let us deal with the important part, which includes theory, exercises, etc. about *HOW* student can achieve the goals above.

Table 13 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- = very important
- = medium important
- = less important

Table 13: Competencies that we measure with the global test model that is proposed

		C1	C2	C3	C4	C5	C6	C7	C8
Subject									
AC7	7. Methods of integration								
AC71	Obtain definite and indefinite integrals of rational functions in partial fraction form								
AC72	Apply the method of integration by parts to indefinite and definite integrals								
AC73	Use the method of substitution on indefinite and definite integrals								
AC74	Solve practical problems which require the evaluation of an integral								
AC75	Recognize simple examples of improper integrals								

AC76	Use the formula for the maximum error in a trapezoidal rule estimate								
AC77	Use the formula for the maximun error in a Simpson's rule estimate								

2 CONTENTS

1. Obtain definite and indefinite integrals of rational functions in partial fraction form.
2. Apply the method of integration by parts to indefinite and definite integrals.
3. Use the method of substitution on indefinite and definite integrals.
4. Solve practical problems which require the evaluation of an integral.
5. Recognise simple examples of improper integrals.
6. Use the formula for the maximum error in a trapezoidal rule estimate.
7. Use the formula for the maximun error in a Simpson's rule estimate.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 14.

Table 14: Learning outcomes (LO) involved in this assessment activity.

Analysis and Calculus	
AC7	7. Methods of integration
AC71	Obtain definite and indefinite integrals of rational functions in partial fraction form
AC72	Apply the method of integration by parts to indefinite and definite integrals
AC73	Use the method of substitution on indefinite and definite integrals
AC74	Solve practical problems which require the evaluation of an integral
AC75	Recognize simple examples of improper integrals
AC76	Use the formula for the maximum error in a trapezoidal rule estimate
AC77	Use the formula for the maximum error in a Simpson's rule estimate

The statement of the exam is the next:

3.1 MULTIPLE-CHOICE QUESTIONS

1. The value of $\int_0^{\frac{\pi}{4}} x\, tan^2\, x\, dx$ is :
   i) $\frac{\pi}{4} - \frac{\pi^2}{32} + ln\frac{\sqrt{2}}{2}$
   ii) $\frac{\pi}{4} - \frac{\pi}{32} + \ln 2$
   iii) $\ln 2$
2. The value of $\int_2^4 |x^2 - 6x + 8| dx$ is :
   i) $\frac{100}{3}$
   ii) $\frac{4}{3}$
   iii) 0
3. Determine the convergence of the integral $\int_0^{\infty} e^{-ax} dx$:
   i) Convergent
   ii) Divergent
   iii) We cannot decide
4. The population of ladybugs in Chloe's ant farm changes at a rate of $f(t) = -15{,}5 \cdot 0{,}8^t$ ladybirds per month (where t is time in months). At time t=0 the ladybirds farm's population is 160 ants. How many ants are in the farm at t=5?
   i) $\int_0^5 f(t)dt$
   ii) 160
   iii) $160 + \int_0^5 \mathrm{f(t)dt}$
5. Using Simpson's formula for the function x^3on the interval [2,10],for *n=4*, then $\int_2^{10} x^3 dx$ is :
   i) 3000
   ii) 2496
   iii) 2400
6. Compute the common area of the parabolas :(P_1): $y^2 = 2px$ and (P_2): $x^2 = 2py$.
   i) $\frac{p^2}{2}$
   ii) ii) $\frac{3p^2}{4}$
   iii) $\frac{4p^2}{3}$

3.2 QUESTIONS 1

Compute the following indefinite and definite integrals:

a) $\int e^x \sin x \, dx$

b) $\int x^3\sqrt{x^2+1}dx$

c) $\int ln^2 x \, dx$

d) $\int tan\, x \, dx$

e) $\int cot\, x \, dx$

f) $\int \frac{8x+4}{x^2+x-5}dx$

g) $\int (tan\, x + tan^3 x)\, dx$

h) $\int x^2 e^{x^3} dx$

i) $\int \frac{sin\, x + cos\, x}{sin\, x - cos\, x}dx$

j) $\int \frac{x+2}{\sqrt{x(x-3)}}dx$

k) $\int_0^2 \frac{1}{(x+1)\sqrt{x^2+x+1}}dx$

l) $\int_0^{\frac{\pi}{4}} tan^3 x \, dx$

m) $\int_0^{ln\, 2} \sqrt{e^x - 1}dx$

n) $\int_0^{\pi} sin^2 x \, cos\, 2\, xdx$

o) $\int_0^{\infty} \frac{1}{x^2+1}dx$

p) $\int_0^{\infty} \frac{x}{x^2+1}dx$

q) $\int_1^{\infty} \frac{1}{x(x^2+1)}dx$

3.3 QUESTION 2

Let $v(t)$ be the velocity of a vehicle and $v(0) = 72km/h$. When breaks are applied, $a = -5m/s^2$. How far did it travel?

3.4 QUESTION 3

Compute the area between the lines $y = 5x, x = 2$ and *Ox* axis.

3.5 QUESTION 4

Using three eight Simpson's rule, approximate $\int_1^5 (1 + x^2)dx$ for $n = 4$.

3.6 QUESTION 5

Using one third Simpson's rule, approximate $\int_0^1 2xdx$ for *n=6.*

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. i)
2. ii)
3. i)
4. iii)
5. ii)
6. iii)

4.2 QUESTIONS 1

a) $\frac{e^x(\sin x+\cos x)}{2}+C$

b) $\frac{1}{5}x^2(x^2+1)^{\frac{3}{2}}-\frac{2}{15}(x^2+1)^{\frac{3}{2}}+C$

c) $x(\ln^2 x-2\ln x+2)+C$

d) $x(\ln^2 x-2\ln x+2)+C$

e) $\ln|\sin x|+C$

f) $4\ln(x^2+x-5)+C$

g) $\frac{\tan^2 x}{2}+C$

h) $\frac{1}{3}e^{x^3}+C$

i) $\ln|\sin x-\cos x|+C$

j) $\frac{7}{2}\ln\left|\frac{1+\sqrt{\frac{x-3}{x}}}{1-\sqrt{\frac{x-3}{x}}}\right|+x\sqrt{\frac{x-3}{x}}+C$

k) $\ln\left(\frac{3-\sqrt{3}}{2}\right)$

l) $\frac{1}{2}-\ln\sqrt{2}$

m) $2-\frac{\pi}{2}$

n) $\frac{7\pi}{8}$

o) $\frac{\pi}{2}, co\,n\,vergent$

p) $\infty, divergent$

q) $\frac{1}{2}\ln 2\,, convergent$

4.3 QUESTION 2

It travelled 40m.

4.4 QUESTION 3

10

4.5 QUESTION 4

40, 125

4.6 QUESTION 5

2

5 PROJECTS

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 14.

5.1 PROJECT 1

1. Compute $\int \frac{1}{cosx} dx$.
2. Determine the cars produced in 7 days if the productivity of the work is characterized by the function $f(t) = 5t - 8$.
3. Determine $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

5.2 PROJECT 2

Using Trapezoidal Method Error Formula, for $n = 6$, estimate

$$\int_{1}^{4} \frac{1}{1+x} dx$$

6 SOLUTIONS OF THE PROJECTS

6.1 PROJECT 1

1. Compute $\int \frac{1}{cosx} dx$.

   $\int \frac{1}{cosx} dx = \int \frac{cosx}{cos^2x} dx = \int \frac{cosx}{1-sin^2x} dx$.

   We make the substitution $sinx = t$.

   Hence, $\int \frac{1}{1-t^2} dt = -\int \frac{1}{t^2-1} dt = -ln\left|\frac{t-1}{t+1}\right| + C = ln\left|\frac{t+1}{t-1}\right| + C$

$$\Rightarrow \int \frac{1}{cosx} dx = ln\left|\frac{sinx+1}{sinx-1}\right| + C$$

2. Determine the cars produced in 7 days if the productivity of the work is characterized by the function $f(t) = 5t - 8$.

$$\int_0^7 (5t-8)dt = (5\frac{t^2}{2} - 8t)\Big|_0^7 = 122{,}5 - 56 = 66{,}5.$$

3. Determine $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

$$= \lim_{\lambda\to-\infty} atanx|_{\lambda}^{0} + \lim_{\lambda\to\infty} atanx|_0^{\lambda} = \lim_{\lambda\to-\infty} -atan\,\lambda + \lim_{\lambda\to\infty} atan|_0^{\lambda} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

6.2 PROJECT 2

Using Trapezoidal Method Error Formula, for $n = 6$, estimate

$$\int_1^4 \frac{1}{1+x} dx$$

For $n = 6$ we have $\Delta x = \frac{4-1}{6}$. Now we compute the values $y_0, y_1, .., y_6$.

$$x = 1 \Longrightarrow y_0 = \frac{1}{2} = 0{,}5$$

$$x = 1{,}5 \Longrightarrow y_1 = \frac{2}{5} = 0{,}4$$

$$x = 2 \Longrightarrow y_2 = \frac{1}{3} = 0,(3)$$

$$x = 2{,}5 \Longrightarrow y_3 = \frac{2}{7} = 0{,}285$$

$$x = 3 \Longrightarrow y_4 = \frac{1}{4} = 0{,}25$$

$$x = 3{,}5 \Longrightarrow y_5 = \frac{2}{9} = 0,(2)$$

$x = 4 \Longrightarrow y_6 = \frac{1}{5} = 0{,}2$

$\int_1^4 \frac{1}{1+x}dx \approx \frac{0{,}5}{2}(0{,}5 + 0{,}4 + 0,(3) + 0{,}285 + 0{,}25 + 0,(2) + 0{,}2) \approx 0{,}54$

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Guide for a Problem

Analysis and Calculus

1.8. Applications of integration

AC8

Ion Mierlus–Mazilu

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 15, to be achieved with the development of this activity guide are the following:

1. Find the length of part of a plane curve.
2. Find the curved surface area of a solid of revolution.
3. Obtain the mean value and root-mean-square (RMS) value of a function in a closed interval.
4. Find the first and second moments of a plane area about an axis.
5. Find the centroid of a plane area and of a solid of revolution.

Table 15 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 15: Competencies that we measure with the global test model that is proposed

		C1	C2	C3	C4	C5	C6	C7	C8
Subjet									
AC8	8. Applications of integration								
AC81	Find the length of part of a plane curve	green	red	green	green	green	green	red	green
AC82	Find the curved surface area of a solid of revolution	green	red	green	green	green	green	red	green
AC83	Obtain the mean value and root-mean-square (RMS) value of a function in a closed interval	green	green	yellow	yellow	green	green	yellow	yellow
AC84	Find the first and second moments of a plane area about an axis	yellow	green	yellow	red	green	green	green	yellow
AC85	Find the centroid of a plane area and of a solid of revolution.	green	green	green	green	green	green	green	green

2 CONTENTS

1. Find the length of part of a plane curve.
2. Find the curved surface area of a solid of revolution.
3. Obtain the mean value and root-mean-square (RMS) value of a function in a closed interval.
4. Find the first and second moments of a plane area about an axis.
5. Find the centroid of a plane area and of a solid of revolution.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 16.

Table 16: Learning outcomes (LO) involved in this assessment activity.

Subjects	
AC8	8. Applications of integration
AC81	Find the length of part of a plane curve
AC82	Find the curved surface area of a solid of revolution
AC83	Obtain the mean value and root-mean-square (RMS) value of a function in a closed interval
AC84	Find the first and second moments of a plane area about an axis
AC85	Find the centroid of a plane area and of a solid of revolution.

The statement of the exam is the next:

3.1 Multiple-Choice Questions

1. Compute the common area of the parabolas: $(P_1): y^2 = 2px$ and $(P_2): x^2 = 2py$.

   i) $\frac{p^2}{2}$

   ii) $\frac{3p^2}{4}$

   iii) $\frac{4p^2}{3}$

2. Determine the length of the orbit of the planet which is in form of a cardioid of equation $\rho = a(1 + cos\,\theta), \theta \in [0,2\pi]8a$

   i) 8

   ii) 8a

   iii) $\frac{8a}{2}$

3. Compute the area of one arc of the cycloid

   i) $3\pi a^2$

   ii) $3\pi a$

   iii) $9\pi a^2$

4. Calculate the volume of the body obtained by rotating around the OX -axis the asteroid of equation: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

   i) $\frac{32\pi a}{105}$

   ii) $\frac{32\pi}{105}$

   iii) $\frac{32a}{105}$

3.2 Question 1

Find the lenght of an arc of the curve $y = \left(\frac{1}{6}\right)x^3 + \left(\frac{1}{2}\right)x^{-1}$ from $x = 2$ to $x = 3$.

3.3 Question 2

Compute the length of the trajectory of an arrow described by an arc in one period of the cycloid $(C): \begin{cases} x = t - sin\,t \\ y = 1 - cos\,t \end{cases}, t \in [0,2\pi]$.

3.4 Question 3

Find the length of the first rotation of the logarithmic spiral $r = e^{\theta}, \theta \in [0,2\pi]$.

3.5 Question 4

Find the length of the curve given by the equation:

$$f(x) = x^2, x \in [1,2]$$

3.6 QUESTION 5

Determine the surface area of the solid obtained by rotating the curve, $f(x) = \sqrt{7 - x^2}$, $-3 \leq x \leq 3$ around the OX -axis.

3.7 QUESTION 6

Determine the surface area of the solid obtained by rotating $x = \frac{1}{3}(y^2 + 1)^{\frac{3}{2}}$ from $1 \leq y \leq 3$ around the OX -axis $1 \leq y \leq 2$.

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. The drawing:

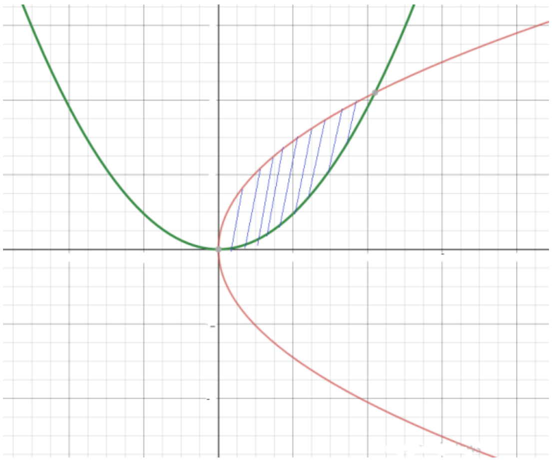

Figure 43

The parabolas intersect in points $O(0,0)$ and $M(2p,2p)$.

Indeed, $A = \int_0^{2p} \sqrt{2px}dx - \int_0^{2p} \frac{x^2}{2p} dx = \frac{2}{3}\sqrt{2p}x^{\frac{3}{2}}\Big|_0^{2p} - \frac{1}{2p}\frac{x^3}{3}\Big|_0^{2p} = \frac{4p^2}{3}$.

2. The drawing:

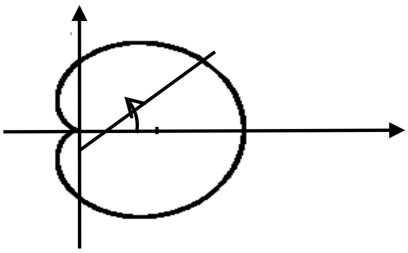

Figure 44

The element of arc in polar coordinates is :

$dx = \sqrt{\rho^2(\theta) + \rho'^2(\theta)}d\theta$

For the figure's symmetry we can take:

$$L = \int_0^{2\pi} \sqrt{\rho^2(\theta) + \rho'^2(\theta)}\, d\theta$$

$\rho'(\theta) = -a \sin\theta$

$\rho^2(\theta) + \rho'^2(\theta) = a^2(1 + 2\cos\theta + \cos^2\theta) + a^2 \sin^2\theta = a^2(2 + 2\cos\theta) =$

$= 2a^2(1 + \cos\theta) = 2a^2 \cdot 2\cos^2\frac{\theta}{2} = 4a^2\cos^2\frac{\theta}{2}$

$$L = 2\int_0^{\pi} \sqrt{4a^2 \cos^2\frac{\theta}{2}}\, d\theta = 2\int_0^{\pi} 2a\left|\cos\frac{\theta}{2}\right| d\theta$$

But $\theta \in [0, \pi] \Rightarrow \frac{\theta}{2} \in \left[0, \frac{\pi}{2}\right] \Rightarrow \cos\frac{\theta}{2} \geq 0 \Rightarrow \left|\cos\frac{\theta}{2}\right| = \cos\frac{\theta}{2}$ hence:

$$L = 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 4a \sin\frac{\sin\frac{\theta}{2}}{\frac{1}{2}}\Big|_0^{\pi} = 8a\left(\sin\frac{\pi}{2} - \sin 0\right) = 8a.$$

3. The drawing:

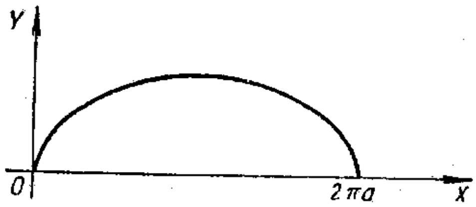

Figure 45

The parametric equations of one arc of the cycloid are:

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, t \in [0, 2\pi]$$

According to (4) we have:

$$S = \int_0^{2\pi} a(1 - \cos t) a(1 - \cos t) dt = a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) dt =$$
$$= a^2\left(t - \sin t + \frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = 3\pi a^2$$

4. The drawing:

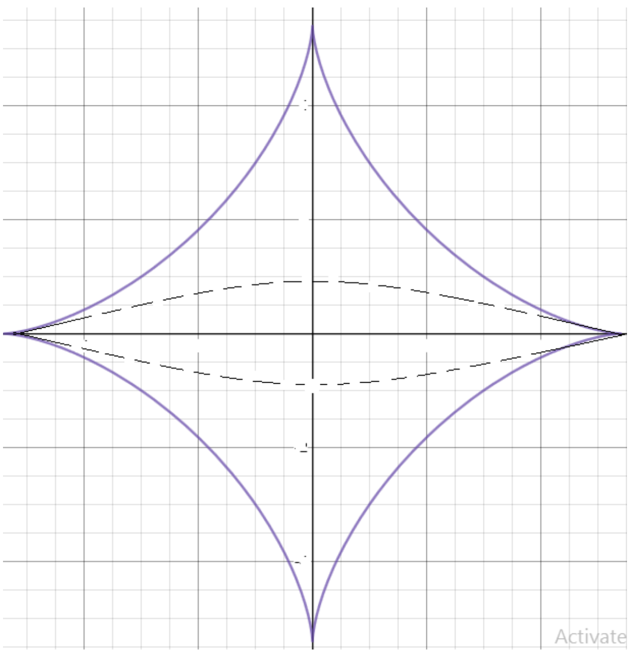

Figure 46

$$V = 2\pi \int_{-a}^{a} \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx = 2\pi \int_{-a}^{a} \left(a^2 - 3a^{\frac{4}{3}}x^{\frac{2}{3}} + 3a^{\frac{2}{3}}x^{\frac{4}{3}} - x^2\right) dx = \frac{32a}{105}$$

$$V = 2\pi \int_{-a}^{a} \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx = 2\pi \int_{-a}^{a} \left(a^2 - 3a^{\frac{4}{3}}x^{\frac{2}{3}} + 3a^{\frac{2}{3}}x^{\frac{4}{3}} - x^2\right) dx = \frac{32\pi a}{105}$$

4.2 QUESTION 1

The curve is $y = \left(\frac{1}{6}\right)x^3 + \left(\frac{1}{2}\right)x^{-1}$.

Since $f'(x) = \left(\frac{1}{2}\right)x^2 - \left(\frac{1}{2}\right)x^{-2}$, we have

$$L = \int_2^3 \sqrt{1 + \left(\left(\frac{1}{2}\right)x^2 - \left(\frac{1}{2}\right)x^{-2}\right)^2}\, dx = \frac{1}{2}\int_2^3 \sqrt{3 + x^4}\, dx = \frac{13}{4}$$

4.3 QUESTION 2

$$L = \int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{x'^2(t) + y'^2(t)}\, dt$$

$x' = 1 - cos\, t$

$y' = sin\, t$

$x'^2 + y'^2 = 1 - cos\, 2\, t + cos^2\, t + sin^2\, t = 2 - 2\, cos\, t =$

$= 2(1 - cos\, t) = 2\left(2\, sin^2 \frac{t}{2}\right) = 4\, sin^2 \frac{t}{2}$

$\sqrt{x'^2 + y'^2} = 2\left|sin \frac{t}{2}\right|$

$t \in [0,2\pi] \Rightarrow \frac{t}{2} \in [0, \pi] \Rightarrow sin \frac{t}{2} \geq 0 \Rightarrow \left|sin \frac{t}{2}\right| = sin \frac{t}{2}$

$\Rightarrow \sqrt{x'^2 + y'^2} = 2\, sin \frac{t}{2}$

$\Rightarrow L = \int_0^{2\pi} 2\, sin \frac{t}{2} dt = 2 \left.\frac{-\, cos \frac{t}{2}}{\frac{1}{2}}\right|_0^{2\pi} = -4\, cos \left.\frac{t}{2}\right|_0^{2\pi} =$

$= -4(cos\, \pi - cos\, 0) = -4(-1 - 1) = 8$

4.4 QUESTION 3

$\sqrt{2}(e^{2\pi} - 1)$

4.5 QUESTION 4

$\frac{3\sqrt{37} + \frac{1}{2} ln(6 + \sqrt{37}) - 2\sqrt{17} - \frac{1}{2} ln(4 + \sqrt{17})}{2}$ = 2,280559

4.6 QUESTION 5

$12\sqrt{7}\pi$

4.7 QUESTION 6

$\frac{21\pi}{2}$

5 PROJECTS

5.1 PROJECT 1

1. Compute the length of the graph of the function:

$$f(x) = \frac{x^2}{2} - 1, x \in [-1,1]$$

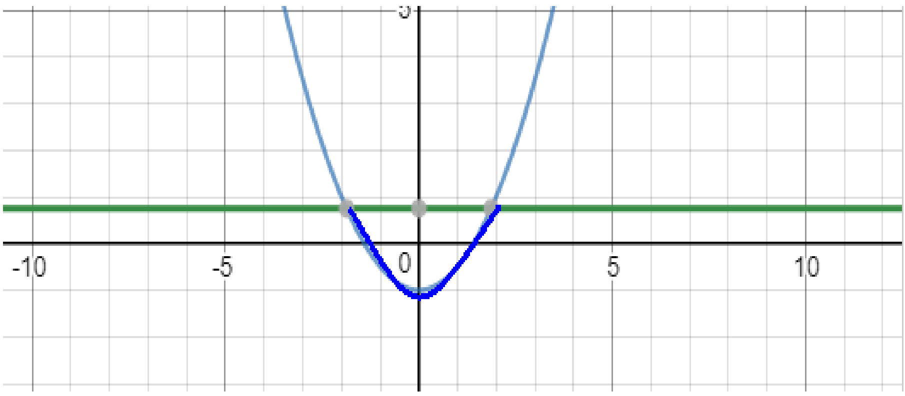

Figure 47

2. Compute the length of the fence needed to cover a hill which can be defined as a cycloid loop of parametrical equations:

$$(C): \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, t \in [0, \pi].$$

3. Calculate the volume of the asteroid obtained by rotating around the OX-axis of the asteroid: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5^{\frac{2}{3}}$.

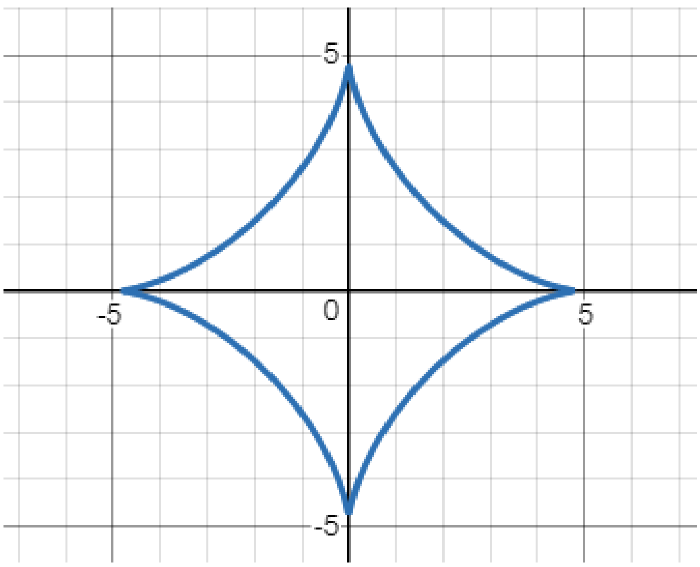

Figure 48

5.2 PROJECT 2

Compute the commune area of the 2 boomerangs in form the parabolas: $(P_1): y^2 = 8x$ and $(P_2): x^2 = 8y$.

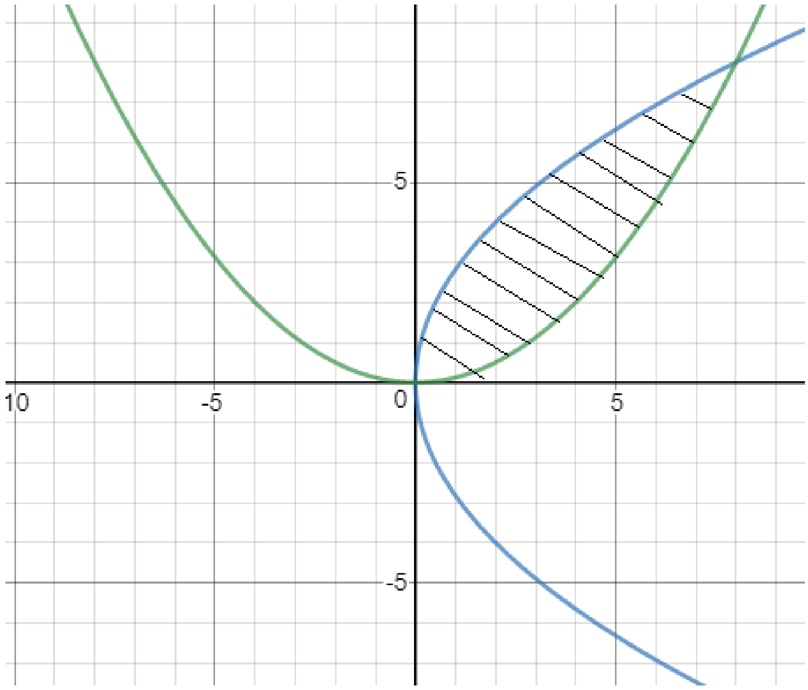

Figure 49

6 SOLUTIONS OF THE PROJECTS

6.1 PROJECT 1

1. Compute the length of the graph of the function:

$$f(x) = \frac{x^2}{2} - 1, x \in [-1,1]$$

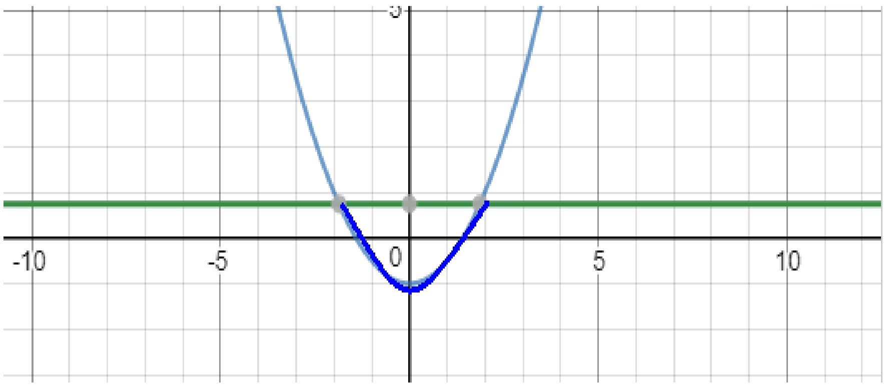

Figure 50

$$L = \int_{-1}^{1} \sqrt{1+x^2}dx = 2\int_{0}^{1} \sqrt{1+x^2}dx = 2\left(\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}ln(x+\sqrt{1+x^2})\right)\Big|_0^1$$
$$= \sqrt{2} + \frac{1}{2}ln(\sqrt{2}+1)$$

2. Compute the length of the fence needed to cover a hill which can be defined as a cycloid loop of parametrical equations:

$$(C): \begin{cases} x = a(t - \sin t) \\ y = a(1-\cos t) \end{cases}, t \in [0,\pi].$$

$$L = \int_0^\pi ds = \int_0^\pi \sqrt{x'^2(t) + y'^2(t)}\,dt$$
$$x' = a(1-\cos t)$$
$$y' = a\sin t$$
$$x'^2 + y'^2 = a^2(1 - \cos 2t + \cos^2 t + \sin^2 t) = a^2(2 - 2\cos t) =$$
$$= 2a^2(1-\cos t) = 2a^2\left(2\sin^2\frac{t}{2}\right) = 4a^2\sin^2\frac{t}{2}$$
$$\sqrt{x'^2+y'^2} = 2a\left|\sin\frac{t}{2}\right|$$
$$t \in [0,\pi] \Rightarrow \frac{t}{2} \in [0,\pi/2] \Rightarrow \sin\frac{t}{2} \geq 0 \Rightarrow \left|\sin\frac{t}{2}\right| = \sin\frac{t}{2}$$
$$\Rightarrow \sqrt{x'^2+y'^2} = 2a\sin\frac{t}{2}$$
$$\Rightarrow L = \int_0^\pi 2a\sin\frac{t}{2}dt = 2a\frac{-\cos\frac{t}{2}}{\frac{1}{2}}\Bigg|_0^\pi = -4a\cos\frac{t}{2}\Big|_0^\pi = 0$$

3. Calculate the volume of the asteroid obtained by rotating around the OX-axis of the asteroid: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5^{\frac{2}{3}}$.

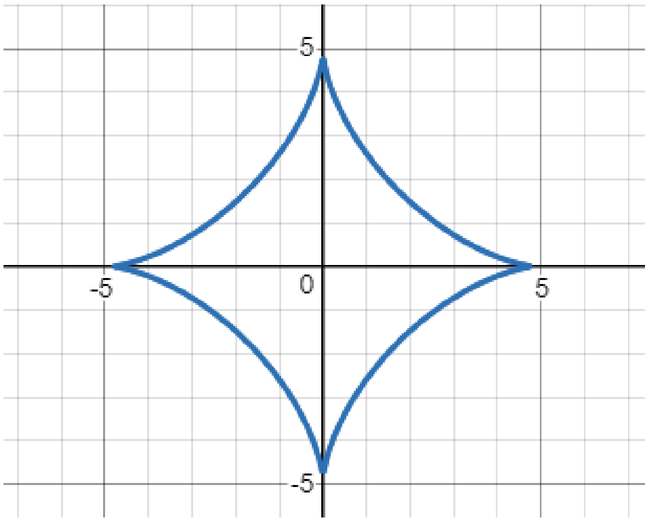

Figure 51

Since $= \left(5^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$, we have:

$$V = \pi \int_{-5}^{5} \left(5^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{3} dx = 2\pi \int_{-5}^{5} \left(5^{2} - 3 \cdot 5^{\frac{4}{3}} \cdot x^{\frac{2}{3}} + 3 \cdot 5^{\frac{2}{3}} \cdot x^{\frac{4}{3}} - x^{2}\right) dx = \frac{32\pi \cdot 125}{105}$$

6.2 PROJECT 2

Compute the commune area of the 2 boomerangs in form the parabolas: $(P_1): y^2 = 8x$ and $(P_2): x^2 = 8y$.

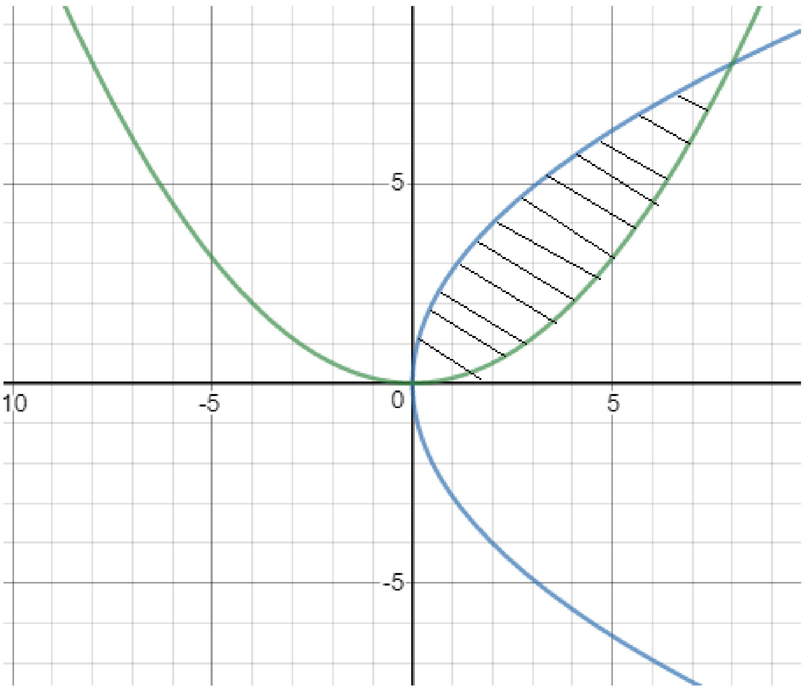

Figure 52

The parabolas intersect in points $O(0,0)$ and $M(8,8)$.

Indeed,

$$A = \int_0^8 \sqrt{8x}dx - \int_0^8 \frac{x^2}{8} dx = \frac{4}{3}\sqrt{2}x^{\frac{3}{2}}\Big|_0^8 - \frac{1}{8}\frac{x^3}{3}\Big|_0^8 = \frac{64}{3}.$$

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Guide for a Problem.

Analysis and Calculus

1.9. Solution of non-linear equations

AC9

Ascensión Hernández Encinas and

Araceli Queiruga-Dios

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 17, to be achieved with the development of this activity guide are the following:

1. To use intersecting graphs to help locate approximately the roots of non-linear equations
2. To use Descartes' rules of signs for polynomial equations
3. To understand the distinction between point estimation and interval reduction methods.
4. To use a point estimation method and an interval reduction method to solve a practical problem.
5. To understand the various convergence criteria.
6. To use appropriate software to solve non-linear equations.

Table 17 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- [green] = very important
- [yellow] = medium important
- [red] = less important

Table 17: Competencies that we measure with the global test model that is proposed.

		C1	C2	C3	C4	C5	C6	C7	C8
Analysis and Calculus									
AC9	9. Solution of non-linear equations								
AC91	Use intersecting graphs to help locate approximately the roots of non-linear equations	[green]	[green]	[green]	[green]	[green]	[green]	[green]	[green]
AC92	Use Descartes' rules of signs por polynomal equations	[yellow]	[green]	[yellow]	[red]	[yellow]	[yellow]	[red]	[red]
AC93	Understand the distinction between point estimation and interval reduction methods	[green]	[green]	[yellow]	[yellow]	[yellow]	[yellow]	[yellow]	[yellow]
AC94	Use a point estimation method and an interval reduction method to solve a practical problem	[green]	[green]	[green]	[green]	[green]	[green]	[green]	[green]
AC95	Understand the various convergence criteria	[yellow]	[green]	[yellow]	[yellow]	[yellow]	[yellow]	[yellow]	[yellow]
AC96	Use appropriate software to solve non-linear equations	[green]	[green]	[green]	[green]	[green]	[green]	[green]	[green]

2 CONTENTS

1. Understands the difference between point estimation and interval reduction.
2. Convergence criteria.
3. Different methods to solve problems with non linear equations: Bisection, Newton, etc.
4. Different software to solve non-linear equations.

3 TEST

The time to solve this exam will be 3 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 2.

Table 2: Learning outcomes involved in this assessment activity.

Analysis and Calculus	
AC9	9. Solution of non-linear equations
AC91	Use intersecting graphs to help locate approximately the roots of non-linear equations
AC92	Use Descartes' rules of signs for polynomial equations
AC93	Understand the distinction between point estimation and interval reduction methods
AC94	Use a point estimation method and an interval reduction method to solve a practical problem
AC95	Understand the various convergence criteria
AC96	Use appropriate software to solve non-linear equations

3.1 Multiple-Choice Questions

Solve the following test, reasoning if it's True or False.

1. Using the rule of Descartes we can know the root of an equation.

   i) True

   ii) False

2. Given the continuous function f: $[a,b]$ -> R, such that $f(a)\ f(b) < 0$. The Bisection method allows approximating a root of $f(x)$ in (a,b) but the Newton's method is not always convergent.

   i) True

   ii) False

3. Let (x_n) a succession generated by Newton's method, which converges to a point c, such that $f(c) = 0$. If $|x_n - x_{n-1}| < \varepsilon$, then we can guarantee $|x_n - c| < \varepsilon$.

   i) True

   ii) False

4. The function $f(x)$, whose graph is shown in the next figure, has two roots c_1 and c_2. Using Newton method with x_0 as initial value, the generated succession converges to c_1, as it is the root closet to x_0.

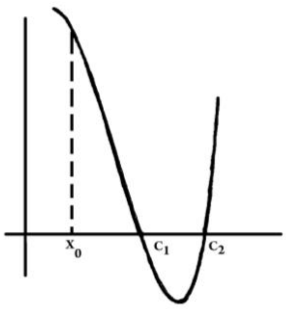

Figure 53

   i) True

   ii) False

5. The solution of the ax+b = 0 ($a \neq 0$) cannot be obtained by Newton's method, since the fuction $f(x) = ax + b$ does not satisfy the convergence conditions.

   i) True

   ii) False

6. If $g(x) = ½ \cos(x)$, the succession (x_n) given by $x_1 = 1$, $x_{n+1} = g(x_n)$ converges to c, where $g(c)=0$.

   i) True

   ii) False

7. If the iterative method $x_{n+1} = g(x_n)$ is applied, with $g(x)$ being a no-constant continuous function, two consecutive values cannot be the same value.

   i) True

   ii) False

3.2 QUESTION 1

1. Use the Descartes rule to determine the number of possible solutions in the following equation: $2x^8 - 5x^6 - x^3 + 4x^2 = 0$.
2. Check that f (x) = e-x – x has as least one root in the interval [0, 1].
3. Separate the roots that the function $F(x) = x^3 - x$ possesses in the interval [-2, 2].
4. Find the root of x3 + 4 x2 – 10 x = 0 near 1.5 with an error dimension of 10-3, using Bisection method and Newton method. Use the appropriate instruction to obtain the solution with the computer.

3.3 QUESTIONS 2

This is an open problem. There are different ways to solve it.

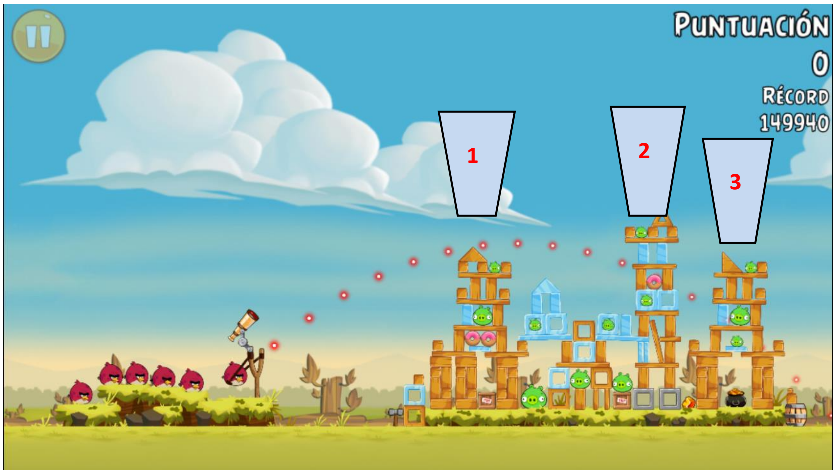

Figure 54. Angry Birds game, developed by Rovio Entertainment Corporation

The Angry Birds game, developed by Rovio Entertainment Corporation, consists of destroying structures of different materials in order to eliminate the pigs inside (green in the Figure 54). With the help of a slingshot we launched an Angry Bird (in red) with the angle and strength necessary to achieve the proposed goal. To prevent the bird from hitting a tower, as you can see in the figure and not reach its destination, we must pass as close to the first two but without touching them.

We assume that the slingshot always shoots from a height of 1 cm, Tower 1 that is 1 cm from the slingshot measures 5 cm, Tower 2 that is 2 cm measures 7 cm and the desired point of impact, in the Tower 3 which is 4 cm from the slingshot, is exactly 5 cm high.

Find the function that describes the Angry Bird in the shot, knowing that we do not risk much and passes 0.5 cm from the Towers.

In this problem we use two "usual" Angry Birds coming from the left and another one who came from the right.

1. Use de rules of Descartes to determine the number of possible solutions in the equation obtained in the trajectory of the fist Angry Bird.
   2. If an Angry Bird comes from the other side, from 10cm away with a trajectory:
   3. f(x)= 10 - x, at what point they found first?
   4. If an Angry Bird starts flying with a trajectory $f(x) = x^3$ will they collide? If so, at what point?

Give an approximated solution using a non linear method with an error dimension of 10^{-3}.

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test, reasoning if the answer it's True or False.

1. Using the rule of Descartes we can know the root of an equation.

   i) True

   ii) False

   Solution:

   False. You can only know how many solutions you can have.

   LO: AC92.

2. Given the continuous function *f*: [*a*,*b*] -> R, such that *f*(*a*) *f*(*b*) < 0. The Bisection method allows approximating a root of *f* in (*a*,*b*) but the Newton's method is not always convergent.

   i) True

   ii) False

   Solution:

   True because some times the Newton's method may not be convergent.

   LO: AC93, AC95.

3. Let (x_n) a succession generated by Newton's method, which converges to a point c, such that *f*(c) = 0. If $|x_n - x_{n-1}| < \varepsilon$, then we can guarantee $|x_n - c| < \varepsilon$.

   i) True

   ii) False

   Solution

   False because ε can take any value and it could be $|x_n - c| > |x_n - x_{n-1}|$.

   LO: AC95.

4. The function *f*(x), whose graph is shown in the Figure 1, has two roots c_1 and c_2. Using Newton's method with x_0 as initial value, the generated succession converges to c_1, as it is the root closet to x_0

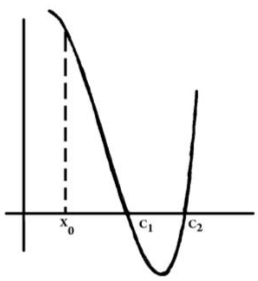

Figure 55

   i) True

   ii) False

Solution:

False because Newton's method converges by tangents to c_2

LO: AC91, AC95.

5. The solution of the ax+b = 0 (a ≠ 0) cannot be obtained by Newton's method, since the fuction *f* (x) = ax + b does not satisfy the convergence conditions.

   i) True

   ii) False

   Solution:

   False. The first iteration of this method gives: $x_1 = x_0 - \frac{ax_0+b}{a} = \frac{-b}{a}$

   LO: AC95.

6. If *g*(x) = ½ cos (x), the succession (x_n) given by $x_1 = 1$, $x_{n+1} = g(x_n)$ converges to c, where *g*(c)=0.

   i) True

   ii) False

   Solution:

   False.

   As $x_{n+1} = g(x_n)$ and *g*(x) is continous, then $\lim_{n\to\infty} g(x_n) = g \lim_{n\to\infty} (x_n)$.

   Then:

   $\lim_{n\to\infty} (x_{n+1}) = \lim_{n\to\infty} g(x_n) = g \lim_{n\to\infty} (x_n) = g(c)$

   $\lim_{n\to\infty} (x_{n+1}) = c$

   Then $g(c) = c$ and that's not true.

   LO: AC95.

7. If the iterative method $x_{n+1} = g(x_n)$ is applied, with *g*(x) being a no-constant continuous function, two consecutive values cannot be the same value.

   i) True

   ii) False

   Solution:

   False.

   For example, if *g*(x) = sin (x) and $x_n = 0$, then $g(x_n) = x_{n+1} = 0$

   LO: AC91, AC92, AC93, AC95.

4.2 QUESTION 1

1. Use the Descartes rule to determine the number of possible solutions in the following equation: $2x^8 - 5x^6 - x^3 + 4x^2 = 0$.

Solution:

The equation has this coefficients: 2, -5, -1, 4 and there are 2 sign changes between -5 and 2, and between -1 and 4. That means that it could have 2 positive roots (or not).

The equation in (-x) is: $2(-x)^8 – 5 (-x)^6 – (-x)^3 + 4(-x)^2 = 2x^8 – 5 x^6 + x^3 + 4x^2$.

The new coeficients are: 2, -5, 1, 4 and there are 2 sign changes between 2 and -5, and between -5 and 1. That means that there are 2 negatives roots or not.

Also, seeing the equation we know that there are 2 roots that are 0.

Then these cases can occur:

Two positive, two negative, two zero and two complex roots.

Two positive, zero negative, two zero and four complex roots.

Zero positive, two negative, two zero and four complex roots.

Zero positive, zero negative, two zero and six complex roots.

LO: AC94.

2. Check that $f(x) = e^{-x} – x$ has as least one root in the interval [0,1].
Solution:

To verify that the function is continuous and changes of signs occur at the end of the interval will be sufficient (Bolzano Theorem)

The function $f(x) = e^{-x} – x$ (Figure 3) is continous in $\mathfrak{R}$ and, and therefore throughout the full range.

The function $f(x)$ takes different signs at the ends:

$f(0) = 1 > 0$,

$f(1) = e^{-1} - 1 < 0$.

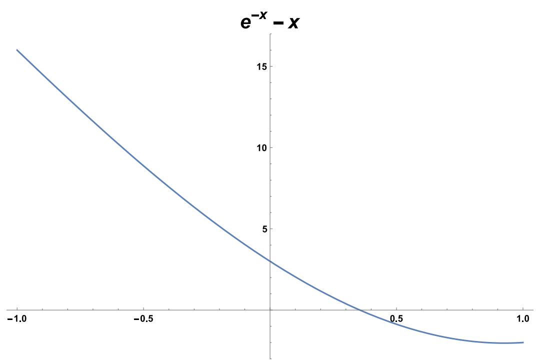

Figure 56

LO: AC93, AC95.

3. Separate the roots that the function $F(x) = x^3 - x$ possesses in the interval [-2,2].

Solution:

As it is a polynomial, it is continous and derivable in $\mathfrak{R}$, and in particular in the interval [-2, 2].

In addition $F(-2) = -6 < 0$ and $F(2) = 6 > 0$, then the Bolzano theorem is satisfied and there are roots of the equation in the interval [-2, 2].

Let see how many extremun have the function in that interval

$$0 = F'(x) = 3x^2 - 1 \to x = \pm\frac{\sqrt{3}}{3}$$

There are two extremes and therefore, as much 3 roots: $\left[-2, -\frac{\sqrt{3}}{3}\right], \left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right], \left[\frac{\sqrt{3}}{3}, 2\right]$

It can be verified that in each subinterval the Bolzano theorem is fulfilled, then there is at least a root in each one.

We calculate the sign of the derivative in each subinterval to see that de roots are unique in every interval:

$F'(-1) > 0$, $F'(0) < 0$, $F'(1) < 0$

In Figure 4 we can see a graph of the function $x^3 - x$

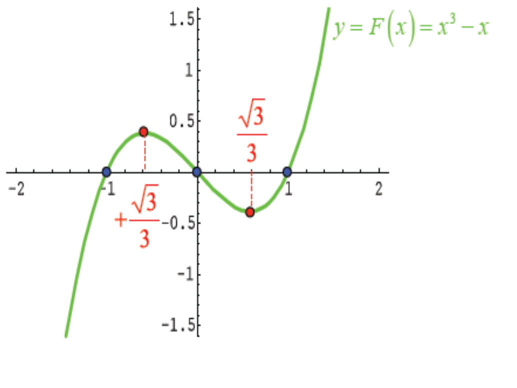

Figure 57. Graph of x3 – x

LO: AC93, AC95.

4. Find the root of $x^3 + 4x^2 - 10x = 0$ near 1.5 with an error dimension of 10^{-3}, using Bisection method and Newton's method. Use the appropriate instruction to obtain the solution with mathematical software (such as Mathematica).

Solution:

First of all we need to separate the interval where there is a solution.

The function $f(x) = x^3 + 4x^2 - 10x + 3$ is C^∞ in $\Re$, because it is a polynomial.

$f(1) = -2$

$f(2) = 7$

There is a change of sign so we can ensure there is any solution (Bolzano Th.), let see if the solution is unique.

$f'(x) = 3x^2 + 8x - 10 > 0$ in the interval [1, 2]. Then in this interval there is only one solution.

We'll find it by the diferent methods.

i) Bisection:

First of all we need to calculate the number of iterations that we need.

$$err \leq \frac{b-a}{2^{n+1}} \leq 10^{-3}$$

$$\frac{2-1}{2^{n+1}} \leq 10^{-3} \Rightarrow n \geq \frac{\text{Log}10^3}{\text{Log }2} - 1 = 8.96$$

Then, we need 9 iterations:

n	a	b	c	*f*(c)
0	1	2	1.5	0.375
1	1	1.5	1.25	-1.29688
2	1.25	1.5	1.375	-0.587891
3	1.375	1.5	1.4375	-0.138916
4	1.4375	1.5	1.46875	0.109833
5	1.4375	1.46875	1.45313	-0.0165825
6	1.45313	1.46875	1.46094	0.0461135
7	1.45313	1.46094	1.45703	0.0146378
8	1.45313	1.45703	1.45508	-0.00100427
9	1.45508	1.45703	1.45605	

The aproximation to the root is 1.45605.

ii) Newton:

$$err \leq \frac{|f(x)|}{m} \leq 10^{-3}, \text{where } m \leq |f'(x)|$$

$f'(x) = 3x^2 + 8x - 10$

$f''(x) = 6x + 8 > 0 \ \forall x \in [1, 2] \Rightarrow f'(x)$ is an increasing function $\Rightarrow$ It's enough to obtain the value at the ends

$$\left.\begin{array}{r}|f'(1)| = 1\\ |f'(2)| = 18\end{array}\right\} \Rightarrow m = 1 \Rightarrow err \leq |f(x)| \leq 10^{-3}$$

Then m = 1.

Since $f''(x) > 0$ then the initial value can be $x_0 = 1.5$ where $f(1.5) = 0.375 > 0$.

$x_0 = 1.5 \Rightarrow |f(x)| = 0.375 > 10^{-3}$

$x_1 = 1.45714 \Rightarrow |f(x)| = 0.0155335 > 10^{-3}$

$x_2 = 1.45521 \Rightarrow |f(x)| = 0.0000313429 < 10^{-3}$

The aproximation to the root is 1. 45521, with 2 interations.

iii) Using Mathematica software:

FindRoot[*f*[x], {x, 1.5}]

{x -> 1. 4552}

LO: AC93, AC94, AC95.

4.3 Question 2

This is an open problem. There are different ways to solve it.

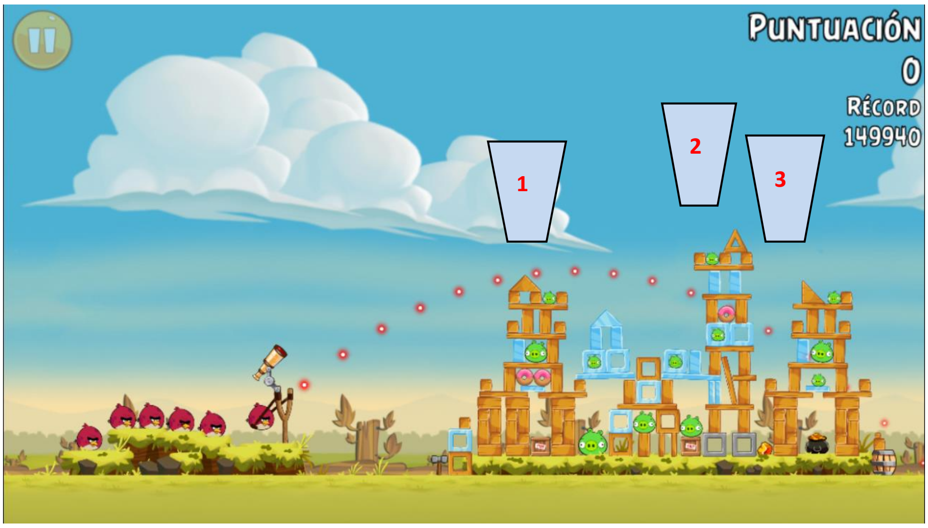

Figure 58. Angry Birds game, developed by Rovio Entertainment Corporation

The Angry Birds game, developed by Rovio Entertainment Corporation (Figure 58), consists of destroying structures of different materials in order to eliminate the pigs inside (green in the Figure 58). With the help of a slingshot we launched an Angry Bird (in red) with the angle and strength necessary to achieve the proposed goal. To prevent the bird from hitting a tower, as you can see in the figure and not reach its destination, we must pass as close to the first two but without touching them.

We assume that the slingshot always shoots from a height of 1 cm, Tower 1 that is 1 cm from the slingshot measures 5 cm, Tower 2 that is 2 cm measures 7 cm and the desired point of impact, in the Tower 3 which is 4 cm from the slingshot, is exactly 5 cm high.

Find the function that describes the Angry Bird in the shot, knowing that we do not risk much and passes 0.5 cm from the Towers.

In this problem we use two “usual” Angry Birds coming from the left and another one who came from the right.

1. Use de rules of Descartes to determine the number of possible solutions in the equation obtained in the trajectory of the fist Angry Bird.
2. If an Angry Bird comes from the other side, from 10cm away with a trajectory:

   $f(x) = 10 - x$, at what point they found first?
3. If an Angry Bird starts flying with a trajectory $f(x) = x^3$ will they collide? If so, at what point?

Give an approximated solution using a non linear method with an error dimension of 10^{-3}.

Solution:

The trajectory of the first Angry Bird can be obtained by different methods (systems of linear equations, interpolation, …) knowing that it passes through the points: (1, 1.5), (2, 7.5), (4, 5.5).

Figure 59. Angry Birds game, developed by Rovio Entertainment Corporation

The solution of this trajectory (Figure 59), using the information that this points give, is:

$$f_1(x) = 1.5 + 5\,x - x^2$$

1. The trajectory of the fist Angry Bird is $f_1(x) = 1.5 + 5\,x - x^2$ then the equation will be:

   $1.5 + 5\,x - x^2 = 0$.

   The coefficients are: - 1, 5, 1.5 and there is 1 sign change between -1 and 5. That means that there is 1 positive root or not.

   The equation in (-x) is: $1.5 + 5\,(-x) - (-x)^2 = 1.5 - 5\,x - x^2$

   The new coeficients are: 1.5, - 5, -1, and again there is 1 sign changes between 1.5 and -5. That means that there is 1 negative root or not.

   At the time these cases can happen:

   i) One positive and one negative.

   ii) Two positive and zero negative.

   iii) Zero positive and two negative.

2. The trajectory of the second Angry Bird is given by the statement: $f_2(x) = 10 - x$

   This is a representation on them (Figure 60):

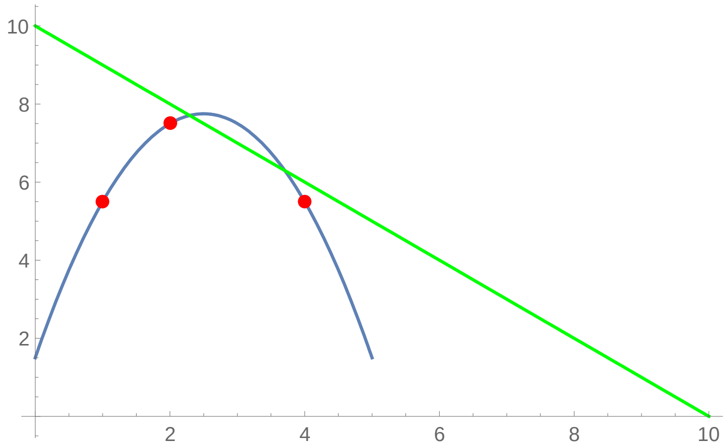

Figure 60. Figure. Graph of the trayectory of the first two Angry Birds

The solution is near 4.

In this case we don't need and aproximation because of:

$f_1(x) = f_2(x) \Rightarrow f(x) = f_1(x) - f_2(x) = 0 \Rightarrow 1.5 + 5x - x^2 - 10 + x = -8.5 + 6x - x^2 = 0$.

Then, solving a second-degree equation we can get the solution: 3.70711, and it is at 6.2929 cm from the ground.

If in spite of everything, we would like to use the numerical methods, we would have:

The function $f(x) = -8.5 + 6x - x^2$ is C^∞ in $\Re$, because it is a polynomial.

$f(3.5) = 0.25$

$f(4.5) = -1.75$

There is a change of sign so we can ensure that there are solutions (Th of Bolzano), let see if it's unique.

$f'(x) = -2x + 6 < 0$ in the interval [3.5, 4.5]. Then in this interval there is only one solution.

We'll find it by the diferent methods.

i) Bisection:

First of all we need to calculate the number of iterations that we need.

$$err \leq \frac{b-a}{2^{n+1}} \leq 10^{-3}$$

$$\frac{4.5-3.5}{2^{n+1}} \leq 10^{-3} \Rightarrow n \geq \frac{\text{Log}10^3}{\text{Log } 2} - 1 = 8.96$$

Then, we need 9 iterations:

n	a	b	c	$f(c)$
0	3.5	4.5	4.	-0.5
1	3.5	4.	3.75	-0.0625
2	3.5	3.75	3.625	0.109375
3	3.625	3.75	3.6875	0.0273438
4	3.6875	3.75	3.71875	-0.0166016
5	3.6875	3.71875	3.70313	0.00561523
6	3.70313	3.71875	3.71094	-0.00543213
7	3.70313	3.71094	3.70703	0.000106812
8	3.70703	3.71094	3.70898	-0.00265884
9	3.70703	3.70898	3.70801	

The approximation to the root is 3.70801

ii) Newton's method:

$$err \leq \frac{|f(x)|}{m} \leq 10^{-3}, \text{where } m \leq |f'(x)|$$

f ' (x) = -2 x + 6

f''(x) = -2 < 0 ∀ x ∈ℜ ⇒ f '(x) is an decreasing function ⇒ It's enough to lood the value at the ends

$$\left.\begin{array}{l}|f'(3.5)| = 1\\|f'(4.5)| = 3\end{array}\right\} \Longrightarrow m = 1 \Longrightarrow err \leq |f(x)| \leq 10^{-3}$$

Then m = 1

f''(x) < 0 then the initial value can be x0= 4.5 where f(4.5) = -1.75 <0.

x0= 4.5 ⇒|f(x)|= 1.75

x1= 3. 91667 ⇒|f(x)|= 0. 340278

x2= 3. 73106 ⇒|f(x)|= 0. 0344496

x3= 3. 7075 ⇒|f(x)|= 0.000555139

The approximation to the root is 3. 7075, with 4 iterations.

iii) Mathematica:

FindRoot[f[x], {x, 4}]

{x -> 3.70711}

The approximation to the root given by Mathematica is 3.70711

3. The trajectory of the second Angry Bird is given by the statement: $f_2(x) = x^3$

This is a representation of them (Figure 61)

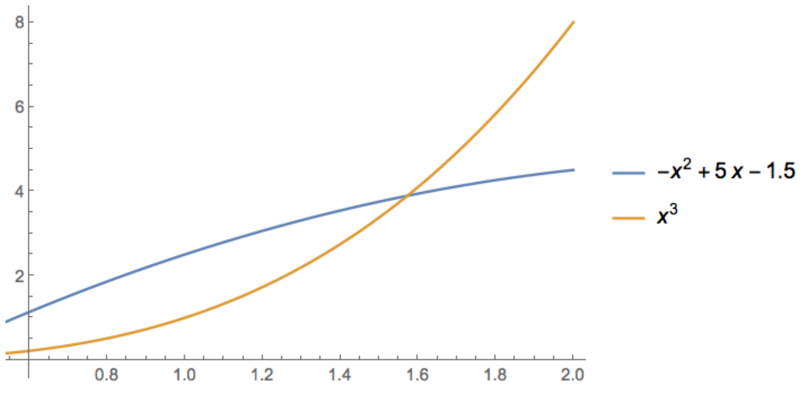

Figure 61. Graph of the trajectory of the first and third Angry Birds

The solution is near 1.5 then we can use a numerical method to obtain the solution of

f1(x) = f2(x) ⇒f(x) = f1(x) - f2(x) = x3 + x2- 5x + 1.5 = 0

The function f(x) = x3 + x2- 5x + 1.5 is C∞ in ℜ, because it is a polynomial.

f(1.5) = - 0.375

f(1.7) = 0.803

There is a change of sign so we can ensure that there are solutions (Theorem of Bolzano), let see if it's unique.

f ' (x) = 2 x -5 < 0 in the interval [1.5, 1.7]. Then in this interval there is only one solution.

We'll find it by the different methods.

i) Bisection:

First of all, we need to calculate the number of iterations that we need.

$$err \leq \frac{b-a}{2^{n+1}} \leq 10^{-3}$$

$$\frac{1.7-1.5}{2^{n+1}} \leq 10^{-3} \Rightarrow n \geq \frac{Log 10^2}{Log\ 2} = 6.64$$

Then, we need 7 iterations

n	a	b	c	*f*(c)
0	1.5	1.7	1.6	0.156
1	1.5	1.6	1.55	-0.123625
2	1.55	1.6	1.575	0.0126094
3	1.55	1.575	1.5625	-0.0563965
4	1.5625	1.575	1.56875	-0.0221165
5	1.56875	1.575	1.57188	-0.00480936
6	1.57188	1.575	1.57344	0.00388604
7	1.57188	1.57344	1.57266	

The approximation to the root is 1. 57266

ii) Newton's method:

$$err \leq \frac{|f(x)|}{m} \leq 10^{-3}, \text{where } m \leq |f'(x)|$$

f ' (x) = 3 x2 + 2 x - 5 + 6

f ''(x) = 6x + 2 > 0 ∀ x ∈[1.5, 1.7] ⇒ f '(x) is an increasing function ⇒ It's enough to lood the value at the ends:

$$\left.\begin{array}{l} |f'(1.5)| = 4.75 \\ |f'(1.7)| = 7.07 \end{array}\right\} \Rightarrow m = 4.75 \Rightarrow err \leq \frac{|f(x)|}{4.75} \leq 10^{-3} \Rightarrow |f(x)| \leq 4.75 \cdot 10^{-3}$$

Since f ''(x) > 0 then the initial value can be x0= 1.7 where f(1.7) = 0.803 <0.

x0= 1.7 ⇒|f(x)|= 0. 803 > 10-3

x1= 1. 58642⇒|f(x)|= 0. 0772253 > 10-3

x2= 1. 57293⇒|f(x)|= 0. 0010462 > 10-3

x3= 1. 57274⇒|f(x)|= 2.0188 · 10-7 < 10-3

The approximation to the root is 1. 57274, with 4 iterations.

iii) Mathematica:

FindRoot[f[x], {x, 1.7}]

{x -> 1. 57274}

The approximation to the root given by Mathematica is 1.57274.

LO: AC91, AC92, AC93, AC94, AC95, AC96.

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Guide for a Problem

Analysis and Calculus

1.10. Solution of non-linear equations

AC9

Snezhana Gocheva-Ilieva and

Atanas Ivanov

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 19, to be achieved with the development of this activity guide are the following:

1. To recognize the type of equation and the number of its roots.
2. To understand and apply different methods for locating real roots.
3. To understand the concept of reducing the localization interval and the difference with iterative processes.
4. To present the equation in a form suitable for an iterative process (Newton method, etc.)
5. Apply basic methods for solving a nonlinear equation (bisection method, iterations by the Newton's method and more).
6. To understand how to stop correctly the calculations of the iterative process using different convergence criteria and depending on the accuracy of the intermediate calculations.
7. To apply the knowledge for the complete solution of practical problems for nonlinear equations.
8. To interpret the result obtained.

Table 19 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- Green = very important
- Yellow = medium important
- Red = less important

Table 19: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Analysis and Calculus									
AC9	5. Solution of non-linear equations								
AC91	Use intersecting graphs to help locate approximately the roots of non-linear equations	Green	Green	Green	Green	Green	Green	Green	Green
AC92	Use Descartes' rules of signs for polynomial equations	Yellow	Green	Yellow	Red	Yellow	Yellow	Red	Red
AC93	Understand the distinction between point estimation and interval reduction methods	Green	Green	Yellow	Yellow	Yellow	Yellow	Yellow	Yellow
AC94	Use a point estimation method and an interval reduction method to solve a practical problem	Green	Green	Green	Green	Green	Green	Green	Green
AC95	Understand the various convergence criteria	Yellow	Green	Yellow	Yellow	Yellow	Yellow	Yellow	Yellow
AC96	Use appropriate software to solve non-linear equations	Green	Green	Green	Green	Green	Green	Green	Green

2 CONTENTS

1. Use of Descartes' rules of signs for determining the possible number of positive and negative real zeros of a single-variable polynomial equation.
2. Understand how to locate the real zero(s) if the function of the equation is continuous.
3. Using plots to represent approximately the location of real roots of the non-linear equation.
4. Understand the idea of the iteration process and convergence for finding an approximate solution to the given equation.
5. Understand the difference between interval reduction and point estimation approaches.
6. Convergence criteria.
7. Determine the necessary working precision of the intermediate calculations for solving the equation with a preset accuracy.
8. Demonstrate classical methods for solving a non-linear equation: bisection method, Newton's method, etc.
9. Apply mathematical software to solve non-linear equations.

3 TEST

The time to solve this exam could be 3 hours (approximately).

The exam includes: Multiple-choice questions; Tasks to solve; Small practical project by implementing mathematical software.

The learning outcomes to be achieved with this procedure are shown in Table 20.

Table 20: Learning outcomes involved in this assessment activity.

Analysis and Calculus	
AC9	5. Solution of non-linear equations
AC91	Use intersecting graphs to help locate approximately the roots of non-linear equations
AC92	Use Descartes' rules of signs for polynomial equations
AC93	Understand the distinction between point estimation and interval reduction methods
AC94	Use a point estimation method and an interval reduction method to solve a practical problem
AC95	Understand the various convergence criteria
AC96	Use appropriate software to solve non-linear equations

3.1 MULTIPLE-CHOICE QUESTIONS

1. Which of the following is not a nonlinear equation?
   i) $3x - 2 = 4x$
   ii) $\sqrt{3x^5 - 2x} = x + 1$
   iii) $sin\left(x - \frac{\pi}{3}\right) + cos(2x + 1) - 0.5 = 0$
   iv) All are non-linear
2. Given the polynomial function $f(x) = 2x^3 - 3x + 4$, how many real zeros posseses the equation $f(x) = 0$ in the interval $(-\infty, 0]$
   i) 1 real zero
   ii) 2 real zeros
   iii) 3 real zeros
   iv) 0 zeros
3. How many positive zeros could have the equation $f(x) = x^4 - x^3 + x - 2 = 0$:
   i) 0 zeros
   ii) 1 or 3 zeros
   iii) 0 or 2 zeros
   iv) 4 zeros
4. Locate analytically at what interval any real root of the equation $f(x) = x^3 - 7x^2 + 10 = 0$ is:
   i) $[-4, -3]$
   ii) $[0,1]$
   iii) $[1,2]$
   iv) $[4,6]$
5. The principle of an iteration method consists in:
   i) Establishing a formula to calculate the root of the equation
   ii) Proving the existence of a root, locating the interval, plotting the function from which you can fix the approximate value of the searched root.
   iii) Using a method for presentation of the equation in the so called normal form.
   iv) Constructing a repeated process until an answer is achieved.
6. The difference between the interval reduction method and point estimation (fixed point iteration) approaches is:
   i) The interval reduction method constructs a sequence of decreasing intervals containing the root of the equation, while the point estimation method computes successive terms using a rule formula and a starting value of the root.
   ii) The interval reduction method is much faster than the point estimation method.
   iii) The interval reduction method finds the exact solution, while the point estimation calculates a solution with a preset accuracy.

iv) The interval reduction method does not differ essentially in speed and accuracy to the fixed point iteration.

7. The convergence criteria for calculating the real root $x*$ with accuracy ε is applied correctly for the iteration process $x_{k+1} = g(x_k), k = 0,1,2,\ldots, x_0$ is given:

   i) $|x_k - x*| < \varepsilon$

   ii) $|x_k - x_0| < \varepsilon$

   iii) $|x_{k+1} - x_k| < \varepsilon$

   iv) $|g(x_k)| < \varepsilon$

8. It is required to find the approximate root of a nonlinear equation with accuracy ε, say $\varepsilon = 0.001$. The intermediate calculations could be performed with accuracy:

   i) $\varepsilon = 0.001$

   ii) $\varepsilon = 0.00001$

   iii) $\varepsilon = 0.0000001$

   iv) $\varepsilon = 0.00000001$

LO: AC4, AC6, AC9.

3.2 Questions 1

1. Using the Newton's method find the first four iterations x_1, x_2, x_3, x_4 with $x_0 = 2$ for calculating the root of the equation $2x^3 - 4x - 3 = 0$. Evaluate the achieved accuracy at each iteration.

   Remark. When appropriate, solve this question using mathematical software.

   LO: AC5, AC6, AC9.

2. Using intersecting graphs locate approximately the unit root of non-linear equation $x^3 - x + 5 = sin(2x)$ in an interval with length 1. *Remark.* When appropriate, solve this question using mathematical software.

   LO: AC6, AC9.

3.3 Question 2

In a chemical reaction $CO + \frac{1}{2}O_2 \rightleftarrows CO_2$, the percent of decomposition x per 1 mole of carbon dioxide (CO_2) is found from the equation $f(x) = \left(\frac{p}{k^2} - 1\right)x^3 + 3x - 2 = 0$, where p (*atm*) is the gas pressure and k is a temperature-dependent equilibrium constant. Find x for $k = 1.648$ (corresponds to a temperature of 2800 Kelvin) and $p = 1$ *atm*. Solve the equation by bisection method using mathematical software with 5 iterations.

LO: AC6, AC9.

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. Which of the following is not a nonlinear equation?

   i) $3x - 2 = 4x$

   ii) $\sqrt{3x^5 - 2x} = x + 1$

   iii) $sin\left(x - \frac{\pi}{3}\right) + cos(2x + 1) - 0.5 = 0$

   iv) All are non-linear

   Solution:

   An equation is said to be nonlinear when it involves terms of degree higher than 1 in the unknown quantity or contains transcendental functions, such as sin, cos, log, exp, etc.

   The right answer is i).

2. Given the polynomial function $f(x) = 2x^3 - 3x + 4$, how many real zeros posseses the equation $f(x) = 0$ in the interval $(-\infty, 0]$

   i) 0 zeros

   ii) 1 real zero

   iii) 2 real zeros

   iv) 3 real zeros

   Solution:

   The function is arranged in descending powers of x. Applying the Descartes' rule of sign for negative x we have expression $f(-x) = -2x^3 + 3x + 4$ with only one change in the sign of the coefficients. So, the equation $f(x) = 0$ has one negative real zero. As $f(0) = 4 \neq 0$ then $x = 0$ is not a root. Hence the equation has exactly one real root in $(-\infty, 0]$.

   The right answer is ii).

3. How many positive zeros could have the equation $f(x) = x^4 - x^2 + x - 2 = 0$:

   i) 0 zeros

   ii) 1 or 3 zeros

   iii) 0 or 2 zeros

   iv) 4 zeros

   Solution:

   For positive x we have three changes in the sign of the coefficients in the given function $f(x) = x^4 - x^2 + x - 2 = 0$. The Descartes´ rule of sign tells us that we have three real positive zeros or less but an odd number of zeros. Consequently, the number of positive zeros must be either three, or one.

   The right answer is ii).

4. Locate analytically at what interval any real root of the equation $f(x) = x^3 - 7x^2 + 10 = 0$ is:

   i) $[-4, -3]$

   ii) $[0,1]$

   iii) $[1,2]$

   iv) $[4,6]$

   Solution:

   Since $f(x)$ is continuous function for each x, we can use the condition for different signs at the ends of the given interval. We have $f(1) = 1 - 7 + 10 = 4 > 0$, $f(2) = 8 - 28 + 10 = -10 < 0$. Therefore, in the interval $[1,2]$ the equation has an odd number of real roots. Following the Descartes' rule, the root is unique. At other intervals this is not fulfilled.

   The right answer is iii).

5. The principle of an iteration method consists in:

   i) Establishing a formula to calculate the root of the equation

   ii) Proving the existence of a root, locating the interval, plotting the function from which you can fix the approximate value of the searched root.

   iii) Using a method for presentation of the equation in the so called normal form.

   iv) Constructing a repeated process until an answer is achieved.

   Solution:

   Iterative algorithms and methods construct a sequence of approximations to the solution sought.

   The right answer is iv).

6. The difference between the interval reduction method and point estimation (fixed point iteration) approaches is:

   i) The interval reduction method constructs a sequence of decreasing intervals containing the root of the equation, while the point estimation method computes successive terms using a rule formula and a starting value of the root.

   ii) The interval reduction method is much faster than the point estimation method.

   iii) The interval reduction method finds the exact solution, while the point estimation calculates a solution with a preset accuracy.

   iv) The interval reduction method does not differ essentially in speed and accuracy to the fixed point iteration.

   Solution:

   Iterative algorithms and methods construct a sequence of approximations to the solution sought.

   The right answer is i).

7. The convergence criteria for calculating the real root $x*$ with accuracy ε is applied correctly for the iteration process $x_{k+1} = g(x_k), k = 0,1,2,\ldots, x_0$ is given:

    i) $|x_k - x*| < \varepsilon$

    ii) $|x_k - x_0| < \varepsilon$

    iii) $|x_{k+1} - x_k| < \varepsilon$

    iv) $|g(x_k)| < \varepsilon$

    Solution:

    The first answer requires knowledge of the solution $x*$. The second answer cannot give a convergence to 0, since x_0 is generally far from the root. The last answer is meaningless. The third answer can guarantee the convergence of the iteration process as a Cauchy sequence.

    The right answer is iii).

8. It is required to find the approximate root of a nonlinear equation with accuracy ε, say $\varepsilon = 0.001$. The intermediate calculations could be performed with accuracy:

    i) $\varepsilon = 0.001$

    ii) $\varepsilon = 0.00001$

    iii) $\varepsilon = 0.0000001$

    iv) $\varepsilon = 0.00000001$

    Solution:

    In order not to accumulate a rounding error, all intermediate calculations are performed with at least 1-2 decimal places more than the set accuracy.

    The right answer is ii).

4.2 QUESTIONS 1

1. Using the Newton's method find the first four iterations x_1, x_2, x_3, x_4 with $x_0 = 2$ for calculating the root of the equation $2x^3 - 4x - 3 = 0$. Evaluate the achieved accuracy at each iteration.

    Remark. When appropriate, solve this question using mathematical software.

    Solution 1:

    The equation $f(x) = 2x^3 - 4x - 3 = 0$ has the only real root. Through checks we find the interval of localization $[a, b] = [1,2]$. Really, $f(a)f(b) = (-5).5 = -25 < 0$. The derivative is $f'(x) = 6x^2 - 4$. The formula of Newtonian method for the given initial approximation has the form:

    $x_0 = 2, x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k = 0,1,2,3 \ldots$

    We will work with intermediate accuracy $\varepsilon = 0.000001$ as we expect a fast convergence of the method. We consistently obtain the results shown in next Table.

Numerical results of the iteration process by the Neuton's method for solving Question 1.

k	x_k	$\lvert x_{k+1} - x_k \rvert$	$f(x_k)$
0	2		5
1	1.75	$\lvert 0.25 \rvert$	0.71875
2	1.7	$\lvert 0.05 \rvert$	0.026
3	1.698051	$\lvert 0.001949 \rvert$	0.000039
4	1.698048	$\lvert 0.000003 \rvert$	$8.64\text{x}10^{-11}$

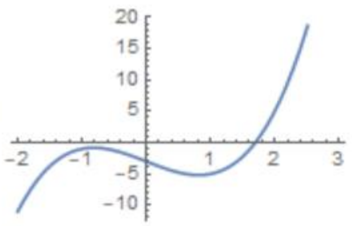
Answer: The approximate value of the root is $x_4 \approx 1.69805$.

Solution 2:

We compile the following code in Wolfram Mathematica:

```
In[1]:= f[x_] := 2 x^3 - 4 x - 3.
        Plot[f[x], {x, -2, 3}]
        fp[x_] := 6 x^2 - 4
        x0 = 2.;
        For[k = 1, k ≤ 4, k++,
          x1 = x0 - f[x0]/fp[x0];
          Print["k=", k, "  x1=", x1, "  Abs(x1-x0)=", Abs[x1 - x0], "  f=", f[x1]];
          x0 = x1
        ]
        x1
```

After execution we get:

Out[2]=



```
k=1  x1=1.75  Abs(x1-x0)=0.25  f=0.71875
k=2  x1=1.7  Abs(x1-x0)=0.05  f=0.026
k=3  x1=1.69805  Abs(x1-x0)=0.00194903  f=0.0000387319
k=4  x1=1.69805  Abs(x1-x0)=2.91212×10^-6  f=8.63993×10^-11
Out[6]= 1.69805

1.698048062394615`
```

Figure 62

Answer: The approximate solution is the same as it was found in Solution 1.

Figure 62

2. Using intersecting graphs locate approximately the unit root of non-linear equation $x^3 - x + 5 = sin(2x)$ in an interval with length 1.

   Remark. When appropriate, solve this question using mathematical software.

   Solution:

   To locate a real root with the help of plots, we draw approximate plots of the two functions on the left and right sides of the equation, which we will denote by $f_1(x) = x^3 - x + 5$ and $f_2(x) = sin(2x)$ respectively. The intersections of the two plots will show the location of the root(s).

   a) Solution without computer

   The left function $f_1(x)$ has one real root. In minus infinity it tends to minus infinity, and for $x = 0$ it is positive. Therefore, the root is negative. We study the first derivative $f_1'(x) = 3x^2 - 1$. It has two roots, i.e. the function $f_1(x)$ has two local extrema, respectively, in points $x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}$. The local maximum is at the point x_1 and is approximately equal to 5.4. The local minimum is at the point x_2 and is approximately equal to 4.6. Also $f_1(-2) = -1$ and $f_1(-1) = 5$ so the function $f_1(x)$ grows in the interval $[-2, -1]$ and is higher than 4 in the interval $[-1, +\infty]$.

   The second function $f_2(x)$ varies with a period $\omega = \pi$ taking values from -1 to +1. Therefore, the root of the given equation is in the interval $[-2, -1]$.

   b) Solution using Wolfram Mathematica software package [3]

   This is comparatively much easier with the formulas and plots parameters being set correctly. The code and results are as follows:

```
In[1]:= f1[x_] := x^3 - x + 5
        f2[x_] := Sin[2 x]
        Plot[{f1[x], f2[x]}, {x, -π, π}, PlotLegends → {"x^3-x+5", "Sin(2x)"}]
```

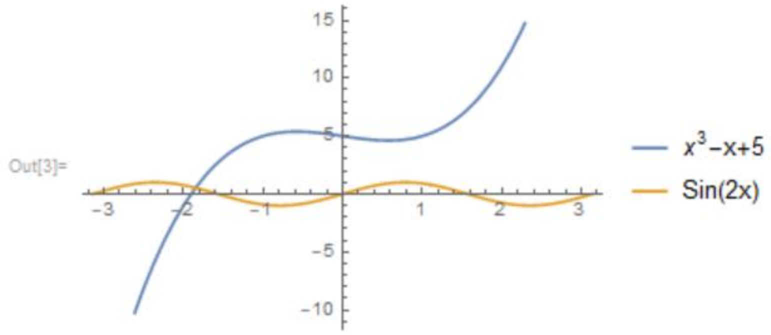


Figure 63

Answer: The root is located in $[-2, -1]$.

4.3 QUESTION 2

In a chemical reaction $CO + \frac{1}{2}O_2 \rightleftarrows CO_2$, the percent of decomposition x per 1 mole of carbon dioxide (CO_2) is found from the equation $f(x) = \left(\frac{p}{k^2} - 1\right)x^3 + 3x - 2 = 0$, where p (atm) is the gas pressure and k is a temperature-dependent equilibrium constant. Find x for $k = 1.648$ (corresponds to a temperature of 2800 Kelvin) and $p = 1\ atm$. Solve the equation by bisection method using mathematical software with 5 iterations.

Solution:

The root must be in the interval $[a, b] = [0,1]$, since the solution sought x is a percentage. The check of signs gives: $f(0)f(1) = (-2)(0.368\ldots) < 0$. If at the ends of the interval the continuous function $f(x)$ has different signs, then there is at least one real root in that interval.

We compile a bisection method algorithm for Wolfram Mathematica in two steps:

a) Localization of the root

```
In[1]:= Clear[k, x, f]
        p = 1; k = 1.648;
        f[x_] := (p/k^2 - 1) * x^3 + 3 x - 2
        f[0]
        f[1]
        Plot[f[x], {x, -3, 3}]
```

```
Out[4]= -2.

Out[5]= 0.368202
```

Out[6]=

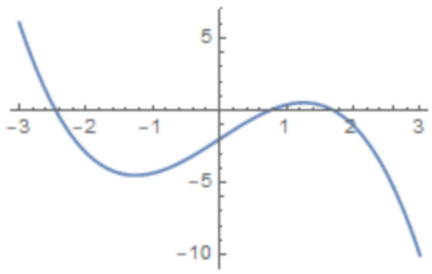


Figure 64

The plot clearly shows that the equation has three real roots, but only one is in the interval $[a, b] = [0,1]$.

b) The bisection method:

```
In[7]:= a = 0; b = 1; eps = 0.001; i = 0;
        While[Abs[b - a] > eps,
          c = (a + b)/2.;
          If[ f[a] * f[c] < 0, b = c, a = c];
          i = i + 1;
          Print[" i=", i, "   a=", a, "   b=", b, "   f[c]=", f[c]]
        ]
```

The results after 10 iterations to reduce the length of the iitial interval 10 times are:

```
i=1  a=0.5  b=1  f[c]=-0.578975
i=2  a=0.75  b=1  f[c]=-0.01654
i=3  a=0.75  b=0.875  f[c]=0.201744
i=4  a=0.75  b=0.8125  f[c]=0.0986179
i=5  a=0.75  b=0.78125  f[c]=0.042485
i=6  a=0.75  b=0.765625  f[c]=0.0133268
i=7  a=0.757813  b=0.765625  f[c]=-0.00151892
i=8  a=0.757813  b=0.761719  f[c]=0.00592597
i=9  a=0.757813  b=0.759766  f[c]=0.00220902
i=10  a=0.757813  b=0.758789  f[c]=0.00034642
```

Answer: The approximate value of the root is $x_{10} \approx \frac{a+b}{2} \approx 0.758$ or $x \approx 75.8\%$ found with absolute accuracy 0.001.

Remark to the solution. To note, this accuracy is satisfactory, since the initial data are given with three decimal places ($k = 1.648$).

5 BIBLIOGRAPHY

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[2] Kulina, H., Gocheva-Ilieva, S., Voynikova, D., Atanasova, P., Ivanov, A., Iliev, A. (2018). Integrating of Competences in Mathematics through Software - Case Study, In *Proceedings of the 2018 IEEE Global Engineering Education Conference (EDUCON)*, 17-20 April 2018, Tenerife, Spain, IEEE, pp. 1586 – 1590. https://doi.org/10.1109/EDUCON.2018.8363424

[3] Wolfram Research, Inc., Mathematica, Version 11.3, Champaign, IL (2018).

CHAPTER 2

Discrete Mathematics

Guide for a Problem

Discrete Mathematics

2.1. Mathematical logic

DM1, DM2

Fatih Yilmaz

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 21, to be achieved with the development of this activity guide are the following: As a result of learning this material you should be able to:

Mathematical Logic

- To recognise a proposition.
- To negate a proposition.
- To form a compound proposition using the connectives AND, OR, IMPLICATION.
- To construct a truth tablefor a compound proposition.
- To construct a truth tablefor an implication.
- To verify the equivalence of two propositions using a truth table.
- To identify a contradiction and a tautology.
- To construct the converse of a proposition.
- To obtain the contrapositive form of an implication.
- To understand the universal quantifier for “all”.
- To understand the existential quantifier “the reexists”.
- To negate propositions with quantifiers.
- To follow simple examples of direct and indirect proof.
- To follow a simple example of a proof by contradiction.

Sets

- To understand the notion of an ordered pair.
- To find the Cartesian product of two sets.
- To define a characteristic function of a subset of a given universe.
- To compare the algebra of switching circuits to that of set algebra and logical connectives.
- To analyse simple logic circuits comprising AND, OR, NAND, NOR and EXCLUSIVE OR gates.
- To understand the concept of a countable set.

Table 21 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

= very important

= medium important

= less important

Table 21: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Mathematical logic									
DM11	Recognise a proposition	Green	Red	Yellow	Green	Green	Green	Yellow	Yellow
DM12	Negate a proposition	Green	Yellow	Red	Yellow	Yellow	Green	Green	Red
DM13	Form a compound proposition using the connectives AND, OR, IMPLICATION	Green	Yellow	Red	Green	Green	Green	Yellow	Yellow
DM14	Construct a truth tablefor a compoundproposition	Green	Green	Green	Green	Green	Yellow	Yellow	Yellow
DM15	Construct a truth tablefor an implication	Yellow	Yellow	Green	Green	Green	Red	Red	Green
DM16	Verify the equivalence of two propositions using a truth table	Green	Yellow	Green	Green	Green	Green	Red	Red
DM17	Identify a contradiction and a tautology	Green	Green	Yellow	Yellow	Red	Green	Yellow	Yellow
DM18	Construct the converse of a proposition	Green	Yellow	Green	Red	Yellow	Yellow	Green	Yellow
DM19	Obtain the contrapositive form of an implication	Yellow	Green	Red	Green	Green	Green	Red	Red
DM20	Understand the universal quantifier 'fo rall'	Green	Yellow	Green	Red	Green	Yellow	Red	Green
DM21	Understand the existential quantifier 'there exists'	Green	Yellow	Green	Yellow	Yellow	Yellow	Green	Red
DM22	Negate propositions with quantifiers	Green	Green	Yellow	Yellow	Yellow	Green	Red	Green
DM23	Follow simple examples of direct and indirect proof	Green	Yellow	Red	Yellow	Green	Green	Yellow	Red
DM24	Follow a simple example of a prof by contradiction	Yellow	Green	Red	Yellow	Yellow	Yellow	Yellow	Yellow
DM2	**Sets**								
DM21	Understand the notion of an ordered pair	Yellow	Yellow	Red	Red	Red	Yellow	Yellow	Yellow
DM22	Find the Cartesian product of two sets	Green	Green	Yellow	Yellow	Red	Green	Green	Green
DM23	Define a characteristic function of a subset of a given universe	Yellow	Yellow	Red	Yellow	Green	Green	Red	Red
DM24	Compare the algebra of switching circuits to that of set algebra and logical connectives	Green	Green	Green	Green	Yellow	Yellow	Yellow	Yellow
DM25	Analyse simple logic circuits comprising AND, OR, NAND, NOR and EXCLUSIVE OR gates	Green	Green	Yellow	Green	Yellow	Yellow	Green	Green
DM26	Understand the concept of a countable set.	Green	Yellow	Green	Green	Green	Green	Red	Green

2 CONTENTS

1. The notion of proposition.
2. The connectives AND, OR, IMPLICATION, truth tables.
3. The definitions contradiction and a tautology.
4. Universal quantifierfor “all”.
5. The existential quantifier “there exists”.
6. The notion of an ordered pair.
7. Cartesian product of twosets.
8. The concept of a countable set.
9. The concept of a countable set.
10. Simple examples of direct and indirect proof.

3 TEST

The time to solve this exam will be 100 minutes (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 22.

Table 22: Learning outcomes (LO) involved in this assessment activity.

Mathematical Logic and Sets	
DM1	
DM11	Recognise a proposition
DM12	Negate a proposition
DM13	Form a compound proposition using the connectives AND, OR, IMPLICATION
DM14	Construct a truth tablefor a compound proposition
DM15	Construct a truth tablefor an implication
DM16	Verify the equivalence of two propositions using a truthtable
DM17	Identify a contradiction and a tautology
DM18	Construct the converse of a proposition
DM19	Obtain the contrapositive form of an implication
DM20	Understand the universal quantifier 'for all'
DM21	Understand the existential quantifier 'there exists'
DM22	Negate propositions with quantifiers
DM23	Follow simple examples of direct and indirect proof
DM24	Follow a simpleexample of a prof by contradiction
DM21	Understand the notion of an orderedpair
DM22	Find the Cartesian product of twosets
DM23	Define a characteristic function of a subset of a given universe
DM24	Compare the algebra of switching circuits to that of set algebra and logical connectives
DM25	Analyse simple logic circuits comprising AND, OR, NAND, NOR and EXCLUSIVE OR Gates
DM26	Understand the concept of a countable set.

3.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test.

1. Here's a sentence: If $x \in S$, then $f(x) = 0$. What is its negation?
   i) If $x \in S$, then $f(x) \neq 0$.
   ii) If $x \notin S$, then $f(x) \neq 0$.
   iii) If $x \notin S$, then $f(x) = 0$.
   iv) $\exists x \in S$such that $x \in S$and $f(x) \neq 0$.
2. If $A = \{a, b\}$and $B = \{1,2,3\}$, then find $A \times B$ and $B \times A$.
   i) $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$,
   $B \times A = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$
   ii) $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$,
   $B \times A = \{(1, a), (2, b), (3, a), (1, b), (2, b), (3, b)\}$
   iii) None of the above
3. For any *p* and *q* propositions, which choice is the equivance of $p \rightarrow q$?
   i) $p \vee q$
   ii) $\overline{p} \vee q$
   iii) $p \vee \overline{q}$
4. For any sets A, B and C, which choice is the equivance of $A \cap (B/C)$?
   i) $A \cap (B/C)$
   ii) $A \cap (B \cap C)$
   iii) $A \cap (B \cup C)$
5. For any sets A and B, get an equality for$\overline{A} \cap \overline{(\overline{A} \cup \overline{B})}$
   i) A
   ii) B
   iii) ∅

3.2 QUESTIONS 1

Solve the following questions.

1. Please verify that $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology via truth table.
2. Please give the following circuit via "and" "or".

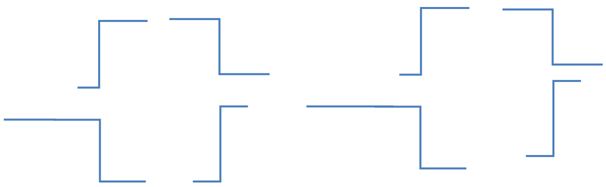


Figure 65

3. Please verify $n! \leq n^n$ for $\forall n \in \mathbb{Z}^+$.
4. Show that $\sqrt{2}$ is irrational .
5. Please find the converse of $\forall x \exists y[(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)$, where $p(x,y), q(x,y)$ and $r(x,y)$ are propositions.
6. Please verify $x^2 - 4x + 19 \neq 0$for $\forall x \in \mathbb{R}$.
7. Please prove that $\bigcup_{i \in I} A_i \times \bigcup_{j \in J} B_j = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \times B_j)$.
8. Represent the following circuit symbolically and write the input-output or switching table.

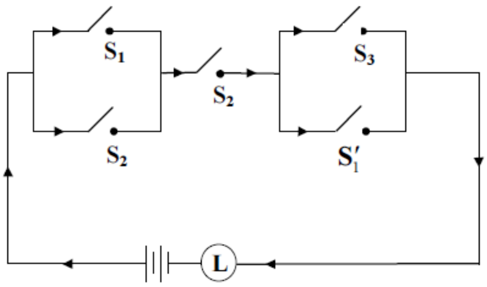


Figure 66

9. Please check whether the following switching circuits are logically equivalent.

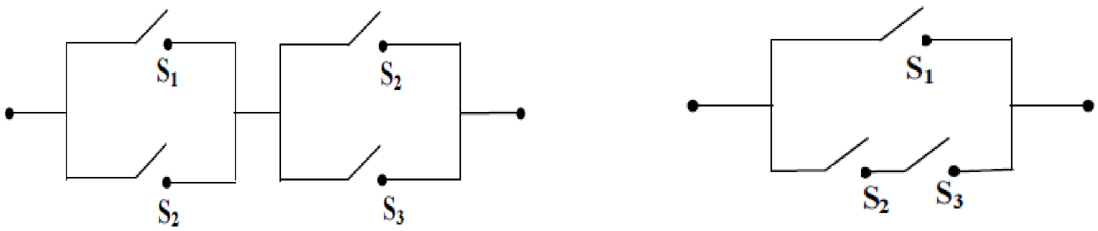


Figure 67

10. Show that × distributes over ∪. That is, for all sets A, B, and C, we have

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

3.3 QUESTIONS 2

1. A construction company considers to start a new project by exploiting some previous projects which has already been done, but wants to make some adjustments to the electrical network. Give an alternative arrangement for the following circuit, so that the new circuit has two switches only. Also write the swithching table.

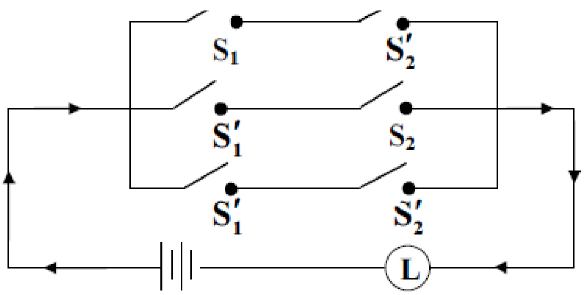
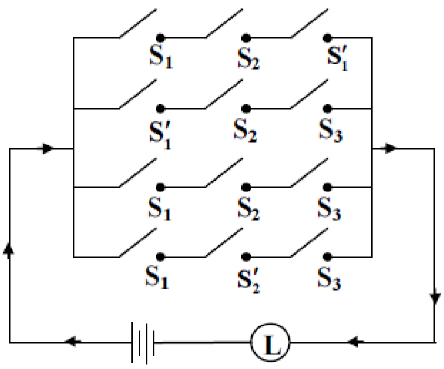


Figure 68

2. The management of a cell-phone company wants to reduce size of their cell-phones to save material and to take up less space.



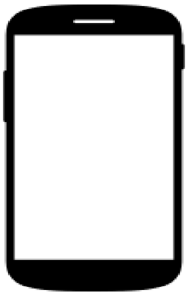


Figure 69

For this, the company wants to make an arragement at circuits, so that the new circuit has minimum switches only. Give an alternative arrangement for the circuit given above.

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test.

1. Here's a sentence: If $x \in S$, then $f(x) = 0$. What is its negation?
   i) If $x \in S$, then $f(x) \neq 0$.
   ii) If $x \notin S$, then $f(x) \neq 0$.
   iii) If $x \notin S$, then $f(x) = 0$.
   iv) $\exists x \in S$such that $x \in S$and $f(x) \neq 0$.

   Solution: iv) is correct answer.

2. If $A = \{a, b\}\text{and}B = \{1,2,3\}$, then find $A \times B$ and $B \times A$.
   i) $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$,
   $B \times A = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$
   ii) $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$,
   $B \times A = \{(1, a), (2, b), (3, a), (1, b), (2, b), (3, b)\}$
   iii) None of the above

   Solution: i) is correct answer.

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$
$$B \times A = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}$$

3. For any p and q propositions, which choice is the equivance of $p \to q$?
   i) $p \vee q$
   ii) $\overline{p} \vee q$
   iii) $p \vee \overline{q}$

   Solution: ii) is correct answer. By using tables:

p	q	$\overline{p}$	$p \to q$	$\overline{p} \vee q$
1 1 0 0	1 0 1 0	0 0 1 1	1 0 1 1	1 0 1 1

4. For any sets A, B and C, which choice is the equivance of $A \cap (B/C)$?
   i) $A \cap (B/C)$
   ii) $A \cap (B \cap C)$
   iii) $A \cap (B \cup C)$

   Solution: i) is correct answer. $A \cap (B/C) = A \cap (B \cap \overline{C}) = (A \cap B) \cap \overline{C} = (A \cap B)/C$.

For $x \in A \cap (B/C)$;

$$\begin{aligned}
&\rightarrow x \in A \wedge x \in (B/C)\\
&\rightarrow x \in A \wedge (x \in B \wedge x \notin C)\\
&\rightarrow (x \in A \wedge x \in B) \wedge (x \in A \wedge x \notin C)\\
&\rightarrow (x \in A \cap B) \wedge (x \notin A \cap C)\\
&\rightarrow x \in (A \cap B)/(A \cap C)
\end{aligned}$$

which means that $A \cap \left(\frac{B}{C}\right) \subseteq \frac{A \cap B}{A \cap C}$. To show the other hand-side:

$$\begin{aligned}
&x \in (A \cap B)/(A \cap C)\\
&\rightarrow (x \in A \wedge x \in B) \wedge x \notin (A \cap C)\\
&\rightarrow (x \in A \wedge x \in B) \wedge (x \notin A \vee x \notin C)\\
&\rightarrow [(x \in A \wedge x \in B) \wedge x \notin A)] \vee [(x \in A \wedge x \in B) \wedge x \notin C]\\
&\rightarrow (x \in A \wedge x \in B) \wedge x \notin C\\
&\rightarrow x \in (A \cap B)/C
\end{aligned}$$

which means that $(A \cap B)/(A \cap C) \subseteq A \cap (B/C)$.

5. For any sets A and B, get an equality for $\overline{A} \cap \overline{(\overline{A} \cup \overline{B})}$
   i) A
   ii) B
   iii) $\emptyset$

   Solution: iii) is correct answer. By exloiting De Morgan rules and $\overline{\overline{A}} = A$:

$$\begin{aligned}
\overline{A} \cap \overline{(\overline{A} \cup \overline{B})} &= \overline{A} \cap (A \cap B)\\
&= (\overline{A} \cap A) \cap B\\
&= \emptyset \cap B\\
&= \emptyset
\end{aligned}$$

4.2 QUESTIONS 1

1. Please verify that $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology via truth table.

   Solution:

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$
1	1	1	1	1	1	1	1	1
1	1	0	0	1	0	0	0	1
1	0	1	1	0	1	1	1	1
1	0	0	1	0	0	0	1	1
0	1	1	1	1	1	1	1	1

0	1	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1

2. Please give the following circuit via "and" "or".

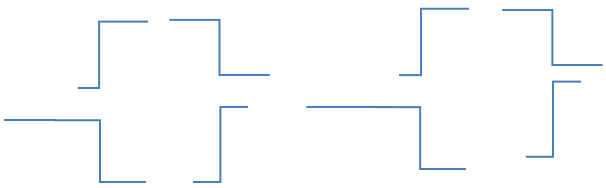


Figure 70

Solution: $[(p \vee q) \wedge r] \wedge (p \vee q)$.

3. Show that $n! \leq n^n$ for $\forall n \in \mathbb{Z}^+$

Solution:

By Principle of Mathematica Induction (PMI), it verifies that $1! \leq 1^1$. Assume that it is true for $k! \leq k^k$ and let's verify that it holds for k+1. In other words, we have to show that $(k+1)! \leq (k+1)^{(k+1)}$. By the following operations, we get:

$$\begin{aligned}(k+1)! &= k!(k+1)\\ &\leq k^k(k+1)\\ &< (k+1)^k(k+1)\\ &< (k+1)^{k+1}\end{aligned}$$

which is desired.

4. Show that $\sqrt{2}$ is irrational .

Solution:

Assume that the hypotesis is wrong. In other words, let $\sqrt{2}$ be rational number. So it can be written in the form$\sqrt{2} = \frac{m}{n}$, where *m* and *n* are integers. By some operation,$2n^2 = m^2$. This means that *m* is an even number and *n* is an odd number. In this case, it can be rewritten that $m = 2k$ and $n = 2l + 1$, where *k* and *l* are integers. So,

$$4k^2 = 2(4l^2 + 4l + 1) = 8l^2 + 8l + 2$$

$$2k^2 = 4l^2 + 4l + 1$$

which is a contradiction, since the left hand side of the equation is an even number but the right hand side is an odd number. This is a contradiction.

5. Please find the converse of $\forall x \exists y[(p(x,y) \wedge q(x,y)) \to r(x,y)]$ where $p(x,y), q(x,y)$ and $r(x,y)$ propositions.

   Solution:

$$\begin{aligned} \neg[\forall x \exists y[(p(x,y) \wedge q(x,y)) \to r(x,y)]] &\Leftrightarrow \exists x \neg[\exists y[(p(x,y) \wedge q(x,y)) \to r(x,y)]] \\ &\Leftrightarrow \exists x \forall y \neg[(p(x,y) \wedge q(x,y)) \to r(x,y)] \\ &\Leftrightarrow \exists x \forall y \neg[\neg[(p(x,y) \wedge q(x,y))] \vee r(x,y)] \\ &\Leftrightarrow \exists x \forall y[\neg\neg[(p(x,y) \wedge q(x,y))] \wedge \neg r(x,y)] \\ &\Leftrightarrow \exists x \forall y[[(p(x,y) \wedge q(x,y))] \wedge \neg r(x,y)] \end{aligned}$$

6. Please verify $x^2 - 4x + 19 \neq 0$ for $\forall x \in \mathbb{R}$.

   Solution:

   $x^2 - 4x + 19 = (x-2)^2 + 15$, so $x^2 - 4x + 19$ is non-negative. For $\forall x \in \mathbb{R}$, $x^2 - 4x + 19 \geq 15$.

7. Please prove that $\bigcup_{i \in I} A_i \times \bigcup_{j \in J} B_j = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \times B_j)$.

   Solution:

$$\begin{aligned} (x,y) \in \bigcup_{i \in I} A_i \times \bigcup_{j \in J} B_j &\Leftrightarrow x \in \bigcup_{i \in I} A_i \text{ and } y \in \bigcup_{j \in J} B_j \\ &\Leftrightarrow \exists i \in I, x \in A_i \text{ and } \exists j \in J, y \in B_j \\ &\Leftrightarrow \exists i \in I \text{ and } \exists j \in J, (x,y) \in A_i \times B_j \\ &\Leftrightarrow \exists i \in I; (x,y) \in \bigcup_{j \in J} A_i \times B_j \\ &\Leftrightarrow (x,y) \in \bigcup_{i \in I} \bigcup_{j \in J} (A_i \times B_j). \end{aligned}$$

8. Represent the following circuit symbolically and write the input-output or switching table.

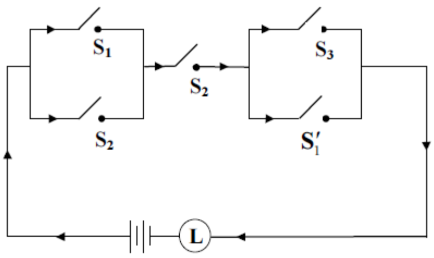


Figure 71

   Solution:

   Let p: The switch S_1 is closed

   q: The switch S2 is closed

   r: The switch S3 is closed

   $\sim p$: The switch S1 is closed

   The symbolic form of the given circuit is

$$(p \vee q) \wedge q \wedge (q \vee \sim p).$$

Input-Output Table

p	q	r	$\sim p$	$p \vee q$	$(p \vee q) \wedge q$	$(r \vee \sim p)$	$(p \vee q) \wedge q \wedge (r \vee \sim p)$
1	1	1	0	1	1	1	1
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
1	0	0	0	1	0	0	0
0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	0	1	1	0	0	1	0
0	0	0	1	0	0	1	0

Switching Table

p	q	r	$\sim p$	$p \vee q$	$(p \vee q) \wedge q$	$(r \vee \sim p)$	$(p \vee q) \wedge q \wedge (r \vee \sim p)$
T	T	T	F	T	T	T	T
T	T	F	F	T	T	F	F
T	F	T	F	T	F	T	F
T	F	F	F	T	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	F	F	T	F
F	F	F	T	F	F	T	F

9. Please check whether the following switching circuits are logically equivalent.

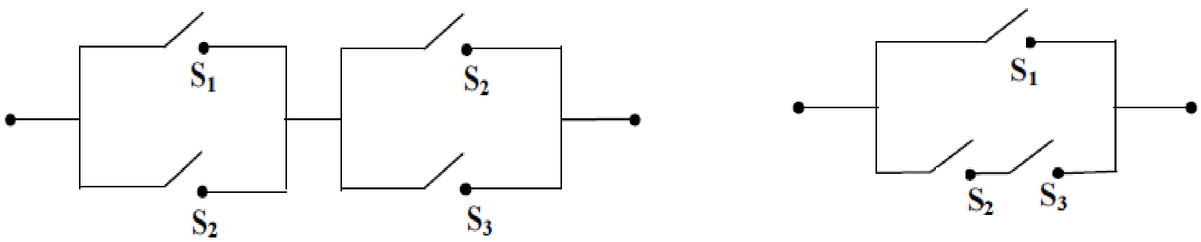


Figure 72

Solution:

Let p: The switch S1 is closed

q: The switch S2 is closed

r: The switch S3 is closed

i) The symbolic form of the given circuit is $(p \vee q) \wedge (q \vee r)$

ii) The symbolic form of the given circuit is $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

From (i) and (ii), the given circuits are not equivalent.

10. Show that × distributes over, for all sets A, B, and C.

Solution:

Let's initially show that the expression$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ holds. Let $(x, y) \in A \times (B \cup C)$, i.e. $x \in A, y \in B \cup$ C. This means$y \in B$or$y \in$ C. Here we will do operations on$y \in$ B. So, $(x, y) \in A \times B \Rightarrow (x, y) \in (A \times B) \cup (A \times C)$. This means $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

On the other hand, let's show $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Let $(x, y) \in (A \times B) \cup (A \times C)$, i.e. $(x, y) \in (A \times B)$or$(x, y) \in (A \times C)$. Here we will do operations on $(x, y) \in (A \times B)$. So $x \in A$and$y \in B \cup C \Rightarrow$ (x,y) $\in A \times (B \cup C)$. This means $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. So the proof is completed.

4.3 QUESTIONS 2

1. A construction company considers to start a new project by exploiting some previous projects which has already been done, but wants to make some adjustments to the electrical network. Give an alternative arrangement for the following circuit, so that the new circuit has two switches only. Also write the swithching table.

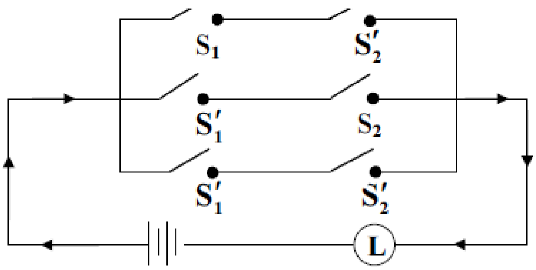


Figure 73

Solution:

Let p: The switch S_1 is closed

q: The switch S_2 is closed

$\sim p$: The switch S_1 is open

$\sim q$: The switch S_2 is open.

The symbolic form of the given circuit is

$$
\begin{aligned}
&(p\wedge\sim q)\vee(\sim p\wedge q)\vee(\sim p\wedge\sim q)\\
&\qquad\equiv(p\wedge\sim q)[\sim p\wedge(q\vee\sim q)]\\
&\qquad\equiv(p\wedge\sim q)\vee(\sim p\wedge T)\\
&\qquad\equiv(p\wedge\sim q)\vee\sim p\\
&\qquad\equiv(p\vee\sim p)\wedge(\sim q\vee\sim p)\\
&\qquad\equiv(p\vee\sim p)\wedge(\sim p\wedge\sim q)\\
&\qquad\equiv T\wedge(\sim p\vee\sim q)\\
&\qquad\equiv\sim p\vee\sim q
\end{aligned}
$$

The alternative arrangement for the given circuit is:

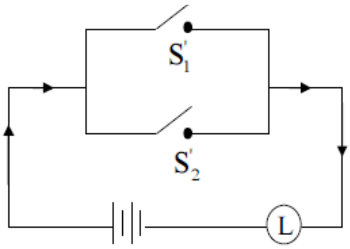


Figure 74

Switching Table

p	q	$\sim p$	$\sim q$	$p\wedge\sim q$	$\sim p\wedge q$	$\sim p\wedge\sim q$	$(p\wedge\sim q)\vee(\sim p\wedge q)\vee(\sim p\wedge\sim q)$
T	T	F	F	F	F	F	F
T	F	F	T	T	F	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	T	T

2. The management of a cell-phone company wants to reduce size of their cell-phones to save material and to take up less space.

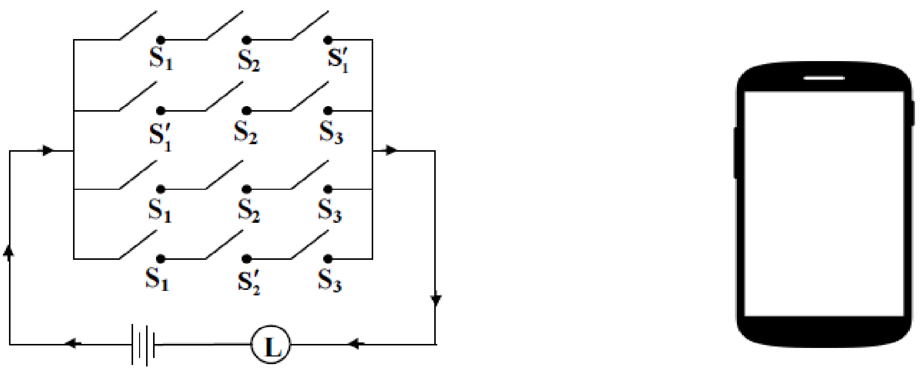


Figure 75

For this, the company wants to make an arragement at circuits, so that the new circuit has minimum switches only. Give an alternative arrangement for the circuit given above.

Solution

Let, p: The switch S_1 is closed

q: The switch S_2 is closed

r: The switch S_3 is closed

$\sim p$: The switch S_1 is open

$\sim q$: The switch S_2 is open.

The symbolic form of the given circuit is:

$$
\begin{aligned}
&(p \wedge q \wedge \sim p) \vee (\sim p \wedge q \wedge r) \vee (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \\
&\equiv (p \wedge \sim p \wedge q) \vee (\sim p \wedge q \wedge r) \vee (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \\
&\equiv F \vee (\sim p \wedge q \wedge r) \vee (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \\
&\equiv (\sim p \wedge q \wedge r) \vee (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \\
&\equiv [(p \vee \sim p) \wedge (q \vee r)] \vee (p \wedge \sim q \wedge r) \\
&\equiv [T \wedge (q \wedge r)] \vee (p \wedge \sim q \wedge r) \\
&\equiv (q \wedge r) \vee (p \wedge \sim q \wedge r) \\
&\equiv [(q \vee (p \wedge \sim q)] \wedge r \\
&\equiv [(q \vee p) \wedge (q \vee \sim q)] \wedge r \\
&\equiv [(q \vee p) \wedge T] \wedge r \\
&\equiv (q \vee p) \wedge r \\
&\equiv (p \vee q) \wedge r
\end{aligned}
$$

The switching circuit corresponding to the given statement is:

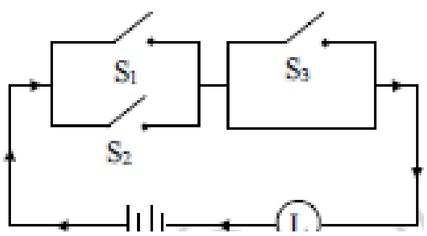


Figure 76

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Guide for a Problem

Discrete Mathematics

2.2. Mathematical induction and recursion

DM3

Marie Demlová and Petr Habala

1 OBJECTIVES

The aim of this activity guide is to achieve certain learning outcomes. As a result of learning this material you should be able to:

- DM31. Recognize whether a given proof is correct. If not, explain where the problem is.
- DM32. Prove a statement by weak induction.
- DM33. Prove an inequality by weak induction.
- DM34. Prove a statement by strong induction.
- DM35. Define a given function by induction (recursive definition).
- DM36. Define a given set using the structural induction.
- DM37. Prove some property using structural induction.
- DM38. Generalize a given property from two to more objects and prove validity.

Table 23 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

Green = very important

Yellow = medium important

Red = less important

Table 23: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Discrete Mathematics									
DM3	3. Mathematical induction and recursion	Green	Green	Red	Red	Yellow	Yellow	Green	Red
DM31	Recognize correct proofs by induction	Green	Green	Yellow	Red	Yellow	Green	Green	Red
DM32	Proof by weak induction	Green	Green	Yellow	Red	Yellow	Green	Green	Red
DM33	Proof of an inequality by induction	Green	Green	Yellow	Red	Yellow	Green	Green	Red
DM34	Proof by strong induction	Green	Green	Yellow	Red	Yellow	Green	Green	Red
DM35	Create a recurrent definition	Green	Green	Yellow	Red	Yellow	Green	Green	Red
DM36	Define a set by structural induction	Green	Yellow	Yellow	Green	Green	Green	Green	Red
DM37	Proof by structural induction	Green	Yellow	Yellow	Green	Green	Green	Green	Red
DM38	Inductive generalization of property	Green	Green	Yellow	Green	Yellow	Green	Green	Red

2 CONTENTS

1. Mathematical induction: weak, strong, and structural.
2. Recursive definition of (mathematical) objects.
3. Inductive definition of sets.

3 EVALUATION

The time to solve this exam will be 180 minutes approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 24.

Table 24: Learning outcomes (LO) involved in this assessment activity.

Discrete Mathematics	
DM3	1. Mathematical induction and recursion
DM31	To recognize whether a given proof is correct. If not, explain where the problem is.
DM32	To prove a statement by weak induction.
DM33	To prove an inequality by weak induction.
DM34	To prove a statement by strong induction.
DM35	To define a given function by induction (recursive definition).
DM36	To define a given set using the structural induction.
DM37	To prove some property using structural induction.
DM38	To generalize a given property from two to more objects and prove validity.

3.1 Multiple-choice Questions

1. Consider the following statement: $n^2 + 1 \geq 1$ for $n \in \boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$.

   Decide which of the following arguments (if any) are correct proofs by induction.

   i) Assume that $n^2 + 1 \geq 1$ for a certain $n \geq 1$. Then

   $$(n + 1)^2 + 1 = n^2 + 2n + 1 + 1 = (n^2 + 1) + 2n + 1 \geq (n^2 + 1) \geq 1.$$

   ii) Assume that $n^2 + 1 \geq 1$ for all $n \geq 0$. Then

   $$(\mathrm{n} \ + 1)^2 + 1 = n^2 + 2n + 1 + 1 = (n^2 + 1) + 2n + 1 \geq (n^2 + 1) \geq 1.$$

   iii) Assume that $n^2 + 1 \geq 1$ for a certain $n \geq 0$. Then

   $$(\mathrm{o} \ + 1)^2 + 1 = n^2 + 2n + 1 + 1 = (n^2 + 1) + 2n + 1 \geq (n^2 + 1) \geq 1.$$

   iv) Assume that $n^2 +1 \geq 1$ for a certain $n \geq 1$. Then using the fact that $(n+1)^2 \geq 0$ for any number $n + 1$ we obtain $(n + 1)^2 + 1 \geq 0 + 1 = 1$.

3.2 QUESTIONS

1. Prove by induction that $2 + 4 + \cdots + (2n) = n(n + 1)$ for all $n \in \boldsymbol{N}$.
2. Prove by induction that $1^2 + 2^2 + \cdots + n^2 \leq n^3$ for all $n \in \boldsymbol{N}$.
3. Consider a function f on $\boldsymbol{N}$ defined recursively by

   $(0)\ f(1) = 1, f(2) = 2;$

   $(1)\ f(n + 1) = n \cdot f(n) + n^2 \cdot f(n - 1), n \geq 2.$

   Prove that $f(n) \geq n!$ for $n \in \boldsymbol{N}$.

4. Consider the function f(n) = n2 defined for n ∈ $\boldsymbol{N}$. Create a recursive definition for this function.
5. Consider the set M of all finite binary strings without successive 1s. Write an inductive definition of this set.
6. Let A be a set of binary words defined by mathematical induction as follows.

   $(0)\ 00 \in M$ and $11 \in M$;

   $(1)\ w \in M \Longrightarrow 0w0 \in M$ and $1w1 \in M.$

   Using structural induction, prove that every binary word which belongs to M has an even length.

7. A well-*known* triangle inequality states that $|x + y| \leq |x| + |y|$ for any two real numbers. Generalize this inequality to arbitrary finite number of real numbers and prove its validity.

3.3 PROJECT

Consider a currency called math credit (MC). It is a hard currency very valued by university students. We start easy by proving an obvious fact.

1. Assume that we have an infinite supply of coins with value 1 math credit. Prove by induction that the following statement is true for any $n \in \boldsymbol{N}$:

   $V(n)$: We can pay out the amount n math credits using the MC1 coins.

Now let's make it more interesting.

2. Assume that we have an infinite supply of coins with value MC2 and coins with value MC5. Prove by induction that the following statement is true for any $n \in \boldsymbol{N}$, $n \geq 4$:

   $V(n)$: We can pay out the amount n math credits using the MC2 and MC5 coins.

Note: By paying out we mean that we just hand over some coins, we do not get anything back.

Theory says that the strong induction principle is equivalent to weak induction principle. In practical terms, the last problem should be solvable also using the weak induction, that is, by going $n \rightarrow (n + 1)$. The trick is in finding the right formulation of the induction statement $W(n)$.

3. Find a statement $W(n)$ that could be proved using weak induction and that, when proved, would imply that we can pay out all amounts starting from 4 using MC2 and MC5.

4. Prove validity of $W(n)$.

Remark: If we allowed for coins to travel both ways, that is, we give some away and the other person can also give some to us, then we can pay any amount starting from $n = 1$. This can be easily proved by weak induction. The main step goes as follows: If we can pay out n, then we also add three MC2 coins, the person hands us back one MC5 coin and we just paid $n + 6 - 5 = n + 1$ as required.

4 SOLUTIONS

4.1 MULTIPLE-CHOICE QUESTIONS

1. Consider the following statement: $n^2 + 1 \geq 1$ for $n \in \boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$.

   Decide which of the following arguments (if any) are correct proofs by induction.

   i) Assume that $n^2 + 1 \geq 1$ for a certain $n \geq 1$. Then

   $$(n + 1)^2 + 1 = n^2 + 2n + 1 + 1 = (n^2 + 1) + 2n + 1 \geq (n^2 + 1) \geq 1.$$

   ii) Assume that $n^2 + 1 \geq 1$ for all $n \geq 0$. Then

   $$(p \ \ + 1)^2 + 1 = n^2 + 2n + 1 + 1 = (n^2 + 1) + 2n + 1 \geq (n^2 + 1) \geq 1.$$

   iii) Assume that $n^2 + 1 \geq 1$ for a certain $n \geq 0$. Then

   $$(q \ \ + 1)^2 + 1 = n^2 + 2n + 1 + 1 = (n^2 + 1) + 2n + 1 \geq (n^2 + 1) \geq 1.$$

   iv) Assume that $n^2 +1 \geq 1$ for a certain $n \geq 1$. Then using the fact that $(n+1)^2 \geq 0$ for any number $n + 1$ we obtain $(n + 1)^2 + 1 \geq 0 + 1 = 1$.

   Solution:

   The solution is iii).

   LO: DM31.

   Remark: Why i) is incorrect: mismatched ranges for the index n. Knowing that the inequality is true for n = 0, we cannot use the induction step (1) to pass to n + 1 = 1, because the induction step was only proved for n ≥ 1.

   Why ii) is incorrect: wrong structure of induction step. We assume that the inequality is true for all n ≥ 1, but in induction we only do our assumption for one specific n (arbitrarily chosen from some range). If we did assume it for all n, then we would not be able to use such an assumption, as the base step only lets us use n = 1. In fact, proving that inequality for all n is exactly our question, so it does not make sense to assume it.

   Why iv) is incorrect: The alleged induction step is no induction at all, as we do not use the assumption in the actual argument. In fact, the argument in (1) that starts with "Then using…" constitutes a valid direct proof of the statement, however, it is not a proof by induction as demanded.

4.2 QUESTIONS

1. Prove by induction that $2 + 4 + \cdots + (2n) = n(n + 1)$ for all $n \in \boldsymbol{N}$.

   Solution:

   (0) $n = 1$: $2 = 1 \cdot (1 + 1)$, checks.

   (1) For a given $n \geq 1$ assume that $2 + 4 + \cdots + 2n = n(n + 1)$. By this assumption then

   $$2 + 4 + \cdots + (2n) + 2(n + 1) = [2 + 4 + \cdots + (2n)] + 2(n + 1)$$
   $$= n(n + 1) + 2(n + 1) = (n + 1)(n + 2) = (n + 1)([n + 1] + 1).$$

LO: DM32.

2. Prove by induction that $1^2 + 2^2 + \cdots + n^2 \le n^3$ for all $n \in \boldsymbol{N}$.

   Solution:

   (0) $n = 1$: $1^2 \le 1^3$ is true.

   (1) For a given $n \ge 1$ assume that $1^2 + 2^2 + \cdots + n^2 \le n^3$. By this assumption then

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = [1^2 + 2^2 + \cdots + n^2] + (n+1)^2 \le n^3 + (n+1)^2 = n^3 + n^2 + 2n + 1 \le n^3 + 3n^2 + 3n + 1 = (n+1)^3.$$

LO: DM33.

3. Consider a function f on $\boldsymbol{N}$ defined recursively by

   (0) $f(1) = 1, f(2) = 2;$

   (1) $f(n+1) = n \cdot f(n) + n^2 \cdot f(n-1),\ n \ge 2.$

   Prove that $f(n) \ge n!$ for $n \in \boldsymbol{N}$.

   Solution:

   (0) $n = 1$: $f(1) = 1 \ge 1!$; $n = 2$: $f(2) = 2 \ge 2!$. Checked.

   (1) We take some $n \ge 2$ and assume that $f(n) \ge n!$ and $f(n-1) \ge (n-1)!$. Using the recursive definition of $f(n+1)$ and our assumption we estimate

$$f(n+1) = n \cdot f(n) + n^2 \cdot f(n-1) \ge n \cdot n! + n^2(n-1)! = 2n \cdot n! \ge (n+1) \cdot n! = (n+1)!.$$

LO: DM34.

4. Consider the function $f(n) = n^2$ defined for $n \in \boldsymbol{N}$. Create a recursive definition for this function.

   Solution

   We see that $f(n+1) = n^2 + 2n + 1$. Thus we can define f as follows:

   (0) $f(1) = 1;$

   (1) $f(n+1) = f(n) + 2n + 1,\ n \ge 1.$

   Other definitions are possible, for instance (1) could read $f(n+1) = \left(\sqrt{f(n)} + 1\right)^2$

   LO: DM35.

5. Consider the set M of all finite binary strings without successive 1s. Write an inductive definition of this set.

   Solution:

   (0) $0 \in M$ and $1 \in M$ and $01 \in M$;

   (1) $w \in M \Longrightarrow w0 \in M$ and $w01 \in M.$

Alternative:

(0) $\varepsilon \in M$ and $1 \in M$;

(1) $w \in M \Longrightarrow w0 \in M$ and $w01 \in M$.

LO: DM36.

6. Let A be a set of binary words defined by mathematical induction as follows.

   (0) $00 \in M$ and $11 \in M$;

   (1) $w \in M \Longrightarrow 0w0 \in M$ and $1w1 \in M$.

   Using structural induction, prove that every binary word which belongs to M has an even length.

   Solution:

   (0) The two words from the basic step have both length 2, hence even.

   (1) Take some w from M and assume that it has even length $2k$. Then the words $0w0$ and $1w1$ have length $2k + 2$, which is also an even number.

   LO: DM37.

7. A well-known triangle inequality states that $|x + y| \leq |x| + |y|$ for any two real numbers. Generalize this inequality to arbitrary finite number of real numbers and prove its validity.

   Solution:

   Statement: $|x_1+x_2+\cdots+x_n| \leq |x_1|+|x_2|+\cdots+|x_n|$ for any $n \in \boldsymbol{N}$ and all $x_1,\ldots,x_n \in \boldsymbol{R}$.

   Proof:

   (0) Case $n = 1$: $|x_1| \leq |x_1|$ is definitely true for any $x_1 \in \boldsymbol{R}$.

   (1) Let $n \geq 1$ be given.

   Assume that $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$ for all $x_1,\ldots,x_n \in \boldsymbol{R}$.

   Now take any $n + 1$ real numbers $x_1,\ldots,x_n,x_{n+1}$. By the triangle inequality we may estimate

   $|x_1 + \cdots + x_n + x_{n+1}| = |(x_1 + \cdots + x_n) + x_{n+1}| \leq |x_1 + \cdots + x_n| + |x_{n+1}|$. Now we apply the induction hypothesis:

   $$|x_1 + \cdots + x_n + x_{n+1}| \leq |x_1 + \cdots + x_n| + |x_{n+1}| \leq |x_1| + \cdots + |x_n| + |x_{n+1}|,$$

   proving the statement.

   LO: DM38.

4.3 PROJECT

1. Assume that we have an infinite supply of coins with value 1 MC. Prove by induction that the following statement is true for any $n \in \boldsymbol{N}$:

   $V(n)$: We can pay out the amount n math credits using the MC1 coins.

Solution:

(0) Let $n = 1$. $V(1)$ reads "we can pay out the amount 1 MC using MC1 coins", which is definitely true, we just give one coin.

(1) Take arbitrary $n \geq 1$ and assume that we can pay out the amount n math credits using those coins. If we add one more, we just paid out the amount $n + 1$.

This proves validity of the implication $V(n) \Longrightarrow V(n + 1)$, completing the inductive proof.

2. Assume that we have an infinite supply of coins with value MC2 and coins with value MC5. Prove by induction that the following statement is true for any $n \in \boldsymbol{N}$, $n \geq 4$:

$V(n)$: We can pay out the amount n math credits using the MC2 and MC5 coins.

Solution:

Analysis: Weak induction jumps by one, $n \to (n + 1)$, but this is not possible in our example, because the smallest coin available has value larger than 1.

We thus have to use the strong induction that allows us to jump more freely. In this particular case it seems like a good idea to jump by 2. This tells us how we will shape the induction step (1), we will go $(n - 1) \to (n + 1)$.

Now we need to set up the basic step. We obviously have to check on $n = 4$, but when we combine this base value and our induction step, we see that we would only prove our statement for even amounts. Thus we also have to check on an odd starting amount, namely $n = 5$, to cover all possibilities.

Proof:

(0) Let $n = 4$. When we give two MC2 coins, we paid out 4 math credits. This case is validated.

Let $n = 5$. When we give one MC5 coin, we paid out 5 math credits. This case is also validated.

(1) Take arbitrary $n \geq 5$ and assume that we can pay out all amounts between 4 and n. We need to show that we can pay out $n + 1$ math credits.

Since $n \geq 5$, then $n - 1 \geq 4$, also $n - 1 \leq n$, so the amount $n - 1$ is included in our induction hypothesis. Thus we assume that we can pay out $n-1$ using our coins. If we add one MC2 coin, we just paid out $n + 1$ math credits as requested.

This completes the proof by induction.

Note: If we decided to use the MC5 coin for our induction (that is, to jump by 5), we would have to add more amounts into the base step (0).

3. Find a statement $W(n)$ that could be proved using weak induction and that, when proved, would imply that we can pay out all amounts starting from 4 using MC2 and MC5.

Solution:

The natural candidate is as follows:

W(n): We can pay out any amount between 4 and n using coins with values MC2 and MC5.

In fact, this trick is the basis of the theoretical proof that the weak and strong principles are equivalent. There are also other possibilities.

Note that if W(n) is true for any n ≥ 4, then we can pay out any amount of 4 or more. However, weak induction would not be possible for this, because the step 4 → 5 cannot be handled without MC1 coins.

Thus we have to try better: Note that if W(n) is true for any n ≥ 5, then we can pay out any amount of 4 or more.

4. Prove validity of $W(n)$.

Solution:

As outlined above, we will prove it for $n \geq 5$.

(0) $n = 5$: We need to prove $W(5)$, that we can pay out 4 and 5 math credits. This is easy, just give two MC2 coins, and one MC5 coin, respectively.

(1) Take some $n \geq 5$ and assume that $W(n)$ is true. This means that we can pay out any amount between 4 and n.

We need to prove W(n + 1), that is, we have to show that we can pay out any amount between 4 and n + 1. For the amount between 4 and n we already know it by our induction hypothesis. It remains to address n + 1.

Since n ≥ 5, it follows that $n - 1 \geq 4$, so we can definitely pay out $n - 1$, we add one MC2 coin and we just paid out $n + 1$ math credits.

The proof is complete.

LO: DM31, DM32, DM34.

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Guide for a Problem

Discrete Mathematics

2.3. Graphs

DM4

Marie Demlova and Petr Habala

1 OBJECTIVES

The aim of this activity guide is to achieve certain learning outcomes. As a result of learning this material you should be able to:

- DM41. Understand the notion of a path and a cycle/circuit.
- DM42. Understand the notion of a tree and a binary tree.
- DM43. Understand the notion of an Euler trail and Euler graph. Recognize an Euler trail in a graph.
- DM44. Understand the notion of a Hamiltonian path, Hamiltonian circuit/cycle and Hamiltonian graph. Recognize a Hamiltonian path/cycle in a graph.
- DM45. Understand the notion of connectivity; find components of connectivity in a directed/undirected graph.
- DM46. Understand the notion of strong connectivity; find strongly connected components in a directed graph.
- DM47. Find a minimal spanning tree of a given connected weighted graph.

Table 25 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:Key: The background color shows how well a particular competence is addressed by a certain LO:

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 25: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Discrete Mathematics									
DM4	4. Graphs								
DM41	Understand paths, circuits, cycles	green	green	yellow	yellow	red	red	yellow	red
DM42	Understand trees	green	green	green	yellow	red	red	yellow	red
DM43	Understand Euler trails and graphs	green	green	yellow	yellow	red	yellow	yellow	green
DM44	Understand Hamiltonian paths	green	green	yellow	yellow	red	yellow	yellow	green
DM45	Understand connectivity and components	green	green	yellow	yellow	red	yellow	yellow	green
DM46	Understand strong connectivity	green	green	yellow	yellow	red	yellow	yellow	green
DM47	Find minimal spanning tree	green	yellow	yellow	yellow	yellow	green	green	green

The topics 3–7 can also incorporate algorithms in Maple or another CAP. This addresses C8.

2 CONTENTS

1. Graphs, paths, cycles.
2. Special paths.
3. Connected graphs.
4. Trees.

3 EVALUATION

The time to solve this exam will be 120 minutes (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 26.

Table 26: Learning outcomes (LO) involved in this assessment activity.

Discrete Mathematics	
DM4	1. Graphs
DM41	Understand the notion of a path and a cycle/circuit.
DM42	Understand the notion of a tree and a binary tree.
DM43	Understand the notion of an Euler trail and Euler graph. Recognize an Euler trail in a graph.
DM44	Understand the notion of a Hamiltonian path, Hamiltonian circuit/cycle and Hamiltonian graph. Recognize a Hamiltonian path/cycle in a graph.
DM45	Understand the notion of connectivity; find components of connectivity in a directed/undirected graph.
DM46	Understand the notion of strong connectivity; find strongly connected components in a directed graph.
DM47	Find a minimal spanning tree of a given connected weighted graph.

3.1 MULTIPLE-CHOICE QUESTIONS

1. Given a simple undirected graph G without loops and with n vertices and m edges. Which of the following assertions is true?
   a) If $m = n$ then G contains a circuit.
   b) If $m < n$ then G does not contain a circuit.
   c) If $m = n - 1$ then G is a tree.
   d) If $m > n$ then G is connected.

2. Given a simple undirected graph G without loops and with n vertices and m edges. Which of the following assertions is true?
   a) G cannot contain a unique vertex with degree 1.
   b) If n is an odd number then m is also an odd number.
   c) If n is an even number then m is also an even number.
   d) If n is odd then there is an odd number of vertices with even degree.

3. Given the following simple directed graph G with 9 vertices and 14 edges:

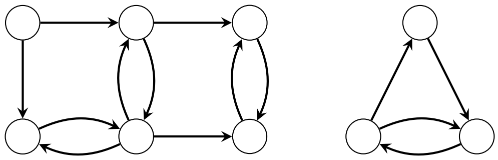

Figure 77

Which of the assertions is correct?

a) G is connected and has 4 components of strong connectivity.

b) G is not connected and has 3 components of strong connectivity.

c) G is connected and has 2 components of strong connectivity.

d) G is not connected and has 4 components of strong connectivity.

4. Given a directed graph G. Consider the following statement.

If there is an undirected path from a vertex u to a vertex v then there is an undirected path from v to u.

a) The statement is true.

b) The statement is false.

3.2 Questions

1. Given a connected simple graph G without loops having 10 vertices and just one circuit. How many edges does G have? Justify your answer.

2. How many non-isomorphic trees exist on a set of 5 vertices? Justify your answer.

3. Is there a rooted binary tree with 6 vertices and the following list of out degrees

$$0, 0, 0, 1, 2, 2?$$

If yes draw an example of such a binary tree; if not show that such a binary tree does not exist.

4. Decide whether the graph G with the set of vertices $V = \{1,\ldots,8\}$ and edges given by the following list contains a closed Euler trail.

IV	1	1	1	2	2	3	3	3	4	5	6	6	7	8
TV	2	3	5	1	3	1	4	6	1	2	3	7	8	6

(Either find an Euler trail or explain why it cannot exist. Explain what is an Euler trail.)

5. Is it true that every strongly connected graph has to be Hamiltonian (i.e. such a graph has to contain a Hamiltonian cycle)? If yes, justify; if not, give a counterexample.

3.3 PROJECT

In many practical situations we need to connect given points/devices as economically as possible. Let us make it more precise.

We are given n different points for which we know which pairs of them can be directly connected and what is the price of such connections. We want to find the set of connections such that any point i is reachable from any point j and the sum of prices of the connections we have chosen is the least possible.

Such a problem can be formulated in the language of graph theory: Given an undirected connected graph $G = (V,E)$ together with a mapping $c: E \to \mathbf{Z}$, find a minimal spanning tree.

1. Consider an undirected weighted graph $G = (V,E)$, where $V = \{v_1,\dots,v_6\}$ and edges E and their weights are given by the following table (an edge $\{v_i,v_j\}$ is denoted by v_iv_j).

e	v_1v_2	v_1v_3	v_1v_4	v_1v_5	v_1v_7	v_2v_3	v_2v_6	v_2v_7	v_3v_4	v_3v_5	v_3v_7
$w(e)$	13	7	14	12	13	9	9	4	3	9	10

e	v_4v_7	v_4v_8	v_5v_6	v_6v_7	v_6v_8	v_7v_8	v_4v_5	v_4v_6
$w(e)$	2	3	3	11	11	2	3	1

Find a minimal spanning tree of G. Describe the method that you used.

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. Given a simple undirected graph G without loops and with n vertices and m edges. Which of the following assertions is true?

   a) If $m = n$ then G contains a circuit.

   b) If $m < n$ then G does not contain a circuit.

   c) If $m = n - 1$ then G is a tree.

   d) If $m > n$ then G is connected.

   Solution:

   a) Correct. A graph without circuits with the biggest number of edges is a tree, and a tree has $n - 1$ edges. Adding a new edge to a tree closes a circuit.

   b) False. One can consider a graph with 4 vertices and 3 edges with the edges forming a triangle.

   c) False. See the example in b).

   d) False. Consider a graph with 5 vertices and 6 edges such that the edges create a complete graph on some 4 vertices and the fifth vertex is isolated.

   LO: DM41, DM42, DM45.

2. Given a simple undirected graph G without loops and with n vertices and m edges. Which of the following assertions is true?

   a) G cannot contain a unique vertex with degree 1.

   b) If n is an odd number then m is also an odd number.

   c) If n is an even number then m is also an even number.

   d) If n is odd then there is an odd number of vertices with even degree.

   Solution:

   a) False. Consider a graph with one circuit x_1,x_2,x_3 (edges $\{x_1,x_2\}$, $\{x_2,x_3\}$, $\{x_3,x_1\}$) and an extra edge $\{x_1,x_4\}$. Then x_4 is the unique vertex of degree 1.

   b) False. Consider a complete graph on 5 vertices (odd). It has 10 edges (even).

   c) False. Consider any graph with even number of vertices $x_1,x_2,\ldots$ and just one edge $\{x_1,x_2\}$.

   d) True. Denote by k the number of vertices with even degree. Then there are $n-k$ vertices of odd degree, and this number must be even for every graph. So $n-k$ is even, $n = (n-k)+k$ is odd, so k must be odd.

   LO: DM41.

3. Given the following simple directed graph G with 9 vertices and 14 edges:

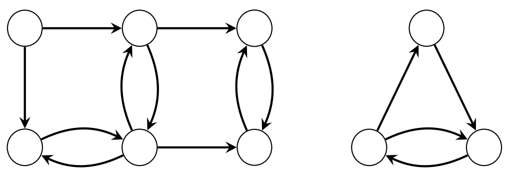

Figure 78

Which of the assertions is correct?

a) G is connected and has 4 components of strong connectivity.

b) G is not connected and has 3 components of strong connectivity.

c) G is not connected and has 2 components of strong connectivity.

d) G is not connected and has 4 components of strong connectivity.

Solution:

a) False.

b) False.

c) False.

d) True. G is not connected and has the following 4 components of strong connectivity: $\{1\}$, $\{2, 8, 9\}$, $\{3, 7\}$, and $\{4, 5, 6\}$.

LO: DM46.

4. Given a directed graph G. Consider the following statement.

If there is an undirected path from a vertex u to a vertex v then there is an undirected path from v to u.

Solution:

Correct answer is a). If $u = u_1, u_2, \ldots, u_n = v$ is an undirected path from u to v, then $u_n, u_{n-1}, \ldots, u_1$ is an undirected path from v to u.

LO: DM41.

4.2 Questions

1. Given a connected simple graph G without loops having 10 vertices and just one circuit. How many edges does G have? Justify your answer.

Solution:

Such graph is a tree together with one more edge; so if it has 10 vertices it must have 10 edges.

LO: DM41, DM42.

2. How many non-isomorphic trees exist on a set with 5 vertices? Justify your answer.

Solution:

There are only 3 different trees on 5 vertices. Indeed, every tree has (at least) two vertices with degree one. Moreover, the sum of all degrees is 8 (because a tree on 5 vertices has 4 edges, and the sum of all degrees is twice the number of edges). Hence, sum of degrees of the remaining vertices is 6. Number 6 must be divided into 3 positive integers and so we have $2 + 2 + 2$, $1 + 2 + 3$, and $1 + 1 + 4$. The graphs are in the following picture:

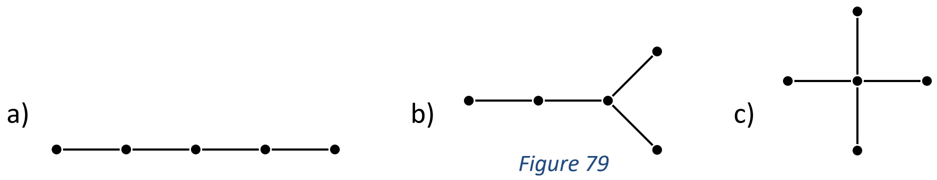

Figure 79

LO: DM42.

3. Is there a rooted binary tree with 6 vertices and the following list of out degrees $0, 0, 0, 1, 2, 2$?

   If yes draw an example of such a binary tree; if not show that such a binary tree does not exist.

   Solution:

   Yes, there is a rooted tree with the sequence of out degrees. Three such rooted trees are in the following picture.

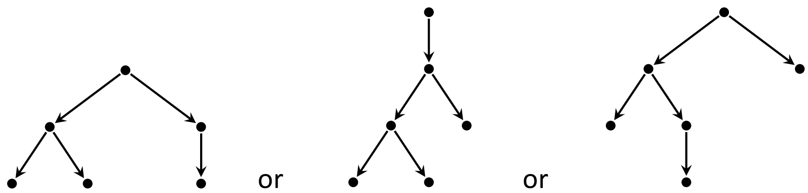

Figure 80

LO: DM42.

4. Decide whether the graph G with the set of vertices $V = \{1,\dots,8\}$ and edges given by the following list contains a closed Euler trail.

IV	1	1	1	2	2	3	3	3	4	5	6	6	7	8
TV	2	3	5	1	3	1	4	6	1	2	3	7	8	6

(Either find an Euler trail or explain why it cannot exist. Explain what is an Euler trail.)

Solution:

Since $d^{+}(v) = d^{-}(v)$ for every vertex $1,\dots,8$, if G is connected then there is a closed directed Euler trail. Let us randomly form a closed trail starting in 1. We have

$$(1, 2), (2, 1), (1, 3), (3, 1), (1, 5), (5, 2), (2, 3), (3, 4), (4, 1).$$

The trail cannot be extended but it does not contain all edges. We find the first vertex in which there is an edge that has not been used yet. Such a vertex is 3. We form another closed trail starting in 3 (and ending in 3)

$$(3, 6), (6, 7), (7, 8), (8, 6), (6, 3).$$

We insert it into the first one

$(1, 2), (2, 1), (1, 3), (3, 6), (6, 7), (7, 8), (8, 6), (6, 3), (3, 1), (1, 5), (5, 2), (2, 3), (3, 4), (4, 1).$

Now, the new *closed* trail contains all edges.

LO: DM43.

5. Is it true that every strongly connected graph has to be Hamiltonian (i.e. such a graph has to contain a Hamiltonian cycle)? If yes, justify; if not, give a counterexample.

Solution:

Not true. The following graph is strongly connected but does not contain a Hamiltonian cycle.

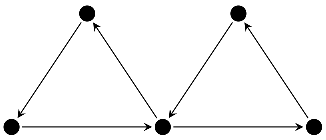

Figure 81

LO: DM44.

4.3 Project

1. Consider an undirected weighted graph $G = (V,E)$, where $V = \{v_1,\ldots,v_6\}$ and edges E and their weights are given by the following table (an edge $\{v_i,v_j\}$ is denoted by v_iv_j).

e	v_1v_2	v_1v_3	v_1v_4	v_1v_5	v_1v_7	v_2v_3	v_2v_6	v_2v_7	v_3v_4	v_3v_5	v_3v_7
$w(e)$	13	7	14	12	13	9	9	4	3	9	10

e	v_4v_7	v_4v_8	v_5v_6	v_6v_7	v_6v_8	v_7v_8	v_4v_5	v_4v_6
$w(e)$	2	3	3	11	11	2	3	1

Find a minimal spanning tree of G. Describe the method that you used.

Solution:

First we sort the edges (we state the weight of an edge in the brackets)

$e_1 = \{5,8\}(1)$, $e_2 = \{3,8\}(2)$, $e_3 = \{2,5\}(4)$, $e_4 = \{3,6\}(4)$, $e_5 = \{4,5\}(4)$, $e_6 =\{1,4\}(5)$,

$e_7 = \{3,5\}(5)$, $e_8 = \{5,6\}(5)$, $e_9 = \{3,4\}(6)$, $e_{10} = \{1,6\}(7)$, $e_{11} = \{7,8\}(7)$,

$e_{12} = \{6,8\}(8)$, $e_{13} = \{1,2\}(9)$, $e_{14} = \{1,8\}(9)$, $e_{15} = \{3,7\}(9)$, $e_{16} = \{1,3\}(13)$,

$e_{17} = \{2,3\}(13)$, $e_{18} = \{2,8\}(14)$, $e_{19} = \{6,7\}(15)$.

Put $T = \emptyset$.

Now we will go through edges in the given order and include e_i into T if and only if it does not close a circuit.

1. e_1 does not close a circuit, hence $T := \{\{5,8\}\}$.
2. e_2 does not close a circuit, hence $T := \{\{5,8\},\{3,8\}\}$.
3. e_3 does not close a circuit, hence $T := \{\{5,8\},\{3,8\},\{2,5\}\}$.
4. e_4 does not close a circuit, hence $T := \{\{5,8\},\{3,8\},\{2,5\},\{3,6\}\}$.
5. e_5 does not close a circuit, hence $T := \{\{5,8\},\{3,8\},\{2,5\},\{3,6\},\{4,5\}\}$.
6. e_6 does not close a circuit, hence $T := \{\{5,8\},\{3,8\},\{2,5\},\{3,6\},\{4,5\},\{1,4\}\}$.
7. e_7 closes a circuit formed by e_1, e_2 and e_7, hence T is the same as in 6.
8. e_8 closes a circuit formed by e_1, e_2, e_4 and e_8, hence T is the same as in 6.
9. e_9 closes a circuit formed by e_5, e_1, e_2 and e_9, hence T is the same as in 6.
10. e_{10} closes a circuit formed by e_6, e_5, e_1, e_2, e_4 and e_{10}, hence T is the same as in 6.
11. e_{11} does not close a circuit, hence $T := \{\{5,8\},\{3,8\},\{2,5\},\{3,6\},\{4,5\},\{1,4\},\{7,8\}\}$.

Since T contains $8 - 1 = 7$ edges, we have

$$T = \{\{5,8\},\{3,8\},\{2,5\},\{3,6\},\{4,5\},\{1,4\},\{7,8\}\}$$

as the set of edges of a minimal spanning tree of G. The weight (price) of T is

$$w(T) = 1 + 2 + 4 + 4 + 4 + 5 + 7 = 27.$$

LO: DM47.

5 BIBLIOGRAPHY

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[3] Lovasz, L., Pelik´an, J., Vesztergombi, K., 2003, Springer, New York, ISBN 978-0-387-95585-8.

CHAPTER 3

Geometry

Guide for a Problem

Geometry

3.1. Conic sections

GE1

Ion Mierlus–Mazilu

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 27, to be achieved with the development of this activity guide are the following:

1. Recognise the equation of an ellipse in standard form and state its foci, semiaxes and directrices
2. Recognise the parametric equations of an ellipse
3. Derive the main properties of an ellipse, including the equation of the tangent at a point
4. Recognise the equation of a hyperbola in standard form and find its foci, semiaxes and asymptotes
5. Recognise the parametric equations of a hyperbola
6. Derive the main properties of a hyperbola, including the equation of the tangent at a point
7. Recognise the equation of a conic section in the general form and classify the type of conic section.

Table 27 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

(green) = very important

(yellow) = medium important

(red) = less important

Table 27: Competencies that we measure with the global test model that is proposed

		C1	C2	C3	C4	C5	C6	C7	C8
Geometry									
GE1	1. Conic sections								
GE11	Recognise the equation of an ellipse in standard form and state its foci, semiaxes and directrices	(green)	(green)	(green)	(green)	(green)	(yellow)	(green)	(green)
GE12	Recognise the parametric equations of an ellipse	(green)	(green)	(green)	(green)	(green)	(green)	(green)	(green)
GE13	Derive the main properties of an ellipse, including the equation of the tangent at a point	(yellow)	(green)	(yellow)	(red)	(yellow)	(yellow)	(red)	(red)
GE14	Recognise the equation of a hyperbola in standard form and find its foci, semiaxes and asymptotes	(green)	(green)	(yellow)	(yellow)	(yellow)	(yellow)	(yellow)	(yellow)
GE15	Recognise the parametric equations of a hyperbola	(green)	(green)	(green)	(green)	(yellow)	(green)	(green)	(green)
GE16	Derive the main properties of a hyperbola, including the equation of the tangent at a point	(yellow)	(green)	(yellow)	(yellow)	(yellow)	(yellow)	(yellow)	(yellow)
GE17	Recognise the equation of a conic section in the general form and classify the type of conic section	(green)	(green)	(green)	(green)	(green)	(green)	(green)	(green)

2 CONTENTS

1. Recognise the equation of an ellipse in standard form and state its foci, semiaxes and directrices
2. Recognise the parametric equations of an ellipse
3. Derive the main properties of an ellipse, including the equation of the tangent at a point
4. Recognise the equation of a hyperbola in standard form and find its foci, semiaxes and asymptotes
5. Recognise the parametric equations of a hyperbola
6. Derive the main properties of a hyperbola, including the equation of the tangent at a point
7. Recognise the equation of a conic section in the general form and classify the type of conic section.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 28.

Table 28: Learning outcomes involved in this assessment activity.

Geometry	
GE1	CONIC SECTIONS
GE11	Recognise the equation of an ellipse in standard form and state its foci, semiaxes and directrices
GE12	Recognise the parametric equations of an ellipse
GE13	Derive the main properties of an ellipse, including the equation of the tangent at a point
GE14	Recognise the equation of a hyperbola in standard form and find its foci, semiaxes and asymptotes
GE15	Recognise the parametric equations of a hyperbola
GE16	Derive the main properties of a hyperbola, including the equation of the tangent at a point
GE17	Recognise the equation of a conic section in the general form and classify the type of conic section

The statement of the exam is the next:

3.1 Multiple-Choice Questions

1. Determine what type of conic the following equation represents:

$$5x^2 + 8xy + 6y^2 - 14x - 18y + 9 = 0$$

   i) ellipse

   ii) hyperbola

   iii) parabole

2. Determine what type of conic the following represents:

$$y^2 - 12x - 4y + 28 = 0$$

   i) ellipse

   ii) hyperbola

   iii) parabole

3. Write the equation for each of the following: Hyperbola with vertical transverse axis with center (0,0) and the asymptotes are

$$y = \pm\frac{2}{3}x.$$

   i) $\frac{y^2}{4} - \frac{x^2}{9} = 1$

   ii) $\frac{y^2}{2} - \frac{x^2}{3} = 1$

   iii) $\frac{y^2}{4} + \frac{x^2}{9} = 1$

4. The following relation represents half of a conic section. Name the type of conic. Also mention its focus/foci, directrix, center, vertex/vertices, axis of symmetry (where applicable), domain and range: $-\sqrt{36 - 4x^2}$.

   i) ellipse

   Center $(0,0)$ Radius : 3 Domain $[-3.3]$ Range : $[-3.0]$

   ii) ellipse

   Center $(1,1)$ Radius : 3 Domain $[-3.3]$ Range : $[-3.0]$

   iii) hyperbole

   Center $(1,1)$ Radius : 3 Domain $[-3.3]$ Range : $[-3.0]$

3.2 QUESTION 1

Determine what type of conic the following represent. Graph it. Find center, focus/foci, vertex/vertices, equations of asymptotes, and directrix for each, where applicable:

$$y^2 - 12x = 4y + 28 = 0$$

3.3 QUESTION 2

Graph each following relation. It represents half of a conic section. Name the type of conic. Also mention its focus/foci, directrix, center, vertex/vertices, axis of symmetry (where applicable), domain and range: $-\sqrt{36 - 4x^2}$.

3.4 QUESTION 3

Write the equation for each of the following: Hyperbola with vertical transverse axis with center (0,0) and the asymptotes are

$$y = \pm\frac{2}{3}x.$$

3.5 QUESTION 4

A cannon shell follows a parabolic path. It reaches a max height of 100m and land at a distance of 50m from the cannon. Write the equation of the parabolic path the shell follows. (Note: your answer will depend on where you locate your coordinate axes. Many correct answers are possible.)

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. Δ = -69, δ = 14⇒ellipsre
2. $y^2 - 12x - 4y + 28 = 0$ *parabola with horizontal axis*

   $y^2 - 4y = 12x - 28$

   $y^2 - 4y + 4 = 12x - 24$

   $(y-2)^2 = 12x - 24$

   $(y-2)^2 = 12(x-2)$
3. Hyperbola with vertical transverse axis, with center $(0,0)$ and asymptotes $y = \pm\frac{2}{3}x$.

   Asymptotes $: m = \pm\frac{a}{b}; a = 2, b = 3$

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

4. $2y = -\sqrt{36 - 4x^2}\,; note : y \leq 0\ (lower\ half\ of\ a\ conic)$

   $4y^2 = 36 - 4x^2$

   $4x^2 + 4y^2 = 36 \;=> x^2 + y^2 = 9$

   Center $(0,0)$

   Radius : 3

   Domain $[-3.3]$

   Range : $[-3.0]$

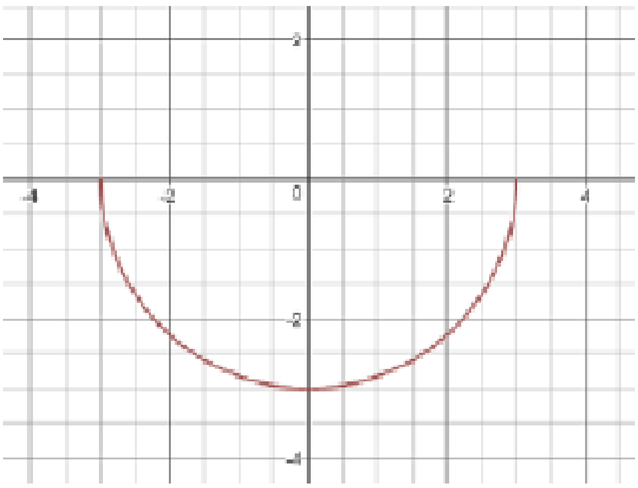

Figure 82

4.2 QUESTION 1

$y^2 - 4y = 12x - 28$

$y^2 - 4y + 4 = 12x - 24$

$(y-2)^2 = 12x - 24$

$(y-2)^2 = 12(x-2)$

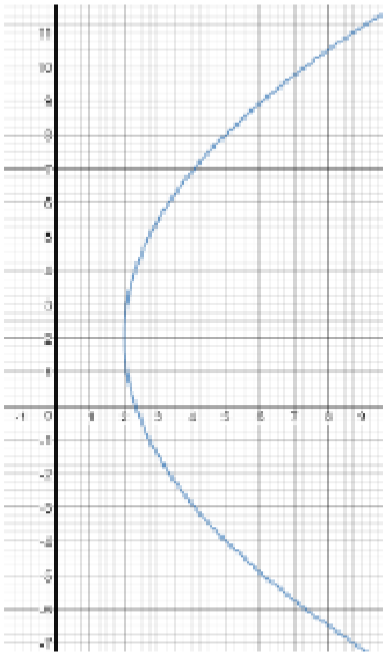

Figure 83

Vertex : $(2,2)$

Focal distance : $p = \frac{12}{4} = 3$

Focus : $(2+3,2) = (5,2)$

Directrix : $x = -1$

4.3 QUESTION 2

$2y = -\sqrt{36 - 4x^2}\,; note : y \leq 0\ (lower\ half\ of\ a\ conic)$

$4y^2 = 36 - 4x^2$

$\mathbf{4}x^2 + 4y^2 = 36\ \ => x^2 + y^2 = 9$

Center $(0,0)$

Radius : 3

Domain $[-3.3]$

Range : $[-3.0]$

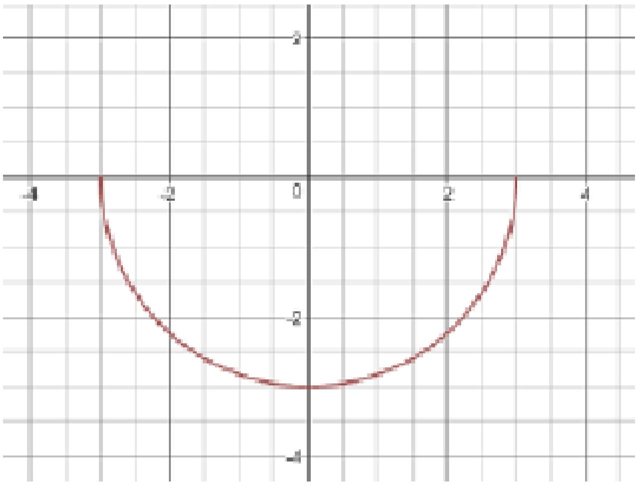

Figure 84

4.4 QUESTION 3

Hyperbola with vertical transverse axis, with center $(0,0)$ and asymptotes $y = \pm\frac{2}{3}x$.

asymptotes $: m = \pm\frac{a}{b}; a = 2, b = 3$

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

4.5 QUESTION 4

The coordinate axes are placed as :

so $(0,100)$ is the vertex and $(50,0)$ is the landing pant.

$$(x-h)^2 = 4p(y-k)$$

$$x^2 = 4p(y-100)$$

passes through $(50,0) => 25^2 = 4p(0-100)$

$$625 = -400\text{p so p} = -\frac{625}{400}$$

$$\text{x}^2 = -\frac{25}{4}(y-100)$$

Note : If axes are set up so vertex $(25,100)$, $(x-25)^2 = -\frac{25}{4}(y-100)$

5 PROJECTS

The time to solve this exam will be 2 hours (approximately

5.1 PROJECT 1

1. Let σ = 3.1 be the true standard deviation of the population from which a random sample is chosen. How large should the sample size be for testing $H_0 : \mu = 5$ versus Ha : μ = 5.5, in order that α = 0.01 and β = 0.05?
2. A machine is considered to be unsatisfactory if it produces more than 6% defectives. It is suspected that the machine is unsatisfactory. A random sample of 130 items produced by the machine contains 16 defectives. Does the sample evidence support the claim that the machine is unsatisfactory? Use $\alpha = 0.01$.
3. In attempting to control the strength of the wastes discharged into a nearby river, an industrial firm has taken a number of restorative measures. The firm believes that they have lowered the oxygen consuming power of their wastes from a previous mean of 400 manganate in parts per million. To test this belief, readings are taken on $n = 20$ successive days. A sample mean of 315.5 and the sample standard deviation 105.23 are obtained. Assume that these 20 values can be treated as a random sample from a normal population. Test the appropriate hypothesis. Use α = 0.05.

5.2 PROJECT 2

In a frequently traveled stretch of the I-75 expressway, where the posted speed is 70 kmph, it is thought that people travel on the average of at least 75 km/h. To check this claim, the following radar measurements of the speeds (in km/h) is obtained for 10 vehicles traveling on this stretch of the interstate highway.

66 74 79 80 69 77 78 65 79 81

Do the data provide sufficient evidence to indicate that the mean speed at which people travel on this stretch of highway is at most 75 km/h? Test the appropriate hypothesis using $\alpha = 0.01$. Draw a box plot and normal plot for this data, and comment.

6 SOLUTIONS OF THE PROJECTS

6.1 PROJECT 1

1. Let σ = 3.2 be the true standard deviation of the population from which a random sample is chosen. How large should the sample size be for testing $H_0 : \mu = 7$ versus Ha : μ = 4.5, in order that α = 0.01 and β = 0.05?

   We are given $\mu_0 = 5$ and $\mu_a = 5.5$. Also, $z_\alpha = z_{0.01} = 2.33$ and $z_\beta = z_{0.05} = 1.645$. Hence, the sample size

$$n = \frac{(z_a+z_\beta)^2\sigma^2}{(\mu_a-\mu_0)^2} = \frac{(2.33+1.645)^2(3.2)^2}{(2.5)^2} = 25.88.$$

   So, *n* = 26 will provide the desired levels. That is, in order for us to test the foregoing hypothesis, we must randomly select 608 observations from the given population.

2. A machine is considered to be unsatisfactory if it produces more than 6% defectives. It is suspected that the machine is unsatisfactory. A random sample of 130 items produced by the machine contains 16 defectives. Does the sample evidence support the claim that the machine is unsatisfactory? Use $\alpha = 0.01$.

   Let *Y* be the number of observed defectives. This follows a binomial distribution. However, because np_0 and nq_0 are greater than 5, we can use a normal approximation to the binomial to test the hypothesis. So we need to test $H_0 : p = 0.06$ versus $H_a : p > 0.06$. Let the point estimate of p be $\hat{p} = \left(\frac{Y}{n}\right) = 0.119$, the sample proportion. Then the value of the TS is

$$z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0 q_0}{n}}} = \frac{0.119-0.06}{\sqrt{\frac{(0.06)(0.92)}{130}}} = 2.92\ .$$

   For α = 0.01, z0.01 = 2.33. Hence, the rejection region is {z > 2.33}.

   Solution:

   Because 2.92 is greater than 2.33, we reject H_0. We conclude that the evidence supports the claim that the machine is unsatisfactory.

3. In attempting to control the strength of the wastes discharged into a nearby river, an industrial firm has taken a number of restorative measures. The firm believes that they have lowered the oxygen consuming power of their wastes from a previous mean of 400 manganate in parts per million. To test this belief, readings are taken on $n = 20$ successive days. A sample mean of 315.5 and the sample standard deviation 105.23 are obtained. Assume that these 20 values can be treated as a random sample from a normal population. Test the appropriate hypothesis. Use α = 0.05.

   Here we need to test the following hypothesis: $H_0 : \mu = 450$ vs. Ha : μ < 450 Given $n = 20$, $\bar{x} = 312.5$, and $s = 106.23$. The observed test statistic is

$$t = \frac{315.5-400}{105.23/\sqrt{20}} = -0.17\ .$$

   The rejection region for α = 0.05 and with 19 degrees of freedom is the set of t-values such that

$$\{t < -t0.05.19\} = \{t < -1.729\}.$$

Solution:

Because t = −0.17 is more than −1.729, do not reject H_0. There is sufficient evidence to confirm the firm's belief. For large random samples, the following procedure is used to perform tests of hypotheses about the population proportion, p.

6.2 PROJECT 2

In a frequently traveled stretch of the I-75 expressway, where the posted speed is 70 kmph, it is thought that people travel on the average of at least 75 km/h. To check this claim, the following radar measurements of the speeds (in km/h) is obtained for 10 vehicles traveling on this stretch of the interstate highway.

66 74 79 80 69 77 78 65 79 81

Do the data provide sufficient evidence to indicate that the mean speed at which people travel on this stretch of highway is at most 75 km/h? Test the appropriate hypothesis using $\alpha = 0.01$. Draw a box plot and normal plot for this data, and comment.

We need to test $H_0 : \mu = 75$ vs. $Ha : \mu > 75$

For this sample, the sample mean is x = 74.8 km/h and the standard deviation is σ = 5.9963 km/h. Hence, the observed test statistic is

$$t = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{74.8 - 75}{5.9963/\sqrt{10}} = -0.10547$$

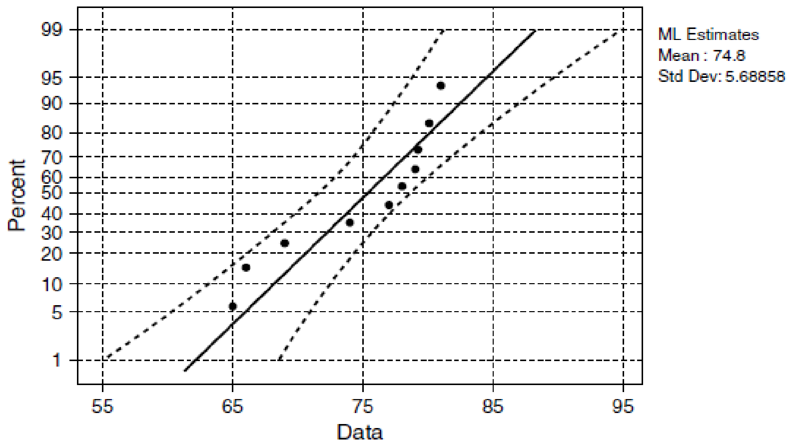

Figure 85

From the t-table, t0.019 = 2.821. Hence, the rejection region is {t > 2.821}. Because, $t = -0.10547$ does not fall in the rejection region, we do not reject the null hypothesis at $\alpha = 0.01$.

Note that we assumed that the vehicles were randomly selected and that collected data follow the normal distribution, because of the small sample size, n < 30, we use the t-test.

7 BIBLIOGRAPHY

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Guide for a Problem

Geometry

3.2. 3D co-ordinate geometry

GE2

Michael Carr, Paul Robinson

and Ciarán O'Sullivan

1 OBJECTIVES

The objectives, to be achieved with the development of this activity guide are the following.

Geometry

- Recognise and use the standard equation of a plane
- Find the angle between two straight lines
- Find where two straight lines intersect
- Find the angle between two planes
- Find the intersection line of two planes
- Find the intersection of a line and a plane
- Find the angle between a line and a plane
- Calculate the distance between two points, a point and a line, a point and a plane
- Calculate the distance between two lines, a line and a plane, two planes
- Recognise and use the standard equation of a singular quadratic surface (cylindrical, conical)
- Recognise and use the standard equation of a regular quadratic surface (ellipsoid, paraboloid, hyperboloid)

Table 29 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

= very important

= medium important

= less important

Table 29: Competencies that we measure with the global test model that is proposed

	3D Co-ordinate geometry	**C1**	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
GE21	Recognise and use the standard equation of a straight line in 3D								
GE22	Recognise and use the standard equation of a plane								
GE23	Find the angle between two straight lines								
GE24	Find where two straight lines intersect								
GE25	Find the angle between two planes								
GE26	Find the intersection line of two planes								
GE27	Find the intersection of a line and a plane								
GE28	Find the angle between a line and a plane								

GE29	Calculate the distance between two points, a point and a line, a point and a plane								
GE30	Calculate the distance between two lines, a line and a plane, two planes								
GE31	Recognise and use the standard equation of a singular quadratic surface (cylindrical, conical)								
GE32	Recognise and use the standard equation of a regular quadratic surface (ellipsoid, paraboloid, hyperboloid)								

2 CONTENTS

1. Co-ordinate geometry of lines and planes.
2. Calculating angles and distances.
3. Introduction to equations for ellipsoid, paraboloid, hyperboloid.

3 EVALUATION

The time to solve this exam will be 100 minutes (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 30.

Table 30: Learning outcomes involved in this assessment activity.

Geometry	
GE2	3D Co-ordinate geometry
GE21	Recognise and use the standard equation of a straight line in 3D
GE22	Recognise and use the standard equation of a plane
GE23	Find the angle between two straight lines
GE24	Find where two straight lines intersect
GE25	Find the angle between two planes
GE26	Find the intersection line of two planes
GE27	Find the intersection of a line and a plane
GE28	Find the angle between a line and a plane
GE29	Calculate the distance between two points, a point and a line, a point and a plane
GE30	Calculate the distance between two lines, a line and a plane, two planes
GE31	Recognise and use the standard equation of a singular quadratic surface (cylindrical, conical)
GE32	Recognise and use the standard equation of a regular quadratic surface (ellipsoid, paraboloid, hyperboloid)

3.1 MULTIPLE CHOICE PROBLEMS

1. How many of the expressions below define the equation of a line through $\underset{\sim}{r}_0$ **and** $\underset{\sim}{r}_1$

   (a) $\underset{\sim}{r} = \underset{\sim}{r}_0 + t(\underset{\sim}{r}_1 - \underset{\sim}{r}_0), t \in \mathbb{R}$

   (b) $\underset{\sim}{r} = \underset{\sim}{r}_1 + s(\underset{\sim}{r}_1 - \underset{\sim}{r}_0), s \in \mathbb{R}$

   (c) $\underset{\sim}{r} = \underset{\sim}{r}_0 + \frac{1}{3}p(\underset{\sim}{r}_0 - \underset{\sim}{r}_1), p \in \mathbb{R}$

   (d) $\underset{\sim}{r} = \frac{1}{2}(\underset{\sim}{r}_0 + \underset{\sim}{r}_1) + q(\underset{\sim}{r}_1 - \underset{\sim}{r}_0), q \in \mathbb{R}$

2. A normal vector to the plane through (-1, 2, 0), (1, 4, 1) and (3, -1, 2) is

   (a) $4\underset{\sim}{\imath} + 2\underset{\sim}{\jmath} - 5\underset{\sim}{k}$

   (b) $2\underset{\sim}{\imath} + \underset{\sim}{\jmath} - 6\underset{\sim}{k}$

   (c) $\underset{\sim}{\imath} - 2\underset{\sim}{k}$

   (d) $-7\underset{\sim}{\imath} - 14\underset{\sim}{k}$

3. The line $\underset{\sim}{r} = 2\underset{\sim}{\imath} + \underset{\sim}{\jmath} + t(3\underset{\sim}{\imath} - \underset{\sim}{\jmath} + \underset{\sim}{k})$ and the plane $2x + y + z = 2$ intersect at

   (a) none of points (b), (c) or (d)

   (b) $-\underset{\sim}{\imath} + 2\underset{\sim}{\jmath} - \underset{\sim}{k}$

   (c) $\frac{1}{2}\underset{\sim}{\imath} + \underset{\sim}{k}$

   (d) $\frac{1}{2}\underset{\sim}{\imath} + \frac{3}{2}\underset{\sim}{\jmath} - \frac{1}{2}\underset{\sim}{k}$

4. How many of the expressions below define the plane through (2, 1, -2), (0, 3, 1) and (2, 3, -1)

   (a) $2x - y + 2z = -1$

   (b) $\underset{\sim}{P} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} s, t \in R$

   (c) $\underset{\sim}{P} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} s, t \in R$

   (d) $\underset{\sim}{P} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} s, t \in R$

5. The acute angle between the line $\underset{\sim}{r} = \underset{\sim}{\imath} + \underset{\sim}{\jmath} - 2\underset{\sim}{k} + t(\underset{\sim}{\imath} + 3\underset{\sim}{\jmath} - \underset{\sim}{k})$ and the plane $x - 2y - 3z = 5$ in radians, and correct to 2 decimal places, is

   (a) 1.73 rad

   (b) 1.41 rad

   (c) 0.99 rad

   (d) $cos^{-1}\left(\frac{5-2t}{\sqrt{14}\sqrt{(1+t)^2+4(1+3t)^2+9(-2-t)^2}}\right)$

3.2 PROBLEMS

1. Let $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{c} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$.

   (a) Find the vector equation of the straight line through the points a and b.

   (b) Find the vector equation of the straight line through the points b and c.

   (c) Find the vector equation of the straight line through the point b and in the direction of a

   (d) Find the scalar equation of the plane through a with normal vector c. Write down a vector form of this plane.

   (e) Find the vector equation of the plane through a , b and c. Hence, find the scalar equation of this plane and write down a normal vector for this plane. Verify that this normal vector is a multiple of $(\underset{\sim}{c} - \underset{\sim}{a}) \times (\underset{\sim}{b} - \underset{\sim}{a})$.

2.

   (a) The vector equation of a line in R^2 is $r = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, where $t \in R$.

   Find the equation of the line in the form y = mx + c.

   (b) The equation of a line in R^2 is y = 3x – 2. Find a vector form of the equation of this line.

3. Write down the coordinates of the points **A**, **B** and **C** in i, j, k form, relative to the origin **0.**

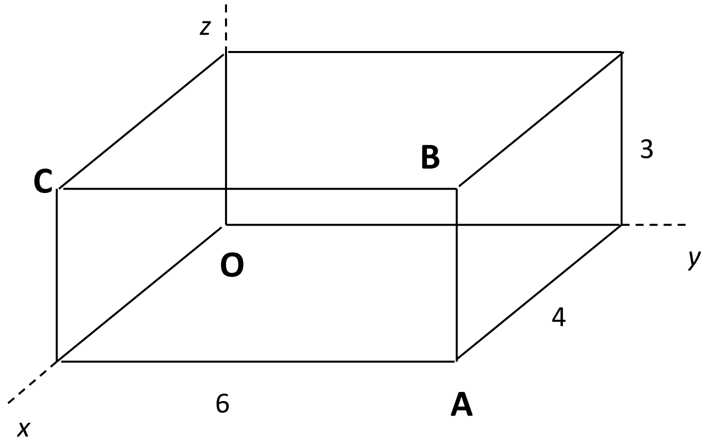

Figure 86

   (a) Find the vector equation of the line through A and B

   (b) Find the vector equation of the line through A and C

   (c) What is the acute angle between the lines (a) and (b)?

   (d) Find the scalar equation of the plane through C, perpendicular to the line through O and B

   (e) Where does the plane in (d) intersect the line between O and A?

   (f) What is the perpendicular distance from the origin to the plane in (d)?

4. The plane through the points $P_0, P_1 and P_2$ is $P = P_0 + s(P_1 - P_0) + t(P_2 - P_0)$ where s and t are parameters in R. The vector equation of the line through the points x_0 and x_1 is $x = x_0 + r(x_1 - x_0)$ where r is a parameter in R.

   (a) Show that the vector equation of the plane through the points $P_0 = \begin{pmatrix}5\\0\\4\end{pmatrix}, P_1 = \begin{pmatrix}1\\6\\-6\end{pmatrix}, P_2 = \begin{pmatrix}3\\-1\\4\end{pmatrix}$ with parameters s and t is $P = \begin{pmatrix}5-4s-2t\\6s-t\\4-10s\end{pmatrix}$

   (b) Show that the vector equation of the line through the points $x_0 = \begin{pmatrix}-1\\-2\\2\end{pmatrix}, x_1 = \begin{pmatrix}-3\\-3\\4\end{pmatrix}$ with parameter r is $x = \begin{pmatrix}-1-2r\\-2-r\\2+2r\end{pmatrix}$

   (c) Putting **P** = **x** show that $\begin{pmatrix}4&2&-2\\-6&1&-1\\10&0&2\end{pmatrix}\begin{pmatrix}s\\t\\r\end{pmatrix} = \begin{pmatrix}6\\2\\2\end{pmatrix}$

   Hence find r and the point of intersection of the plane and line.

5. (a) Write down the vector equation of the plane through the points below with parameters s_1 and s_2

$$P_0 = \begin{pmatrix}5\\0\\4\end{pmatrix}, P_1 = \begin{pmatrix}1\\6\\-6\end{pmatrix}, P_2 = \begin{pmatrix}3\\-1\\4\end{pmatrix}$$

   (b) Write down the vector equation of the plane through the points below with parameters s_3 and s_4

$$P_0 = \begin{pmatrix}-1\\-2\\2\end{pmatrix}, P_1 = \begin{pmatrix}-3\\-3\\2\end{pmatrix}, P_2 = \begin{pmatrix}1\\-4\\5\end{pmatrix}$$

   (c) Hence find s_3 and s_4 in terms of a parameter t and the equation of the line of intersection of these planes in terms of t.

6. (a) What is a vector equation of the plane through O, A and B?

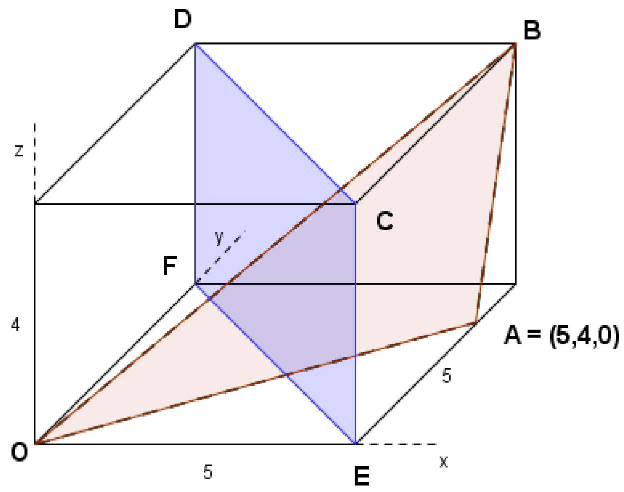

Figure 87

(b) What is a vector equation of the plane through C, D and E?

(c) Find the vector form of the line of intersection of the planes in parts (a) and (b)

(d) What is the scalar form of the two planes and the acute angle between them? (this is the same as the angle between their normal vectors)

(e) Find the vector equation of the lines through O, B and D, F. What is the perpendicular distance between these lines and the points of closest approach on OB and on DF?

3.3 PROJECT

Helpful Websites

Geogebra Homepage: http://www.geogebra.org

Download GeoGebra Classic (you can change language in Options) and run it.

Geogebra Resources: https://www.geogebra.org/materials

Collection of re-usable resources.

Construction 1

We will visualize the shortest distance between 2 lines in 3D.

Click View, 3D Graphics. In the input line type A = (0, 1, 3) and B = (-2, 0, 2) pressing return after each. Click the Line Tool then A and B in turn to see the Line 1 between them.

Repeat for points C = (1, 3, 0), D = (-2, -2, 1) to create Line 2. Click the selection arrow to rotate the graphic and the mouse scroll to zoom.

Click the Point Tool and Line 1 to put a point on it (point E) and repeat for a point (F) on Line 2. Right click each point and in properties make them red.

Click the Segment Tool then points E and F to make a line between them. Click the selection arrow then drag points E and F to where you think the shortest distance is between them. Click the Angle Tool then A, E, F in turn to see the angle between Line 1 and the segment. Repeat for C, F, E. When the distance is a minimum these angles will both be 90^0.

Problem

Calculate the points of closest contact between the two lines (their equations are in the Algebra Window). Do they agree with your approximate E, F? Change the coordinates E and F to the calculated values to see if both angles are now 90^0.

Construction 2

Create a new GeoGebra File and input the point A = (0, 0, 0), B = (2, 2, 2) and C = (-2, 1, 2). Construct the plane through these points using the Plane Tool. Repeat for the plane through the points D = (0, -2, 1), E = (2, 1, -3) and

F = (-1, -2, 2). Make the planes different colours.

Problems

1. Note the **scalar** equations of each plane in the Algebra Window. Calculate the vector equation of each plane and then each scalar equation as comparison with GeoGebra.

2. Use the Point Tool to mark two points on the line of intersection of the planes. Use the Line Tool to see the vector equation of the line through the two points. Use the scalar (or vector) equations of the planes to calculate the line of intersection – how does it compare to your constructed line?

4 SOLUTION

4.1 MULTIPLE CHOICE SOLUTIONS

Q1 (iv) Q2 (c) Q3 (d) Q4 (iii) Q5 (b)

4.2 PROBLEMS SOLUTIONS

1. (a) $\underset{\sim}{r} = \underset{\sim}{a} + t(\underset{\sim}{b} - \underset{\sim}{a}) = \underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k} + t(\underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k}), t \in R$

   (b) $\underset{\sim}{r} = \underset{\sim}{b} + t(\underset{\sim}{c} - \underset{\sim}{b}) = 2\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k} + t(2\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}), t \in R$

   (c) $\underset{\sim}{r} = \underset{\sim}{b} + t(\underset{\sim}{a}) = 2\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k} + t(\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}), t \in R$

   (d) $(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}).(4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) = (\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}).(4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$

   $$\Rightarrow 4x + 2y - z = 4 + 6 - 5 = 5$$

   For a vector form put x = s, y = t then $z = 4s + 2t - 5$ so that (one vector form is)

   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ t \\ 4s + 2t - 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, s, t \in R$$

   (e) $\underset{\sim}{p} = \underset{\sim}{a} + s(\underset{\sim}{b} - \underset{\sim}{a}) + t(\underset{\sim}{c} - \underset{\sim}{a})$

   $$= \underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k} + s(\underset{\sim}{i} - 5\underset{\sim}{j} - 2\underset{\sim}{k}) + t(3\underset{\sim}{i} - \underset{\sim}{j} - 6\underset{\sim}{k})s, t \in R$$

   $$\begin{aligned} &\Rightarrow x = 1 + s + 3t(1) \\ &y = 3 - 5s - t(2) \\ &z = 5 - 2s - 6t(3) \\ &(1) * 2 \Rightarrow 2x = 2 + 2s + 6t \\ &(3) z = 5 - 2s - 6t \end{aligned}$$

   $adding\ 2x + z = 7$

   a normal vector to the plane is $\underset{\sim}{n} = 2\underset{\sim}{i} + \underset{\sim}{k}$, the coefficients of x, y, z in the plane.

   Also,

   $$(\underset{\sim}{c} - \underset{\sim}{a}) \times (\underset{\sim}{b} - \underset{\sim}{a}) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & -1 & -6 \\ 1 & -5 & -2 \end{vmatrix} = -28\underset{\sim}{i} - 14\underset{\sim}{k} = -14\underset{\sim}{n}$$

2. (a) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow x = 1 + t and y = 2 - t$

   $\Rightarrow x + y = 3 so that y = 3 - x$

   (b) $y = 3x - 2 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3x - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + x\begin{pmatrix} 1 \\ 3 \end{pmatrix} x \in R$

3. $\underset{\sim}{A} = 4\underset{\sim}{i} + 6\underset{\sim}{j}, \underset{\sim}{B} = 4\underset{\sim}{i} + 6\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{C} = 4\underset{\sim}{i} + 3\underset{\sim}{k}$

   (a) $\underset{\sim}{r} = 4\underset{\sim}{i} + 6\underset{\sim}{j} + t\underset{\sim}{k}, t \in R[\underset{\sim}{r} = 4\underset{\sim}{i} + 6\underset{\sim}{j} + 3t\underset{\sim}{k} is the same line]$

   (b) $\underset{\sim}{r} = A + t(C - A) = 4\underset{\sim}{i} + 6\underset{\sim}{j} + t(-6\underset{\sim}{j} + 3\underset{\sim}{k}) t \in R$

   (c) Unit vector in the direction of line (a) is $\underset{\sim}{k}$

   Unit vector in the direction of line (b) is $\frac{1}{\sqrt{6^2+3^2}}(-6\underset{\sim}{j} + 3\underset{\sim}{k}) = \frac{(-2\underset{\sim}{j}+\underset{\sim}{k})}{\sqrt{5}}$

   $\Rightarrow cos(\theta) = \underset{\sim}{k}.\frac{(-2\underset{\sim}{j}+\underset{\sim}{k})}{\sqrt{5}} = \frac{1}{\sqrt{5}} \Rightarrow \theta = 1.107\ rad$

   (d) OB line is $\underset{\sim}{r} = t(4\underset{\sim}{i} + 6\underset{\sim}{j} + 3\underset{\sim}{k}), t \in R$

For plane,

$$\underset{\sim}{r}_0 = 4\underset{\sim}{\imath} + 3\underset{\sim}{k} and \underset{\sim}{n} = 4\underset{\sim}{\imath} + 6\underset{\sim}{\jmath} + 3\underset{\sim}{k}$$
$$\Rightarrow (x\underset{\sim}{\imath} + y\underset{\sim}{\jmath} + z\underset{\sim}{k}).(4\underset{\sim}{\imath} + 6\underset{\sim}{\jmath} + 3\underset{\sim}{k}) = (4\underset{\sim}{\imath} + 3\underset{\sim}{k}).(4\underset{\sim}{\imath} + 6\underset{\sim}{\jmath} + 3\underset{\sim}{k})$$
$$\Rightarrow 4x + 6y + 3z = 16 + 9 = 25$$

OA line is $\underset{\sim}{r} = t(4\underset{\sim}{\imath} + 6\underset{\sim}{\jmath}), t \in R$

If line meets the plane

$$\Rightarrow 4(4t) + 6(6t) + 3(0) = 25$$
$$\Rightarrow 52t = 25 \Rightarrow t = \frac{25}{52}$$

and the intersect point is $\underset{\sim}{r} = t(4\underset{\sim}{\imath} + 6\underset{\sim}{\jmath}) = \frac{100}{52}\underset{\sim}{\imath} + \frac{150}{52}\underset{\sim}{\jmath} = \frac{1}{26}(50\underset{\sim}{\imath} + 75\underset{\sim}{\jmath})$

4. (a) $\underset{\sim}{P} = \begin{pmatrix}5\\0\\4\end{pmatrix} + s\left[\begin{pmatrix}1\\6\\-6\end{pmatrix} - \begin{pmatrix}5\\0\\4\end{pmatrix}\right] + t\left[\begin{pmatrix}3\\-1\\4\end{pmatrix} - \begin{pmatrix}5\\0\\4\end{pmatrix}\right]$

$= \begin{pmatrix}5\\0\\4\end{pmatrix} + s\begin{pmatrix}-4\\6\\-10\end{pmatrix} + t\begin{pmatrix}-2\\-1\\0\end{pmatrix} = \begin{pmatrix}5-4s-2t\\6s-t\\4-10s\end{pmatrix} s, t \in R$

(b) $\underset{\sim}{x} = \begin{pmatrix}-1\\-2\\2\end{pmatrix} + r\begin{pmatrix}-2\\-1\\2\end{pmatrix} = \begin{pmatrix}-1-2r\\-2-r\\2+2r\end{pmatrix} r \in R$

(c) $\begin{pmatrix}5-4s-2t\\6s-t\\4-10s\end{pmatrix} = \begin{pmatrix}-1-2r\\-2-r\\2+2r\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}-4s-2t+2r\\6s-t+r\\-10s-2r\end{pmatrix} = \begin{pmatrix}-6\\-2\\2\end{pmatrix} \Rightarrow \begin{pmatrix}-4 & -2 & 2\\6 & -1 & 1\\-10 & 0 & -2\end{pmatrix}\begin{pmatrix}s\\t\\r\end{pmatrix} = \begin{pmatrix}-6\\-2\\2\end{pmatrix}$$

Writing in augmented form and performing row reduction we arrive at

$$\begin{pmatrix}1 & 0.5 & -0.5 & -1.5\\0 & 1 & -1 & -1.75\\0 & 0 & 1 & 2.125\end{pmatrix}$$

So that r = 2.125 and the intersection point is $\underset{\sim}{r} = \begin{pmatrix}-1-2r\\-2-r\\2+2r\end{pmatrix} = \begin{pmatrix}-5.25\\-4.125\\6.25\end{pmatrix}$

5. (a) $\underset{\sim}{P}_1 = \begin{pmatrix}5\\0\\4\end{pmatrix} + s_1\begin{pmatrix}-4\\6\\-10\end{pmatrix} + s_2\begin{pmatrix}-2\\-1\\0\end{pmatrix} = \begin{pmatrix}5-4s_1-2s_2\\6s_1-s_2\\4-10s_1\end{pmatrix} s_1, s_2 \in R$

(b) $\underset{\sim}{P}_2 = \begin{pmatrix}-1\\-2\\2\end{pmatrix} + s_3\begin{pmatrix}-2\\-1\\0\end{pmatrix} + s_4\begin{pmatrix}2\\-2\\3\end{pmatrix} = \begin{pmatrix}-1-2s_3+2s_4\\-2-s_3-2s_4\\2+3s_4\end{pmatrix} s_3, s_4 \in R$

(c) For the line of intersection $\underline{P}_1 = \underline{P}_2$

$$\Rightarrow \begin{pmatrix} 5-4s_1-2s_2 \\ 6s_1-s_2 \\ 4-10s_1 \end{pmatrix} = \begin{pmatrix} -1-2s_3+2s_4 \\ -2-s_3-2s_4 \\ 2+3s_4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4s_1-2s_2+2s_3-2s_4 \\ 6s_1-s_2+s_3+2s_4 \\ -10s_1-3s_4 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 & 2 & -2 \\ 6 & -1 & 1 & 2 \\ -10 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ -2 \end{pmatrix}$$

Writing this equation in augmented form and performing row reduction we arrive at

$$\begin{pmatrix} 1 & 0.5 & -0.5 & 0.5 & 1.5 \\ 0 & 1 & -1 & 0.25 & 2.75 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

So that s_3 = t is free and s_4 = -1. Substituting these values into $\underset{\sim}{P}_2$ the line of intersection is

$$\underset{\sim}{x} = \begin{pmatrix} -1-2t-2 \\ -2-t+2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} t \in R$$

6. (a) $\underset{\sim}{P} = \underset{\sim}{O} + s_1(\underset{\sim}{A} - \underset{\sim}{O}) + s_2(\underset{\sim}{B} - \underset{\sim}{O})$

$$=> \underset{\sim}{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s_1\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + s_2\begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} s_1, s_2 \in R$$

(b) $\underset{\sim}{P} = \underset{\sim}{C} + s_3(\underset{\sim}{D} - \underset{\sim}{C}) + s_4(\underset{\sim}{E} - \underset{\sim}{C})$

$$=> \underset{\sim}{P} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} + s_3\left[\begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}\right] + s_4\left[\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}\right]$$

$$=> \underset{\sim}{P} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} + s_3\begin{pmatrix} -5 \\ 5 \\ 0 \end{pmatrix} + s_2\begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} s_3, s_4 \in R$$

(c) Plane (a) is $\begin{pmatrix} 5s_1+5s_2 \\ 4s_1+5s_2 \\ 4s_2 \end{pmatrix}$ and Plane (b) is $\begin{pmatrix} 5-5s_3 \\ 5s_3 \\ 4-4s_4 \end{pmatrix}$

For line of intersection, $\begin{pmatrix} 5s_1+5s_2 \\ 4s_1+5s_2 \\ 4s_2 \end{pmatrix} = \begin{pmatrix} 5-5s_3 \\ 5s_3 \\ 4-4s_4 \end{pmatrix} => \begin{pmatrix} 5s_1+5s_2+5s_3 \\ 4s_1+5s_2-5s_3 \\ 4s_2+4s_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$

Augmented form,

$$\begin{pmatrix} 5 & 5 & 5\ 05 \\ 4 & 5 & -500 \\ 0 & 4 & 0\ 44 \end{pmatrix} \begin{matrix} \div 5 \\ \\ \div 4 \end{matrix} \rightarrow \begin{pmatrix} 1 & 1 & 1\ 01 \\ 4 & 5 & -500 \\ 0 & 1 & 0\ 11 \end{pmatrix} -4R1 \rightarrow \begin{pmatrix} 1 & 1 & 1\ 0\ 1 \\ 0 & 1 & -90-4 \\ 0 & 1 & 0\ 1\ 1 \end{pmatrix}$$

$$\underset{-R2}{\rightarrow} \begin{pmatrix} 1 & 1 & 1\ 0\ 1 \\ 0 & 1 & -90-4 \\ 0 & 0 & 9\ 1\ 5 \end{pmatrix}$$

$$s_4 = t\text{is free}$$

$$=> 9s_3 + t = 5$$

$$=> s_3 = \frac{1}{9}(5-t) = \frac{5}{9} - \frac{1}{9}t$$

Then the line is,

$$\underline{r} = \begin{pmatrix} 5-5s_3 \\ 5s_3 \\ 4-4s_4 \end{pmatrix} = \begin{pmatrix} 5-5\left(\frac{5}{9}-\frac{1}{9}t\right) \\ 5\left(\frac{5}{9}-\frac{1}{9}t\right) \\ 4-4t \end{pmatrix} = \begin{pmatrix} \frac{20}{9}+\frac{5}{9}t \\ \frac{25}{9}-\frac{5}{9}t \\ 4-4t \end{pmatrix}$$

$$\underline{r} = \frac{1}{9}\begin{pmatrix} 20 \\ 25 \\ 36 \end{pmatrix} + \frac{1}{9}\begin{pmatrix} 5 \\ -5 \\ -36 \end{pmatrix} t, t \in R$$

(d)Plane (a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5s_1+5s_2 \\ 4s_1+5s_2 \\ 4s_2 \end{pmatrix}$

$$x = 5s_1 + 5s_2(1)$$
$$y = 4s_1 + 5s_2(2)$$
$$z = 4s_2(3)$$

$(1)-(2) \Rightarrow (\quad)x - y = s_1$
Substitute in$(1) \Rightarrow x = 5(x-y) + 5s_2$
$\Rightarrow 4x = 20(x-y) + 20s_2(4)$
$(3) * 5 \Rightarrow 5z = 20s_2(5)$
$(4)-(5) \Rightarrow 4x - 5z = 20(x-y)$
$\Rightarrow -16x + 20y - 5z = 0$
So normal vector $\underline{n}_1 = -16\underline{\imath} + 20\underline{\jmath} - 5\underline{k}$

Plane (b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5-5s_3 \\ 5s_3 \\ 4-4s_4 \end{pmatrix}$
$x = 5 - 5s_3(1)$
$y = 5s_3(2)$
$z = 4 - 4s_4(3)$
$(1)+(2) \Rightarrow x + y = 5$
So normal vector $\underline{n}_2 = \underline{\imath} + \underline{\jmath}$

Angle between planes,

$$cos(\theta) = \frac{\underline{n}_1 . \underline{n}_2}{|\underline{n}_1||\underline{n}_2|} = \frac{-16+20+0}{\sqrt{(16^2+20^2+5^2)}\sqrt{(1^2+1^2)}} = 4$$

(e) Line O,B

$\underline{r}_1 = t(5\underline{\imath} + 5\underline{\jmath} + 4\underline{k}), t \in R$
and hence will be in the direction of the vector $5\underline{\imath} + 5\underline{\jmath} + 4\underline{k}$

Line D,F

$\underline{r}_2 = 5\underline{\jmath} + s\underline{k}, s \in R$
and hence will be in the direction of the vector $\underline{k}$

Vector between points on each line is

$$\underline{w} = \underline{r}_2 - \underline{r}_1 = -5t\underline{\imath} + (5-5t)\underline{\jmath} + (s-4t)\underline{k}$$

This vector $\underline{w}$is perpendicular to the line $\underline{r}_1$ and so will be perpendicular to $5\underline{\imath} + 5\underline{\jmath} + 4\underline{k}$

$$\Rightarrow (5\underline{\imath} + 5\underline{\jmath} + 4\underline{k}).[-5t\underline{\imath} + (5-5t)\underline{\jmath} + (s-4t)\underline{k}] = 0$$
$$\Rightarrow -25t + 5(5-5t) + 4(5-4t) = 0$$
$$\Rightarrow 25 - 50t + 4s - 16t = 0$$

$$\Rightarrow 4s - 66t = -25(1)$$

This vector $\underline{w}$is also perpendicular to the line $\underline{r}_2$ and so will be perpendicular to $\underline{k}$

$$\Rightarrow (\underline{k}).\left[-5t\underline{i} + (5-5t)\underline{j} + (s-4t)\underline{k}\right] = 0$$
$$\Rightarrow s - 4t = 0$$
$$\Rightarrow s = 4t(2)$$

Therefore, substituting from (2) into (1):

$$\Rightarrow 4(4t) - 66t = -25$$
$$\Rightarrow -50t = -25$$
$$\Rightarrow t = 0.5 \text{ and hence } s = 2$$

Closest approach on line OB is at position vector

$$\underline{u} = 0.5(5\underline{i} + 5\underline{j} + 4\underline{k}) = 2.5\underline{i} + 2.5\underline{j} + 2\underline{k}$$

Closest approach on line DF is at position vector

$$\underline{v} = 5\underline{j} + 2\underline{k}$$

and so the vector between these points is

$$\underline{v} - \underline{u} = (5\underline{j} + 2\underline{k}) - (2.5\underline{i} + 2.5\underline{j} + 2\underline{k}) = -2.5\underline{i} + 2.5\underline{j} + 0\underline{k}$$

Hence the closest distance between lines is

$$|\underline{v} - \underline{u}| = \sqrt{(-2.5)^2 + 2.5^2} = 2.5\sqrt{2}$$

4.3 PROJECT

Solution:

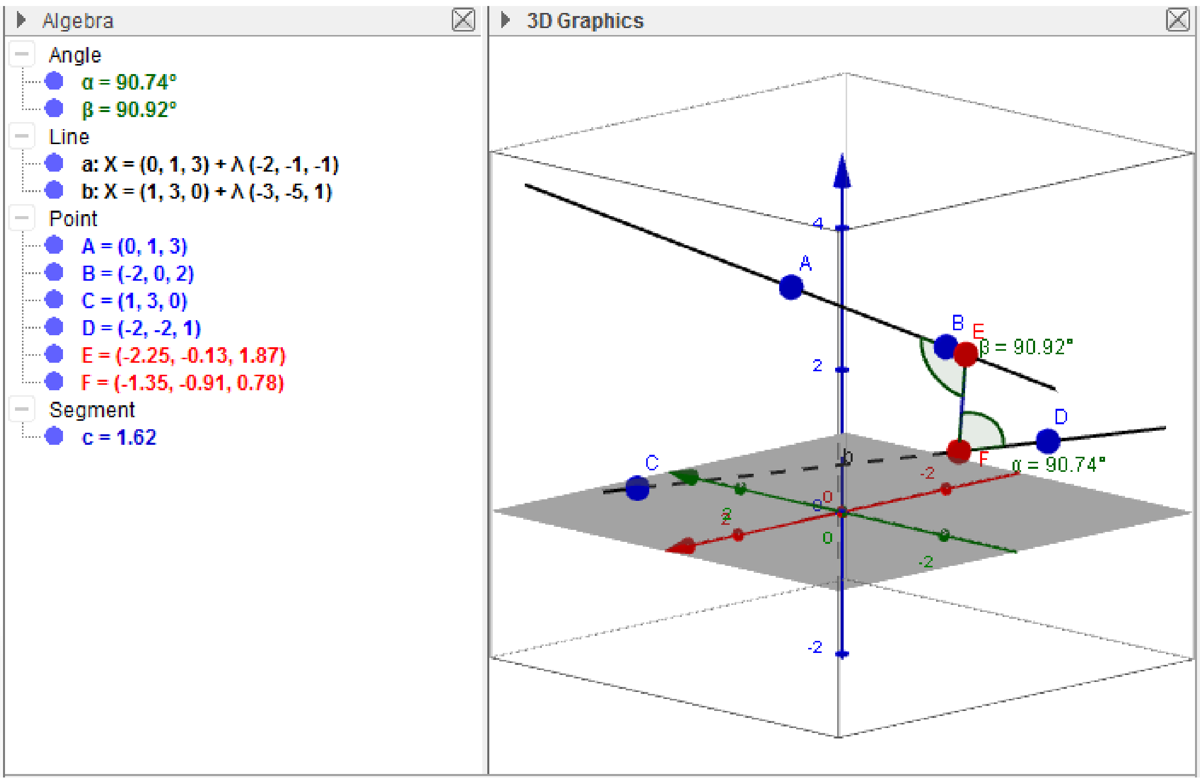

Figure 88

The points on the two lines are

$$\underline{x}_1 = \begin{pmatrix}0\\1\\3\end{pmatrix} + t\begin{pmatrix}-2\\-1\\-1\end{pmatrix} \text{ direction } \underline{e}_1 = \begin{pmatrix}-2\\-1\\-1\end{pmatrix} \text{ and } \underline{x}_2 = \begin{pmatrix}1\\3\\0\end{pmatrix} + s\begin{pmatrix}-3\\-5\\1\end{pmatrix} \text{ direction } \underline{e}_2 = \begin{pmatrix}-3\\-5\\1\end{pmatrix}$$

Solving $(\underline{x}_2 - \underline{x}_1).\underline{e}_1 = 0 and (\underline{x}_2 - \underline{x}_1).\underline{e}_2 = 0$ for t and s we get $t = \frac{75}{66} and s = \frac{43}{55}$ so that

$$\underline{x}_1 = \begin{pmatrix}0\\1\\3\end{pmatrix} + \frac{75}{66}\begin{pmatrix}-2\\-1\\-1\end{pmatrix} = \begin{pmatrix}-2.27\\-0.14\\1.86\end{pmatrix} \text{ and } \underline{x}_2 = \begin{pmatrix}1\\3\\0\end{pmatrix} + \frac{43}{55}\begin{pmatrix}-3\\-5\\1\end{pmatrix} = \begin{pmatrix}-1.35\\-0.91\\0.78\end{pmatrix}$$

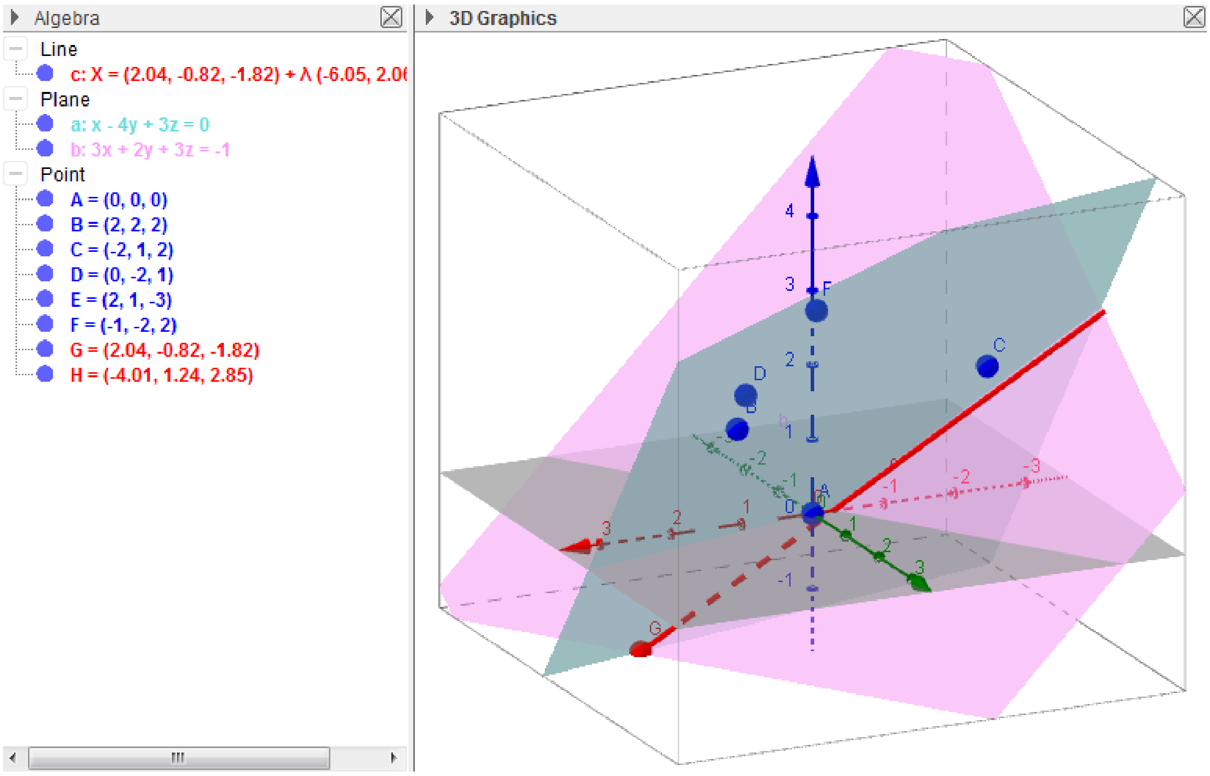

Figure 89

Scalar equations of the planes are

$$x - 4y + 3z = 0 \text{ and } 3x + 2y + 3z = -1 \qquad (*)$$

Vector equation of first plane is

$$\underline{P} = A + s_1(B - A) + s_2(C - A)$$

$$=> \underline{P} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} + s_1\begin{pmatrix}2\\2\\2\end{pmatrix} + s_2\begin{pmatrix}-2\\1\\2\end{pmatrix} = \begin{pmatrix}2s_1 - 2s_2\\2s_1 + s_2\\2s_1 + 2s_2\end{pmatrix} s_1, s_2 \in R$$

Then

$$x + z = 4s_1 and 2y - z = 2s_1 \Rightarrow x + z = 2(2y - z)$$
$$\Rightarrow x - 4y + 3z = 0$$

Similarly, for the second plane, in agreement with GeoGebra.

The equations (*) are 2 equations in 3 unknowns. Put z = t and solving for x and y in terms of t gives

$$x = \frac{-4-18t}{14} and y = \frac{-1+6t}{14} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} + \frac{t}{14}\begin{pmatrix} -18 \\ 6 \\ 14 \end{pmatrix}$$

You will probably need to plot this line to see that it matches your Geogebra line as it may look quite different algebraically.

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CHAPTER 4

Linear Algebra

Guide for a Problem

Linear Algebra

4.1. Vector arithmetic

LA1

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1 OBJECTIVES

LA1 – Vector arithmetic

The objectives of this part of the guide are addressed to the learning outcomes (LO) to be achieved and assessed in subjects, detailed in Table 31.

1. Understand the notion of a scalar quantity.
2. Understand the notion of a vector quantity
3. Understand the graphic representation of a vector in the plane
4. Define and comprehend how to carry out the multiplication by a scalar, addition, subtraction of vectors using the parallelogram rule.
5. Determine the unit vector in a specified direction
6. Demonstrate simple examples for using vectors for engineering purposes
7. Define and understand the component form for representation of vectors (two and three components only)
8. Develop the ability of the learners to design their own examples using vectors

Table 31 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- green = very important
- yellow = medium important
- red = less important

Table 31: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

Linear Algebra		Competencies							
		C1	C2	C3	C4	C5	C6	C7	C8
LA1	1. Vector arithmetic								
LA1/1	Distinguish between vector and scalar quantities	green	green	green	green	green	green	green	green
LA1/2	Understand and use vector notation	green	green	green	green	green	green	green	green
LA1/3	Represent a vector pictorially	red	red	green	green	yellow	yellow	yellow	green
LA1/4	Carry out addition and scalar multiplication and represent them pictorially	red	red	green	green	yellow	yellow	yellow	green
LA1/5	Determine the unit vector in a specified direction	yellow	yellow	green	green	yellow	yellow	yellow	green
LA1/6	Represent a vector in component form (two and three components only)	green	green	green	green	green	green	green	green

2 CONTENTS

In this guide, we will use the following type of vector notation: $\vec{a}$ and $\overrightarrow{AB}$, where the second notation clearly defines the beginning and end as well as the direction of the vector.

The content includes basic knowledge of vectors, their presentation pictorially and by the coordinates in 2– and 3–dimensional cases only. The selected problems are oriented for engineering students. The content covers all points, included in LA1, Core Level 1:

1. Distinguish between vector and scalar quantities.
2. Understand and use vector notation.
3. Represent a vector pictorially.
4. Carry out addition and scalar multiplication and represent them pictorially.
5. Determine the unit vector in a specified direction.
6. Represent a vector in component form (two and three components only).

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 32.

Table 32: Learning outcomes involved in this assessment activity.

Linear Algebra	
LA1	1. Vector arithmetic
LA1/1	Distinguish between vector and scalar quantities
LA1/2	Understand and use vector notation
LA1/3	Represent a vector pictorially
LA1/4	Carry out addition and scalar multiplication and represent them pictorially
LA1/5	Determine the unit vector in a specified direction
LA1/6	Represent a vector in component form (two and three components only)

3.1 MULTIPLE-CHOICE QUESTIONS

1. Which of these is scalar quantity?
   i) – 5.2
   ii) 15 km
   iii) A positive number
   iv) All are true
2. What of these could not be considered as a vector quantity?
   i) Two forces acting on a point object
   ii) Runner running northeast at a speed of 7 km/h
   iii) Position of an object
   iv) Two boys running straight towards with given speed from two points that are 100 m apart.
3. Vector $\vec{b}$ has length 1 and the oposite direction as vector $\vec{a} = (12, -5)$. The coordinates of vector $\vec{b}$ are:
   i) $\vec{b} = (-12,5)$
   ii) $\vec{b} = \left(\frac{12}{\sqrt{12}}, \frac{-5}{\sqrt{5}}\right)$
   iii) $\vec{b} = \left(-\frac{12}{13}, \frac{5}{13}\right)$
   iv) $\vec{b} = (5, -12)$
4. Two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are acting on a body, placed at the origin point $O(0,0,0)$. Given vectors $\overrightarrow{F_1} = \overrightarrow{OA} = (1,2,3)$ and $\overrightarrow{F_2} = \overrightarrow{OB} = (4, -1,3)$, then the half of the resultant force $\overrightarrow{AB}$ is:
   i) $\frac{1}{2}\overrightarrow{AB} = \left(-1, \frac{5}{2}, 1\right)$
   ii) $\frac{1}{2}\overrightarrow{AB} = \left(\frac{3}{2}, -\frac{3}{2}, 0\right)$
   iii) $\frac{1}{2}\overrightarrow{AB} = \left(\frac{1}{2}, \frac{7}{2}, \frac{3}{2}\right)$
   iv) $\frac{1}{2}\overrightarrow{AB} = \left(-\frac{3}{2}, \frac{3}{2}, 0\right)$

LO: LA1.

3.2 QUESTION 1

1. A car goes 5.0 km east, 3.0 km south, 2.0 km west, and 1.0 km north. If the starting point has coordinates (0,0) and the unit vectors are i = 1 km, j = 1 km, represent the car moving using planar vectors. Draw the resultant vector and find the coordinates of the endpoint and the angle the resultant concludes with the $\overrightarrow{Ox}$ axis.

   LO: LA1.

2. A car whose weight is w is on a ramp which is inclined to the horizontal by an angle θ. Find how large a perpendicular force must the ramp withstand so it is not to break under the car weight.

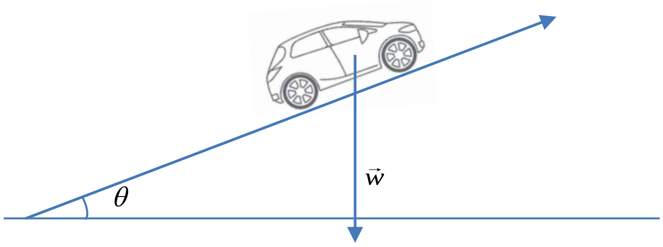

Figure 90

LO: LA1.

3.3 QUESTION 2

A robot starts from the beginning of the Cartesian coordinate system and moves 2 m at an acute angle with Ox. It then turns left at an obtuse angle and moves to a distance of 4 m. If the coordinates of its final position are $(-1{,}5)$, determine the resultant vector, the trajectory and the angles of motion of the robot.

Remark. When appropriate, use mathematical software.

LO: LA1, AC9.

4 SOLUTIONS

4.1 MULTIPLE-CHOICE QUESTIONS

1. Which of these is scalar quantity?

   i) – 5.2

   ii) 15 km

   iii) A positive number

   iv) All are true

   Solution:

   A scalar quantity has only a magnitude. It may have units after it, e.g. temperature=-10 C^o.

   The right answer is iv).

2. What of these could not be considered as a vector quantity?

   i) Two forces acting on a point object

   ii) Runner running northeast at a speed of 7 km/h

   iii) Position of an object

   iv) Two boys running straight towards with given speeds from two points that are 100 m apart.

   Solution:

   A vector quantity has both magnitude and direction. Case iii) has no direction.

   The right answer is iii).

3. Vector $\vec{b}$ has length 1 and the oposite direction as vector $\vec{a} = (12, -5)$. The coordinates of vector $\vec{b}$ are:

   i) $\vec{b} = (-12,5)$

   ii) $\vec{b} = (5, -12)$

   iii) $\vec{b} = \left(-\frac{12}{13}, \frac{5}{13}\right)$

   iv) $\vec{b} = \left(\frac{12}{\sqrt{12}}, \frac{-5}{\sqrt{5}}\right)$

   Solution:

   The length of a vector is the distance between the initial and tip points. For vector $\vec{a} = (12, -5)$ we apply the formula $|\vec{a}| = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$. The oposite of $-\vec{a} = -(12, -5) = (-12,5)$. Normalizing this to obtain a length 1, we obtain the answer $\vec{b} = \left(-\frac{12}{13}, \frac{5}{13}\right)$.

   The right answer is iii).

4. Two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are acting on a body, placed at the origin $O(0,0,0)$. Given vectors $\overrightarrow{F_1} = \overrightarrow{OA} = (1,2,3)$ and $\overrightarrow{F_2} = \overrightarrow{OB} = (4,-1,3)$, then the half of the resultant force $\overrightarrow{AB}$ is:
   i) $\frac{1}{2}\overrightarrow{AB} = \left(-1, \frac{5}{2}, 1\right)$
   ii) $\frac{1}{2}\overrightarrow{AB} = \left(\frac{3}{2}, -\frac{3}{2}, 0\right)$
   iii) $\frac{1}{2}\overrightarrow{AB} = \left(\frac{1}{2}, \frac{7}{2}, \frac{3}{2}\right)$
   iv) $\frac{1}{2}\overrightarrow{AB} = \left(-\frac{3}{2}, \frac{3}{2}, 0\right)$

   Solution:

   For the resultant force vector we get: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (4,-1,3) - (1,2,3) = (3,-3,0)$. Then for a half vector we obtain: $\frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(3,-3,0) = \left(\frac{3}{2}, \frac{-3}{2}, 0\right)$.

   The right answer is ii).

4.2 Question 1

1. A car goes 5.0 km east, 3.0 km south, 2.0 km west, and 1.0 km north. If the starting point has coordinates (0,0) and the unit vectors are i = 1 km, j = 1 km, represent the car moving using planar vectors. Draw the resultant vector and find the coordinates of the endpoint and the angle the resultant concludes with the $\overrightarrow{Ox}$ axis.

   Solution:

   Let us denote the sought coordinates of the endpoint by(x, y). The horizontal moving is 5 km to east to point (5,0). Then the car goes to south vertically down to (5, -3), back to west horizontally to point (3,-3) and finally to north to the endpoint (x, y) =(3, -2). Figure 2 shows the displacement vectors and the resultant vector with the endpoint.

   The angle α between $\overrightarrow{Ox}$ and resultant $\vec{R}$ could be found from the Pytagorean theorem, using the legs x, y, i.e. $\mathrm{Tan}\alpha = \frac{y}{x} = -\frac{2}{3}$, so $\alpha = \mathrm{ArcTan}\left(-\frac{2}{3}\right) = -\mathrm{ArcTan}\left(\frac{2}{3}\right) \approx 0.588$.

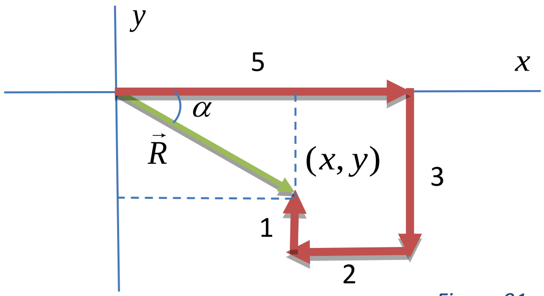

Figure 91

Answer: The coordinates of the endpoint are $(x, y) = (3, -2)$. The angle is approximately 0.588.

2. A car whose weight is w is on a ramp which is inclined to the horizontal by an angle θ. Find how large a perpendicular force must the ramp withstand so it is not to break under the car weight.

   Solution:

   The car's weight is a force $\vec{w}$ that pulls straight down on the car. Consider the perpendicular $\vec{b}$ and the other component $\vec{a}$ of the weight $\vec{w}$ in the inclined ramp (see Figure 3). By the theorem for angles with mutually perpendicular shoulders, the angle between $\vec{b}$ and $\vec{w}$ is equal to θ. This way $b = w.cos(\theta)$.

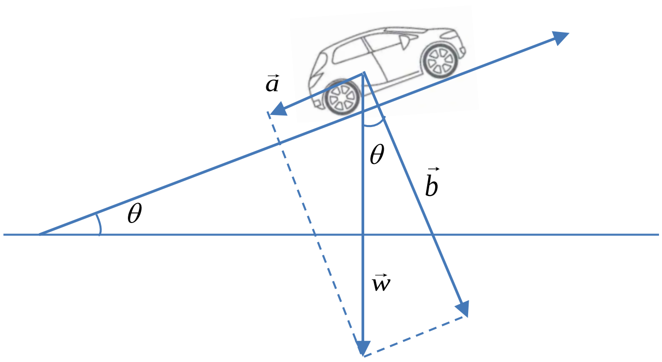

Figure 92

   Interpretation of the result: The rump must balance the force component $\vec{b}$so that the ramp will not crash under the weight of the car.

   Answer: The perpendicular is: $b = w.cos(\theta)$

4.3 QUESTION 2

A robot starts from the beginning of the Cartesian coordinate system and moves 2 m at an acute angle with Ox. It then turns left at an obtuse angle and moves to a distance of 4 m. If the coordinates of its final position are $(-1{,}5)$, determine the resultant vector, the trajectory and the angles of motion of the robot.

Remark. When appropriate, use mathematical software.

Solution:

Let (x_1, y_1) are coordinates of the first displacement of the robot and $(x_2, y_2) = (-1{,}5)$ are the coordinates of its final position (see Figure 4). We denote by θ_1, θ_2 the acute and obtuse angles, respectively. The resultant vector $\vec{R}$ is a hypotenuse in a Pythagorean triangle with legs (x_2, y_2), from where we find its length $|\vec{R}| = \sqrt{(-1)^2 + 5^2} = \sqrt{26}$. From the same triangle $\tan(\theta_3) = \frac{5}{|-1|} = 5$, so $\theta_3 = \arctan(5) \approx 1.3734$, or $\theta_3 = 78.69^o$. This way we have for the angle of the final displacement $\measuredangle(Ox, \vec{R}) = 180^o$-$78.69^o$=$101.31^o$.

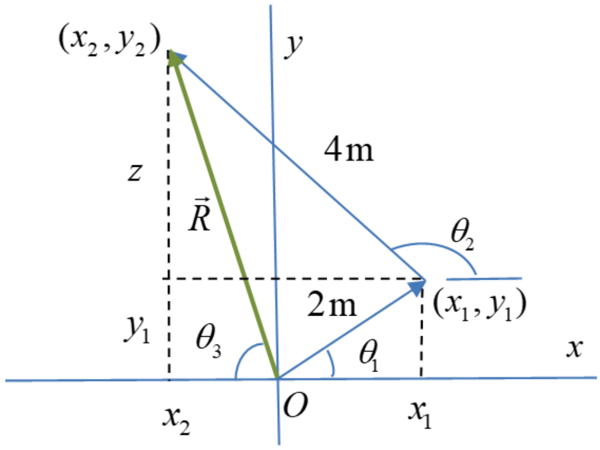

Figure 93

We need to find the coordinates of the first displacement of the robot (x_1, y_1). For this aim we use the Pythagorean triangles with hypotenuses 2 and 4. We obtain the following nonlinear algebraic system of equations:

$$\left| \begin{array}{c} x_1^2 + y_1^2 = 4 \\ (x_1 - x_2)^2 + z^2 = 16, \\ z = y_2 - y_1 \end{array} \right.$$

where $x_2 = -1, y_2 = 5$. After some simplification, we reduce this system to the quadratic equation

$$x_1^2 + \left(\frac{d - 2x_1x_2}{2y_2}\right)^2 = 4, \qquad (1)$$

where $d = x_2^2 + y_2^2 - 12, y_1 = \frac{d-2x_1x_2}{2y_2}$. In our case, only the positive solution of (1) is essential. Replacing the given coordinates $x_2 = -1, y_2 = 5$ and solving (1) we find

$x_1 \approx 1.16m, y_1 \approx 1.63m, z = 3.37m.$

For the first angle we have: $sin\,\theta_1 = \frac{y_1}{2}, \theta_1 = arcsin\left(\frac{y_1}{2}\right), \theta_1 = 54.66^o$.

For the obtise angle $sin(\pi - \theta_2) = \frac{z}{4}, \pi - \theta_2 = arcsin\left(\frac{z}{4}\right), \theta_2 = 122.63^o$.

Solution using software

The solution by Wolfram Mathematica is given by the following code with results:

```
In[2]:= R = √26.
        θ3 = ArcTan[5.]; θ3 = 180 / π * θ3; θ3 = 180 - θ3

Out[2]= 5.09902

Out[3]= 101.31
```

```
In[1]:=
x2 = -1; y2 = 5; d = x2^2 + y2^2 - 12.
w = Solve[x1^2 + ((d - 2 x1 * x2)/(2 y2))^2 == 4, x1]
x1 = w[[2,1,2]]
y1 = (d - 2 x1 * x2)/(2 y2)
z = 5 - y1
Out[1]= 14.
Out[2]= {{x1 -> -1.69542}, {x1 -> 1.15696}}
Out[3]= 1.15696
Out[4]= 1.63139
Out[5]= 3.36861

In[6]:= t = ArcSin[y1/2]; θ = 180 t/π
t1 = ArcSin[z/4]; θ1 = π - t1; 180 * t1 / π;
θ2 = 180 - 180 t1/π
Out[6]= 54.6563
Out[8]= 122.632
```

Answer: $x_1 \approx 1.16m$, $y_1 \approx 1.63m$, $\theta_1 \approx 54.56^o$, $\theta_2 \approx 122.63^o$, $|\vec{R}| = \sqrt{26}m \approx 5.1m$, $\measuredangle(Ox, \vec{R}) = 101.31^o$

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[4] Kulina, H., Gocheva-Ilieva, S., Voynikova, D., Atanasova, P., Ivanov, A., Iliev, A. (2018). Integrating of Competences in Mathematics through Software - Case Study, In *Proceedings of the 2018 IEEE Global Engineering Education Conference (EDUCON)*, 17-20 April 2018, Tenerife, Spain, IEEE, pp. 1586 – 1590. https://doi.org/10.1109/EDUCON.2018.8363424

Guide for a Problem

Linear Algebra

4.2. Vector algebra and applications

LA2, LA3, LA4

Fatih Yilmaz

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 34,to be achieved with the development of this activity guide are the following: As a result of learning this material you should be able to:

Solution of Simultaneous Linear Equations:

- To represent a system of linear equations in matrix form
- To understand in homogeneous linear systems
- To recognise the different possibilities for the solution of a system of linear equations
- To use the inverse matrix to find the solution
- To understand relationship between the rank of the coefficient matrix and the augmented matrix of a linear system
- To use appropriate software to solve simultaneous linear equations

Vector Algebra and Applications:

- To solve simple problems in geometry using vectors
- To define the scalar product of two vectors and use it in simple applications
- To understand the geometric interpretation of the scalar product
- To define the vector product of two vectors and use it in simple applications
- To define the scalar triple product of three vectors and use it in simple applications
- To understand the geometric interpretation of the scalar triple product

Table 33 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

= very important

= medium important

= less important

Table 33: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Solution of simultaneous linear equations									
LA41	Represent a system of linear equations in matrix form								
LA42	Understand how the general solution of an inhomogeneous linear system of m equations in n								

	unknowns is obtained from the solution of the homogeneous system and a particular solution
LA43	Recognise the different possibilities for the solution of a system of linear equations
LA44	Give a geometrical interpretation of the solution of a system of linear equations
LA45	Understand how and why the rank of the coefficient matrix and the augmented matrix of a linear system can be used to analyse its solution
LA46	Use the inverse matrix to find the solution of 3 simultaneous linear equations when possible
LA47	Understand the term 'ill-conditioned'
LA48	Apply the Gauss elimination method and recognise when it fails
LA49	Understand the Gauss-Jordan variation
LA50	Use appropriate software to solve simultaneous linear equations
Vector algebra and applications	
LA22	Solve simple problems in geometry using vectors
LA23	Solve simple problems using the component form (for example, in mechanics)
LA24	Define the scalar product of two vectors and use it in simple applications
LA25	Understand the geometric interpretation of the scalar product
LA26	Define the vector product of two vectors and use it in simple applications
LA27	Understand the geometric interpretation of the vector product
LA28	Define the scalar triple product of three vectors and use it in simple applications

LA29	Understand the geometric interpretation of the scalar triple product								

2 CONTENTS

1. The notion of vector and its geometry.
2. The definitions scalar product, vector product, and scalar triple product and their connectives.
3. The matrix representation of system of linear equations.
4. The geometric interpretation of the solution of a system of linear equation.
5. Gauss elimination method and Gauss Jordan variation.
6. The use of inverse matrix in solution of linear equations.

3 TEST

The time to solve this exam will be 100 minutes (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 34.

Table 34: Learning outcomes(LO)involved in this assessment activity.

Linear Algebra	
LA2	2. Vector Algebra and Applications
LA22	Solve simple problems in geometry using vectors
LA23	Solve simple problems using the component form (for example, in mechanics)
LA24	Define the scalar product of two vectors and use it in simple applications
LA25	Understand the geometric interpretation of the scalar product
LA26	Define the vector product of two vectors and use it in simple applications
LA27	Understand the geometric interpretation of the vector product
LA28	Define the scalar triple product of three vectors and use it in simple applications
LA29	Understand the geometric interpretation of the scalar triple product
LA4	3. Solution of Simultaneous Linear Equations
LA41	Represent a system of linear equations in matrix form
LA42	Understand how the general solution of an inhomogeneous linear system of m equations in n unknowns is obtained from the solution of the homogeneous system and a particular solution
LA43	Recognise the different possibilities for the solution of a system of linear equations
LA44	Give a geometrical interpretation of the solution of a system of linear equations
LA45	Understand how and why the rank of the coefficient matrix and the augmented matrix of a linear system can be used to analyze its solution
LA46	Use the inverse matrix to find the solution of 3 simultaneous linear equations when possible
LA47	Understand the term 'ill-conditioned'
LA48	Apply the Gauss elimination method and recognise when it fails
LA49	Understand the Gauss-Jordan variation
LA50	Use appropriate software to solve simultaneous linear equations

3.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test.

1. The angle between $u = [3,2], v = [1,5]$ is
   i) 30
   ii) 45
   iii) 60
2. The vector product of the vectors $u = 4i + 3j + 7k$ and $v = 2i + 5j + 4k$ is
   i) $-23i - 2j + 14k$
   ii) $-23i + 2j + 14k$
   iii) $-23i - 2j - 14k$
3. In the diagram M is the midpoint of VW. Please find the expression $\overrightarrow{OV}, \overrightarrow{VW}, \overrightarrow{OM}$ in terms of e, f, g.

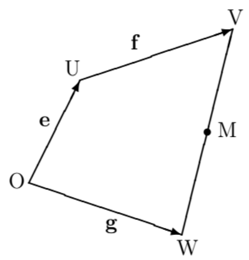

Figure 94

   i) $\overrightarrow{OV}: e + f, \overrightarrow{VW}: f - e + g, \overrightarrow{OM}: \frac{1}{2}(e + f + g)$
   ii) $\overrightarrow{OV}: e - f, \overrightarrow{VW}: f - e - g, \overrightarrow{OM}: \frac{1}{2}(e + f + g)$
   iii) $\overrightarrow{OV}: e + f, \overrightarrow{VW}: f - e + g, \overrightarrow{OM}: \frac{1}{2}(e - f + g)$
4. Let consider the vectors: $u = (2,1,3), v = (1,0,1), w = (1,2,-3)$. Determine the volume of the parallelopiped formed by the given vectors.
   i) 2
   ii) 4
   iii) 6
5. Get a vector which is perpendicular to vectors $u = (1,5,-3) \text{and} v = (2,-1,0)$.
   i) $(-3,-6,-11)$
   ii) $(-2,-4,-11)$
   iii) $(-3,-6,-10)$
6. $\left.\begin{matrix} -x + 3y = 0 \\ 2x + 4y = 8 \end{matrix}\right\}$ choose the representation of the system of linear equations in matrix form.
   $i)$ $\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$
   ii) $\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$
   iii) $\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

7. $\left.\begin{array}{l} x + 2y = 4 \\ 6x + 5y = 6 \end{array}\right\}$

Solve the system of linear equations finding a pair (x_0, y_0) which satisfies both equations.

i) $\left(\frac{-8}{7}, \frac{18}{7}\right)$

ii) $\left(\frac{-8}{7}, \frac{8}{7}\right)$

iii) $\left(\frac{8}{7}, \frac{18}{7}\right)$

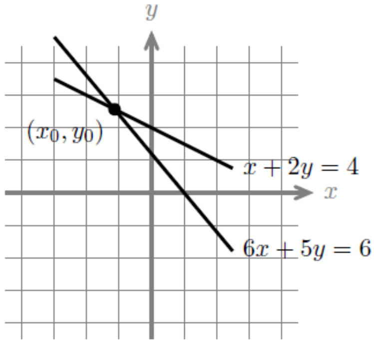

Figure 95

8. $\left.\begin{array}{l} x + 2y = 4 \\ x + 2y = 0 \end{array}\right\}$

Solve the system of linear equations finding a pair (x_0, y_0) which satisfies both equations.

i) $(2,1)$

ii) $(2, -1)$

iii) No solution

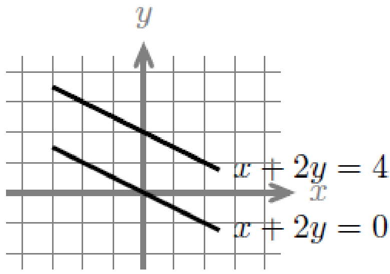

Figure 96

3.2 QUESTIONS 1

1. Please find an expression for $\overrightarrow{AM}$ in terms of p and q. Note that M is the mid point of $\overrightarrow{BC}$.

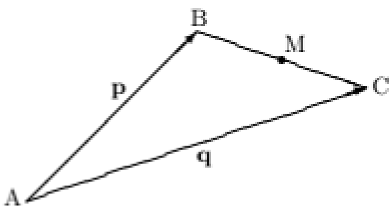

Figure 97

2. If $a$is a vector and h is a scalar, show that $|ha| = |h||a|$.
3. Let $u = [1,4], v = [2, -3]$ then $u \cdot v = ?$
4. Let $u = [1,1,2], v = [1,1,-1]$ then $u \cdot v = ?$
5. Find the volume of the parallelepiped with edges $u = i + 2j + 3k, v = i + j + 2k$and $w = i + 3j + 4k$.
6. For the following equation system:

7. $$\left.\begin{array}{l} 2x+4y=a \\ 3x+6y=b \\ 2x+9y=b-3 \\ x+2y=a-2 \end{array}\right\}$$

8. Please find for which values of a and b, the system has solution.

9. For the following equation system:

$$\left.\begin{array}{l} x+y-z=1 \\ 2x+3y+az=3 \\ x+ay+3z=2 \end{array}\right\}$$

Please find for which values of *a* and *b*, the system has solution.

4 QUESTIONS 2

1. Let's take a look at any part of a city where there is no traffic light at the intersections. Let us illustrate the roads of the intersections and the *amounts* of vehicles passing by the following diagram. As it is seen in the diagram, let the number of vehicles passing through time in some ways is known. Please give mathematically model the flow of traffic in the region by using linear equation system.

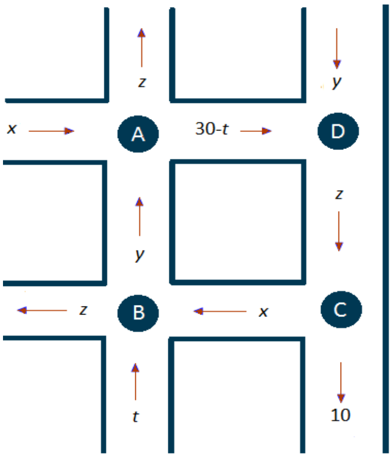

Figure 98

2. The management of Alexander Rent-A-Car plans to expand its fleet of rental cars for the next quarter by purchasing compact and full-size cars. The average cost of a compact car is \$10,000, and the average cost of a full-size car is \$24,000.

Figure 99

   a) If a total of 800 cars is to be purchased with a budget of \$12 million, how many cars of each size will be acquired?
   b) If the predicted demand calls for a total purchase of 1000 cars with a budget of \$14 million, how many cars of each type will be acquired?

5 SOLUTION

5.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test.

1. The angle between $u = [3,2], v = [1,5]$ is

   i) 30

   ii) 45

   iii) 60

   Solution:

   If $u = [3,2], v = [1,5] \Rightarrow u \cdot v = 3 \cdot 1 + 2 \cdot 5 = 13, |u| = \sqrt{13}, |v| = \sqrt{26}$.

   Substituting these into the formula $uv = |u||v| \cos\theta$;

$$13 = \sqrt{13}\sqrt{26} \cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}; i.e. \theta = 45°.$$

   The solution is ii).

2. The vector product of the vectors $u = 4i + 3j + 7k$ and $v = 2i + 5j + 4k$ is

   i) $-23i - 2j + 14k$

   ii) $-23i + 2j + 14k$

   iii) $-23i - 2j - 14k$

   Solution:

   We write down a determinant, which is an array of numbers: in the first row we write the three unit vectors *i, j* and *k*. In the second and third rows we write the three components of *u* and *v* respectively:

$$u \times v = \begin{vmatrix} i & j & k \\ 4 & 3 & 7 \\ 2 & 5 & 4 \end{vmatrix}.$$

   By considering the similar way in Sarrus Rule, we get

$$u \times v = \begin{vmatrix} i & j & k \\ 4 & 3 & 7 \\ 2 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 7 \\ 5 & 4 \end{vmatrix} i + \begin{vmatrix} 4 & 7 \\ 2 & 4 \end{vmatrix} j + \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} k = -23i - 2j + 14k.$$

   The solution is i).

3. In the diagram M is the midpoint of VW. Please find the expression $\overrightarrow{OV}, \overrightarrow{VW}, \overrightarrow{OM}$ in terms of e, f, g.

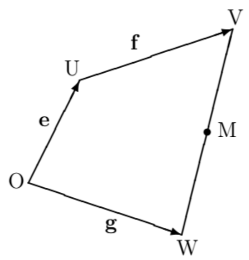

Figure 100

i) $\overrightarrow{OV}: e + f, \overrightarrow{VW}: f - e + g, \overrightarrow{OM}: \frac{1}{2}(e + f + g).$

ii) $\overrightarrow{OV}: e - f, \overrightarrow{VW}: f - e - g, \overrightarrow{OM}: \frac{1}{2}(e + f + g)$

iii) $\overrightarrow{OV}: e + f, \overrightarrow{VW}: -f - e + g, \overrightarrow{OM}: \frac{1}{2}(e + f + g)$

Solution:

By exploiting elementary row operations, we get:

$$\begin{aligned} \overrightarrow{OV} &= e + f \\ \overrightarrow{VW} &= -f - e + g \\ \overrightarrow{OM} &= e + f + \frac{1}{2}[-f - e + g] = \frac{1}{2}e + \frac{1}{2}f + \frac{1}{2}g \end{aligned}$$

The solution is iii).

4. Let consider the vectors: $u = (2,1,3), v = (1,0,1), w = (1,2,-3)$. Determine the volume of the parallelopiped formed by the given vectors

i) 2

ii) 4

ii) 6

Solution:

The volume is equal to determinant of triple product of the vectors, i.e:

$$\langle u, v \times w \rangle = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & -3 \end{vmatrix} = 6.$$

The solution is iii).

5. Get a vector which is perpendicular to vectors $u = (1,5,-3) \text{and} v = (2,-1,0)$.

i) $(-3,-6,-11)$

ii) $(-2,-4,-11)$

ii) $(-3,-6,-10)$

Solution:

Since the cross product of two non-parallel vectors is a vector that is perpendicular to both of them, we have compute the cross product:

$$w = u \times v = \begin{vmatrix} \overrightarrow{e_1} & \overrightarrow{e_1} & \overrightarrow{e_1} \\ 1 & 5 & -3 \\ 2 & -1 & 0 \end{vmatrix} = (-3,-6,-11).$$

The solution is i).

6. $\left.\begin{array}{r}-x+3y=0\\2x+4y=8\end{array}\right\}$ choose the representation of the system of linear equations in matrix form.

i) $\begin{pmatrix}-1 & 3\\ 2 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 8\end{pmatrix}$

ii) $\begin{pmatrix}-1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 8\end{pmatrix}$

ii) $\begin{pmatrix}-1 & 3\\ 2 & 4\end{pmatrix}\begin{pmatrix}0\\ 8\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$

Solution:

The solution is i).

7. $\left.\begin{array}{r}x+2y=4\\6x+5y=6\end{array}\right\}$

Solve the system of linear equations finding a pair (x_0, y_0) which satisfies both equations.

i) $\left(\frac{-8}{7}, \frac{18}{7}\right)$

ii) $\left(\frac{-8}{7}, \frac{8}{7}\right)$

ii) $\left(\frac{8}{7}, \frac{18}{7}\right)$

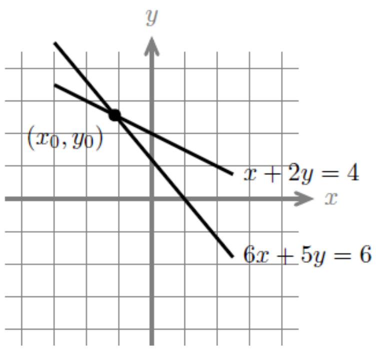

Figure 101

Solution: The solution is **i)**.

It is obvious that if two lines intersect at a point there is one solution.

8. $\left.\begin{array}{r}x+2y=4\\x+2y=0\end{array}\right\}$

Solve the system of linear equations finding a pair (x_0, y_0) which satisfies both equations.

i) $(2,1)$

ii) $(2,-1)$

iii) No solution

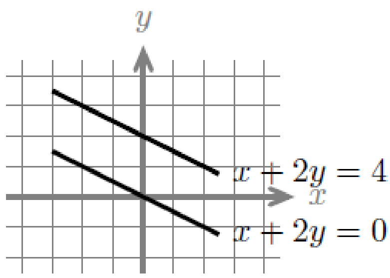

Figure 102

Solution:

The solution is iii).

It is known that if two lines are parallel (not the same), there is no solution.

5.2 QUESTIONS 1

1. Please find an expression for $\overrightarrow{AM}$ in terms of p and q. Note that M is the mid point of $\overrightarrow{BC}$.

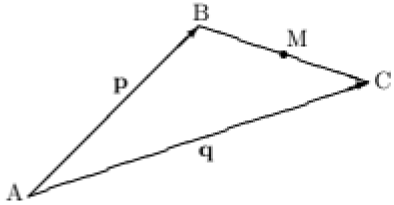

Figure 103

Solution:

The diagram shows that

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \boldsymbol{p} + \overrightarrow{BM}$$

and since M is the midpoint of BC, you can see that

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \boldsymbol{p} + \overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\boldsymbol{q} - \boldsymbol{p})$$

Substituting this into the previous equation:

$$\overrightarrow{AM} = \boldsymbol{p} + \frac{1}{2}(\boldsymbol{q} - \boldsymbol{p}) = \frac{1}{2}(\boldsymbol{q} + \boldsymbol{p})$$

2. If a is a vector and $h$is a scalar, showthat $|ha| = |h||a|$.

   Solution:

   Let $a = [x, y]$ then $ha = [hx, hy]$ and $|ha| = \sqrt{(hx)^2 + (hy)^2} = h\sqrt{x^2 + y^2} = |h||a|$.

3. Let $u = [1,4], v = [2, -3]$ then $u \cdot v =?$

   Solution:

$$\text{If} u = [1,4], v = [2, -3] \Rightarrow u \cdot v = (1 \times 2) + (4 \times -3) = -10$$

4. Let $u = [1,1,2], v = [1,1,-1]$ then $u \cdot v =?$

   Solution:

$$\text{If } u = [1,1,2], v = [1,1,-1] \Rightarrow u \cdot v = (1 \times 1) + (1 \times 1) + (2 \times -1) = 0$$

5. Find the volume of the parallelepiped with edges $u = i + 2j + 3k, v = i + j + 2k$ and $w = i + 3j + 4k$.

Solution:

Let's first evaluate the vector product $v \times w$. That is:

$$v \times w = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 3 & 4 \end{vmatrix} = -2i - 2j + 2k.$$

Then we need to find the scalar product of u and $v \times w$.

$$u \cdot (v \times w) = (i + 2\mathrm{j} + 3k) \cdot (-2i - 2j + 2k) = 0.$$

6. For the following equation system:

$$\left.\begin{array}{l} 2x + 4y = a \\ 3x + 6y = b \\ 2x + 9y = b - 3 \\ x + 2y = a - 2 \end{array}\right\}$$

Please find for which values of *a* and *b*, the system has solution.

Solution:

By exploiting elementary row operations, we get:

$$\begin{bmatrix} 2 & 4 & a \\ 3 & 6 & b \\ 2 & 9 & b-3 \\ 1 & 2 & a-2 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & \frac{7a-18}{5} \\ 0 & 1 & \frac{4-a}{5} \\ 0 & 0 & \frac{b-2a+4}{2} \\ 0 & 0 & \frac{4-a}{2} \end{bmatrix}$$

For the solution, it must be:

$$\frac{b-2a+4}{2} = 0 \text{and} \frac{4-a}{2} = 0.$$

That means, we find $a = 4 \text{and} b = 6$.

7. For the following equation system:

$$\left.\begin{array}{l} x + y - z = 1 \\ 2x + 3y + az = 3 \\ x + ay + 3z = 2 \end{array}\right\}$$

Please find for which values of a and b, the system has solution.

Solution:

By remebering the well-known property: the system which is the form $Ax = b$, to have solutions, it must be $\text{rank}[A{:}\, b] = \text{rank}[A]$. Namely;

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & a & 3 \\ 1 & a & 3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & a-1 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & 6-a-a^2 & 2-a \end{array}\right)$$

Here let's interpret the situation:

i) If $a = 2$: it will be $\text{rank}[A{:}\,b] = \text{rank}[A] = 2$. This means that the system will have infinite solutions depended on one parameter.

ii) If $a = -3$: it will be $\text{rank}[A{:}\,b] = 3$,$\text{rank}[A] = 2$. This means that the system will have no solutions.

iii) If $a \neq 2, a \neq -3$: it will be $\text{rank}[A{:}\,b] = \text{rank}[A] = 3$. This means that the system will have solutions which is $x = A^{-1}b$.

5.3 QUESTIONS 2

1. Let's take a look at any part of a city where there is no traffic light at the intersections. Let us illustrate the roads of the intersections and the amounts of vehicles passing by the following diagram. As it is seen in the diagram, let the number of vehicles passing through time in some ways is known. Please give mathematically model the flow of traffic in the region by using linear equation system.

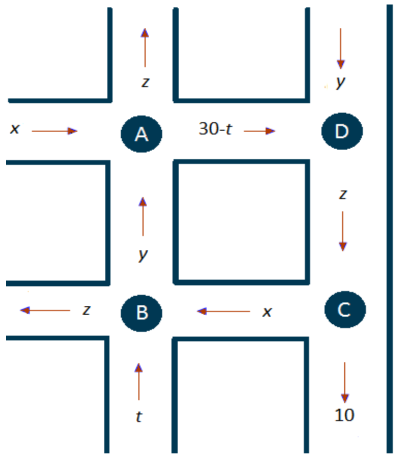

Figure 104

Solution:

Considering that the number of vehicles entering and leaving to the A, B, C and D junctions will be equal, we get:

$$\begin{gathered} x + y = z + 30 - t \\ 30 - t + y = z \\ z = x + 10 \\ x + t = y + z \end{gathered}$$

This system models the flow of traffic shown in the diagram above. After some operations, we get:

$$\begin{gathered} x + y - z + t = 30 \\ -y + z + t = 30 \\ -x + z = 10 \\ x - y - z + t = 0 \end{gathered}$$

Writing the equation system in matrix form,

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}, b = \begin{bmatrix} 30 \\ 30 \\ 10 \\ 0 \end{bmatrix}$$

Which can be represented by $AX = b$. The determinant of the coeffiecents matrix, $\det(A) = -2 \neq 0$ so the system has solution. By matrix method, we can rewrite:

$$[A:b] = \begin{bmatrix} 1 & 1 & -1 & 1 & :30 \\ 0 & -1 & 1 & 1 & :30 \\ -1 & 0 & 1 & 0 & :10 \\ 1 & -1 & -1 & 1 & :\ 0 \end{bmatrix}$$

and $rank[A:b] = rank(A) = 4$. So the system has aunique solution. Using the inverse matrix method:

$$A^{-1} = \begin{bmatrix} 0 & 1 & -2 & -1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & -1 \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

The solution is:

$$X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = A^{-1}b = \begin{bmatrix} 10 \\ 15 \\ 20 \\ 25 \end{bmatrix}$$

i.e., $x = 10, y = 15, z = 20, t = 25$. By writing these at the traffic flow chart :

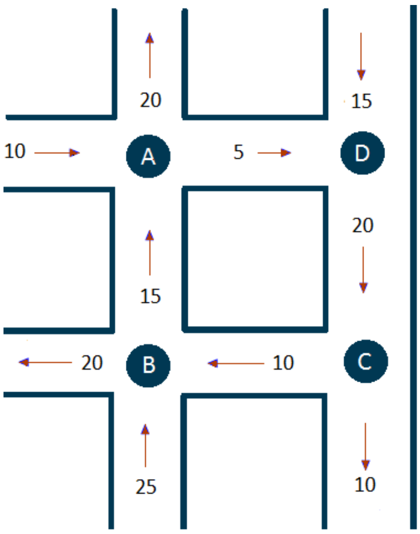

Figure 105

The diagram shows the number of vehicles passing through all roads in unit time. Thus, we get the necessary information to place traffic light at the intersections.

2. The management of Alexander Rent-A-Car plans to expand its fleet of rental cars for the next quarter by purchasing compact and full-size cars. The average cost of a compact car is $10,000, and the average cost of a full-size car is $24,000.

Figure 106

a) If a total of 800 cars is to be purchased with a budget of $12 million, how many cars of each size will be acquired?

b) If the predicted demand calls for a total purchase of 1000 cars with a budget of $14 million, how many cars of each type will be acquired?

Solution:

a) Let x and y denote the number of compactand full size cars to be purchased. Futhermore, let n denote the total number of cars to be acquired and b the amount of money budgeted for the purchase of these cars. Then,

$$x + y = n$$
$$10000x + 24000y = b$$

This system is in two variables and may be written in the matrix form

$AX = B.$

where $A = \begin{bmatrix} 1 & 1 \\ 10000 & 24000 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} n \\ b \end{bmatrix}.$

Therefore $X = A^{-1}b$. The inverse of the matrix A is:

$A^{-1} = \frac{1}{14000}\begin{bmatrix} 24000 & -1 \\ -10000 & 1 \end{bmatrix}.$

Thus

$$X = A^{-1}B = \begin{bmatrix} \frac{12}{7} & \frac{-1}{14000} \\ \frac{-5}{7} & \frac{1}{14000} \end{bmatrix}\begin{bmatrix} 800 \\ 12000000 \end{bmatrix} = \begin{bmatrix} 514.3 \\ 285.7 \end{bmatrix}.$$

Therefore 514 compact cars and 286 full size cars will be acquired.

b) Here n= 1000 and b=14000000, so

$$X = A^{-1}B = \begin{bmatrix} \frac{12}{7} & \frac{-1}{14000} \\ \frac{-5}{7} & \frac{1}{14000} \end{bmatrix}\begin{bmatrix} 1000 \\ 14000000 \end{bmatrix} = \begin{bmatrix} 714.3 \\ 285.7 \end{bmatrix}.$$

Therefore, 714 compact cars and 286 full size cars will be purchaesd in this case.

6 BIBLIOGRAPHY

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Guide for a Problem

Linear Algebra

4.3. Matrices and determinants

LA3

Snezhana Gocheva-Ilieva

1 OBJECTIVES

The objectives of this part of the guide are addressed to the learning outcomes to be achieved and assessed in subjects, detailed in Table 35. The shows the list of the eight mathematical competencies that we measure with the proposed global test model, where

- ■ (G) = very important
- ■ (Y) = medium important
- ■ (R) = less important

Table 35: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

Linear Algebra 3		Competencies							
		C1	C2	C3	C4	C5	C6	C7	C8
LA3	3. Matrices and determinants								
LA3/1	Understand what is meant by a matrix	G	R	R	Y	G	R	R	G
LA3/2	Recall the basic terms associated with matrices (for example, diagonal, trace, square, triangular, identity)	G	Y	Y	Y	G	Y	Y	G
LA3/3	Obtain the transpose of a matrix	Y	Y	G	Y	R	Y	R	G
LA3/4	Determine any scalar multiple of a matrix	Y	Y	G	Y	R	Y	R	G
LA3/5	Recognise when two matrices can be added and find, where possible, their sum	G	G	G	Y	Y	Y	Y	G
LA3/6	Recognise when two matrices can be multiplied and find, where possible, their product	G	G	G	Y	Y	Y	Y	G
LA3/7	Calculate the determinant of 2 x 2 and 3 x 3 matrices	Y	G	G	G	R	Y	R	G
LA3/8	Understand the geometric interpretation of 2 x 2 and 3 x 3 determinants	G	Y	Y	G	Y	Y	Y	G
LA3/9	Use the elementary properties of determinants in their evaluation	G	G	G	Y	R	Y	R	G
LA3/10	State the criterion for a square matrix to have an inverse	G	Y	Y	R	R	Y	Y	G
LA3/11	Write down the inverse of a 2 x 2 matrix when it exists	G	Y	G	Y	Y	Y	Y	Y
LA3/12	Determine the inverse of a matrix, when it exists, using row operations	Y	G	G	Y	Y	Y	Y	Y
LA3/13	Calculate the rank of a matrix	Y	G	G	Y	Y	Y	Y	G
LA3/14	Use appropriate software to determine inverse matrices	Y	Y	Y	Y	Y	Y	Y	G

2 CONTENTS

The content is part from the linear algebra topic, concerning the repesentation of a matrix and basic matrix algebra operations. There are considered only matrices whose elements are real numbers.

We will use the following notations for matrices and matrix operations:

A, $A(n \times m)$	- Matrix with n rows and m columns
$\|A\|$ or $\det(A)$	- Determinant of a square matrix
A^T	- Transpose matrix of A
A^{-1}	- Inverse matrix of A

Content includes multiple choice questions, one question to solve and a small practical problem. They can be solved both by hand and using software when appropriate. The content of the proposed assessment is based on the following elements:

1. Representation of different type of matrices.
2. Algebraic operations with matrices: summation, subtraction, multiplication by a scalar, multiplication of matrices. Carry out these operations for 2 x 2 and 3 x 3 matrices.
3. Obtain the transpose of a matrix.
4. Calculate the determinant of 2 x 2 and 3 x 3 matrices.
5. Understand the geometric interpretation of 2 x 2 and 3 x 3 determinants.
6. Use the elementary properties of determinants in their evaluation.
7. Understand the case when the inverse matrix exists.
8. Use appropriate software to carry out algebraic operations, transpose matrix and determinants.
9. Find the inverse of a 2 x 2 matrix when it exists.
10. Calculate the rank of a matrix.
11. Use software to find inverse matrices and the rank of a matrix and to check the results.

3 TEST

The time to solve this exam will be 3 hours (approximately).
The learning outcomes to be achieved with this procedure are detailed in Table 36.

Table 36: Learning outcomes involved in this assessment activity.

Linear Algebra	
LA3	3. Matrices and determinants
LA31	Understand what is meant by a matrix
LA32	Recall the basic terms associated with matrices (for example, diagonal, trace, square, triangular, identity)
LA33	Obtain the transpose of a matrix
LA34	Determine any scalar multiple of a matrix
LA35	Recognise when two matrices can be added and find, where possible, their sum
LA36	Recognise when two matrices can be multiplied and find, where possible, their product
LA37	Calculate the determinant of 2 x 2 and 3 x 3 matrices
LA38	Understand the geometric interpretation of 2 x 2 and 3 x 3 determinants
LA39	Use the elementary properties of determinants in their evaluation
LA310	State the criterion for a square matrix to have an inverse
LA311	Write down the inverse of a 2 x 2 matrix when it exists
LA312	Determine the inverse of a matrix, when it exists, using row operations
LA313	Calculate the rank of a matrix
LA314	Use appropriate software to determine inverse matrices

3.1 Multiple-Choice Questions

Solve the following test.

1. Given the matrices $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ 5 & -1 & 4 \end{pmatrix}$, find the matrix $C = 2A - 3B$. The result is:

   i) $D = \begin{pmatrix} 10 & 11 & 13 \\ 12 & -10 & 14 \end{pmatrix}$

   ii) $D = \begin{pmatrix} 2 & -3 & -2 \\ -1 & 1 & -4 \end{pmatrix}$

   iii) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

   iv) $D = \begin{pmatrix} -3 & -4 & 6 \\ -11 & 1 & -4 \end{pmatrix}$

2. The determinant $|A|$ of matrix of $A = \begin{pmatrix} 5 & 3 \\ 2 & -4 \end{pmatrix}$ is equal to:

   i) -26

   ii) 7

   iii) -73

   iv) $|A|$ does not exits

3. If $R = U.V$, where $U = \begin{pmatrix} 2 & 3 \\ -2 & 4 \end{pmatrix}$, $V = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, R^{-1} is equal to:

   i) $R^{-1} = \begin{pmatrix} 9 & 6 \\ -2 & 8 \end{pmatrix}$

   ii) $R^{-1} = \begin{pmatrix} \frac{1}{14} & -\frac{2}{21} \\ \frac{1}{14} & \frac{1}{14} \end{pmatrix}$

   iii) $R^{-1} = \begin{pmatrix} 9 & 6 \\ -2 & 8 \end{pmatrix}$

   iv) R^{-1}does not exist

4. Apply the row operations to determine the rank of a matrix $Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{pmatrix}$. The rank of Q is:

   i) 1

   ii) 2

   iii) 3

   iv) 4

3.2 QUESTION 1

Using matrices and determinants, establish linear dependency or independency of each group of vectors. Find the dependence when it exists.

(a) $v1 = (2,3,-1,5), v2 = (-4,-6,2,-10)$

(b) $w1 = (0,2,-3), w2 = (1,-1,1), w3 = (2,-8,11)$.

3.3 QUESTION 2

Consider the electrical circuit given in Figure 1. The problem is to determine the currents in the branches, using Kirchhoff's laws. The parameters of the circuit elements are as follows:

$$R_1 = 45Om, R_2 = 15Om, R_3 = 45Om, R_4 = 75Om, E_1 = 60V, E_2 = 450V.$$

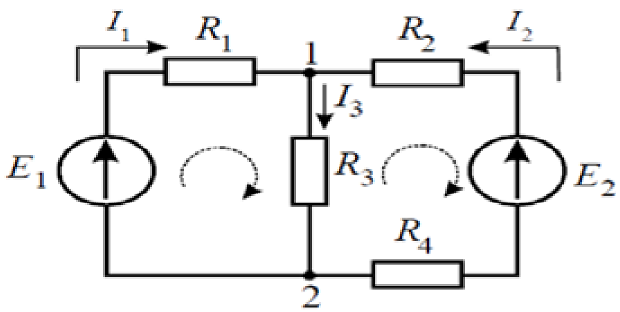

Figure 107. Electrical circuit diagram.

To solve this problem, use the system of equations obtained for currents I_1, I_2, I_3 in the two nodes 1, 2 according to the first and second Kirchhoff laws:

$$\left| \begin{array}{c} I_1 + I_2 - I_3 = 0 \\ I_1R_1 + I_3R_3 = E_1 \\ -I_2(R_2 + R_4) - I_3R_3 = -E_2 \end{array} \right.$$

The solution to this system can be found by determinants using the following Cramer's rule:

$$I_1 = \frac{\Delta_1}{\Delta}, I_2 = \frac{\Delta_2}{\Delta}, I_3 = \frac{\Delta_3}{\Delta},$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 45 & 0 & 45 \\ 0 & -90 & -45 \end{vmatrix}, \Delta_1 = \begin{vmatrix} 0 & 1 & -1 \\ 60 & 0 & 45 \\ -450 & -90 & -45 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & -1 \\ 45 & 60 & 45 \\ 0 & -450 & -45 \end{vmatrix}, \Delta_3 = \begin{vmatrix} 1 & 1 & 0 \\ 45 & 0 & 60 \\ 0 & -90 & -450 \end{vmatrix}$$

Determine the currents by calculating determinants and applying the Cramer's rule. When appropriate use mathematical software.

LO: LA3.

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. Given the matrices $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ 5 & -1 & 4 \end{pmatrix}$, find the matrix $C = 2A - 3B$. The result is:

   i) $C = \begin{pmatrix} 10 & 11 & 13 \\ 12 & -10 & 14 \end{pmatrix}$

   ii) $C = \begin{pmatrix} 2 & -3 & -2 \\ -1 & 1 & -4 \end{pmatrix}$

   iii) $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

   iv) $C = \begin{pmatrix} -3 & -4 & 6 \\ -11 & 1 & -4 \end{pmatrix}$

Solution 1:

First we find $C_1 = 2A = 2\begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 6 \\ 4 & -2 & 8 \end{pmatrix}$. The second term is $C_2 = 3B = 3\begin{pmatrix} 1 & 2 & 0 \\ 5 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 0 \\ 15 & -3 & 12 \end{pmatrix}$. The substraction gives the desired solution $C = C_1 - C_2 = \begin{pmatrix} 0 & 2 & 6 \\ 4 & -2 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 6 & 0 \\ 15 & -6 & 24 \end{pmatrix} = \begin{pmatrix} -3 & -4 & 6 \\ -11 & 1 & -4 \end{pmatrix}$

*The right answer is **iv***

Solution 2:

Using Wolfram Mathematica we obtain the same result:

```
In[1]:= A = (0  1 3)
            (2 -1 4); B = (1  2 0)
                          (5 -1 4);
        c1 = 2 A; MatrixForm[c1]
        c2 = 3 B; MatrixForm[c2]
        c = c1 - c2; MatrixForm[c]

Out[2]//MatrixForm=
        (0  2 6)
        (4 -2 8)

Out[3]//MatrixForm=
        (3   6  0)
        (15 -3 12)

Out[4]//MatrixForm=
        (-3  -4  6)
        (-11  1 -4)
```

2. The determinant $|A|$ of matrix of $A = \begin{pmatrix} 5 & 3 \\ 2 & -4 \end{pmatrix}$ is equal to:

   i) -26

   ii) 7

   iii) -73

   iv) $|A|$ does not exits

Solution:

The determinant of a square matrix is a number. In the case of a 2×2 matrix this formula is: $|A| = \left|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right| = a_{11}a_{22} - a_{12}a_{21}$. For our matrix we obtain: $|A| = \left|\begin{pmatrix} 5 & 3 \\ 2 & -4 \end{pmatrix}\right| =$ $5.(-4) - 2.3 = -20 - 6 = -26$

The right answer is i).

3. If $R = U.V$, where $U = \begin{pmatrix} 2 & 3 \\ -2 & 4 \end{pmatrix}$, $V = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, R^{-1} is equal to:

i) $R^{-1} = \begin{pmatrix} 9 & 6 \\ -2 & 8 \end{pmatrix}$

ii) $R^{-1} = \begin{pmatrix} \frac{1}{14} & -\frac{2}{21} \\ \frac{1}{14} & \frac{1}{14} \end{pmatrix}$

iii) $R^{-1} = \begin{pmatrix} 9 & 6 \\ -2 & 8 \end{pmatrix}$

iv) R^{-1}does not exist

Solution 1:

Given $U = \begin{pmatrix} 2 & 3 \\ -2 & 4 \end{pmatrix}$, $V = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ we carry out the multiplication "row by column" to obtain $R = \begin{pmatrix} 2*3+3*0 & 2*1+3*2 \\ -2*3+4*0 & -2*1+4*2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ -6 & 6 \end{pmatrix}$. To find the inverse matrix we need to know it's a determinant. In our case $det(R) = 36 + 48 = 84$. Since it is different from 0, the inverse matrix of R exists. By the theory, if $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse matrix is $R^{-1} = \frac{1}{det(R)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Substituting the elements of R, we find $R^{-1} = \frac{1}{det(R)}\begin{pmatrix} 6 & -8 \\ 6 & 6 \end{pmatrix} =$ $\frac{1}{84}\begin{pmatrix} 6 & -8 \\ 6 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{14} & -\frac{2}{21} \\ \frac{1}{14} & \frac{1}{14} \end{pmatrix}$

The right answer is ***ii***

Solution 2:

The same solution by using Wolfram Mathematica coud be found automatically as it is shown below:

```
In[5]:= U = (2  3; -2 4); V = (3 1; 0 2);
        R = U.V; MatrixForm[R]
        R1 = Inverse[R]; MatrixForm[R1]
Out[6]//MatrixForm=
```

$$\begin{pmatrix} 6 & 8 \\ -6 & 6 \end{pmatrix}$$

```
Out[7]//MatrixForm=
```

$$\begin{pmatrix} \frac{1}{14} & -\frac{2}{21} \\ \frac{1}{14} & \frac{1}{14} \end{pmatrix}$$

4. Apply the row operations to determine the rank of a matrix $Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{pmatrix}$. The rank of Q is:

   i) 1

   ii) 2

   iii) 3

   iv) 4

   Solution 1:

   We will use the fact that the row operartions do not change the rank of the matrix. We obtain

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{pmatrix} Q_3 + 3Q_2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 0 & 1 & -1 & 2 \end{pmatrix} Q_2 + Q_1 \sim$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \end{pmatrix} Q_3 - Q_2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   The last matrix above has the tiangular form with two non-zero elements on the main right diagonal. This means, that the rank of Q is 2.

   The right answer is ii).

   Solution 2:

   By Wolfram Mathematica this example could be solved by using the add-on function MatrixRank[]:

```
In[1]:= Q = ( 1  2  3  4
             -1 -1 -4 -2
              3  4 11  8 );

        MatrixRank[Q]

Out[2]= 2
```

LO: LA3.

4.2 QUESTION 1

Using matrices and determinants, establish linear dependency or independency of each group of vectors. Find the dependence when it exists.

(a) $v1 = (2,3,-1,5), v2 = (-4,-6,2,-10)$

(b) $w1 = (0,2,-3), w2 = (1,-1,1), w3 = (2,-8,11)$.

Solution 1:

(a) Both vectors are 4-dimensional. To establish a linear dependence or independence we look for the rank of the matrix $A = \begin{pmatrix} v1 \\ v2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 & 5 \\ -4 & -6 & 2 & -10 \end{pmatrix}$. By row operations we have

$$A = \begin{pmatrix} 2 & 3 & -1 & 5 \\ -4 & -6 & 2 & -10 \end{pmatrix} A_2 = A_2 + 2A_1 \sim \begin{pmatrix} 2 & 3 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We obtained a rank equal to 1 and $v2 - 2.v1 = \vec{0}$. Consequently, the two vectors are linearly dependent, where the dependence is expressed by the last equality.

(b) Consider the matrix $B = \begin{pmatrix} w1 \\ w2 \\ w3 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -3 \\ 1 & -1 & 1 \\ 2 & -8 & 11 \end{pmatrix}$. For its determinant we have $det(B) = \begin{vmatrix} 0 & 2 & -3 \\ 1 & -1 & 1 \\ 2 & -8 & 11 \end{vmatrix} = 4 + 24 - 6 - 22 = 0$. If determinant is equal to zero, then the vectors are linearly dependent. For finding the coefficients of dependency we tray to express one of the vectors as a linear combination of the two others. For the given vectors let find the dependence

$$w3 = a_1.w1 + b_1.w2.$$

This could be written in the form

$$a_1.(0,2,-3) + b_1.(1,-1,1) = (2,-8,11)$$

or as the following equivalent system of three linear equations with two unknowns:

$$\begin{vmatrix} 0.a_1 + b_1 = 2 \\ 2a_1 - b_1 = -8 \\ -3a_1 + b_1 = 11 \end{vmatrix}.$$

By solving this system from the two first equations we obtain $a_1 = -3, b_1 = 2$, which also satisfies the third equation.

Answer: (a) the two vectors are linearly dependent, $v2 - 2.v1 = \vec{0}$; (b) the three vectors are linearly dependent and $w3 = -3w1 + 2w2$.

LO: LA3, LA1.

Solution 2:

Using Wolfram Mathematica the solution is:

(a)

```
In[1]:= v1 = {2, 3, -1, 5};  v2 = {-4, -6, 2, -10};
        MatrixRank[{v1, v2}]
        Solve[v2 == k*v1, k]

Out[2]= 1

Out[3]= {{k → -2}}
```

(b)

```
In[4]:= w1 = {0, 2, -3}; w2 = {1, -1, 1}; w3 = {2, -8, 11};
        b = {w1, w2, w3}; MatrixForm[b]
        Det[b]
        Solve[w3 == a1 * w1 + b1 * w2, {a1, b1}]

Out[5]//MatrixForm=
        ( 0   2  -3 )
        ( 1  -1   1 )
        ( 2  -8  11 )

Out[6]= 0

Out[7]= {{a1 → -3, b1 → 2}}
```

4.3 QUESTION 2

Consider the electrical circuit given in Figure 1. The problem is to determine the currents in the branches, using Kirchhoff's laws. The parameters of the circuit elements are as follows:

$$R_1 = 45Om, R_2 = 15Om, R_3 = 45Om, R_4 = 75Om, E_1 = 60V, E_2 = 450V.$$

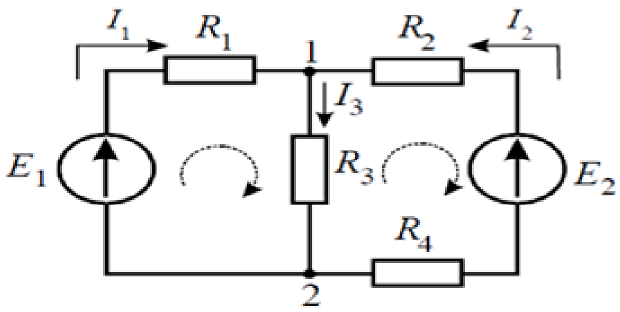

Figure 108. Electrical circuit diagram.

To solve this problem, use the system of equations obtained for currents I_1, I_2, I_3 in the two nodes 1, 2 according to the first and second Kirchhoff laws:

$$\left| \begin{array}{c} I_1 + I_2 - I_3 = 0 \\ I_1 R_1 + I_3 R_3 = E_1 \\ -I_2(R_2 + R_4) - I_3 R_3 = -E_2 \end{array} \right.$$

The solution to this system can be found by determinants using the following Cramer's rule:

$$I_1 = \frac{\Delta_1}{\Delta}, I_2 = \frac{\Delta_2}{\Delta}, I_3 = \frac{\Delta_3}{\Delta},$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 45 & 0 & 45 \\ 0 & -90 & -45 \end{vmatrix}, \Delta_1 = \begin{vmatrix} 0 & 1 & -1 \\ 60 & 0 & 45 \\ -450 & -90 & -45 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & -1 \\ 45 & 60 & 45 \\ 0 & -450 & -45 \end{vmatrix}, \Delta_3 = \begin{vmatrix} 1 & 1 & 0 \\ 45 & 0 & 60 \\ 0 & -90 & -450 \end{vmatrix}$$

Determine the currents by calculating determinants and applying the Cramer's rule. When appropriate use mathematical software.

LO: LA3

Solution:

Using Wolfram Mathematica the solution is:

```
In[1]:= d = {{1., 1, -1}, {45, 0, 45}, {0, -90, -45}};
        d1 = {{0., 1, -1}, {60, 0, 45}, {-450, -90, -45}};
        d2 = {{1., 0, -1}, {45, 60, 45}, {0, -450, -45}};
        d3 = {{1., 1, 0}, {45, 0, 60}, {0, -90, -450}};
        MatrixForm[d]
        MatrixForm[d1]
        MatrixForm[d2]
        MatrixForm[d3]
```

Out[5]//MatrixForm=

$$\begin{pmatrix} 1. & 1 & -1 \\ 45 & 0 & 45 \\ 0 & -90 & -45 \end{pmatrix}$$

Out[6]//MatrixForm=

$$\begin{pmatrix} 0. & 1 & -1 \\ 60 & 0 & 45 \\ -450 & -90 & -45 \end{pmatrix}$$

Out[7]//MatrixForm=

$$\begin{pmatrix} 1. & 0 & -1 \\ 45 & 60 & 45 \\ 0 & -450 & -45 \end{pmatrix}$$

Out[8]//MatrixForm=

$$\begin{pmatrix} 1. & 1 & 0 \\ 45 & 0 & 60 \\ 0 & -90 & -450 \end{pmatrix}$$

```
In[14]:= Δ = Det[d]
         Δ1 = Det[d1]
         Δ2 = Det[d2]
         Δ3 = Det[d3]
```

$$I1 = \frac{\Delta 1}{\Delta}; \quad I2 = \frac{\Delta 2}{\Delta}; \quad I3 = \frac{\Delta 3}{\Delta};$$

```
         Print[" Currents are: I1=", I1, "A,  I2=", I2, "A,  I3=", I3, "A"]

Out[14]= 10125.

Out[15]= -12150.

Out[16]= 37800.

Out[17]= 25650.

 Currents are: I1=-1.2A,  I2=3.73333A,  I3=2.53333A
```

5 BIBLIOGRAPHY

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[3] Blomhøj, M., Jensen, T.H. (2007). What's all the fuss about competencies? In *Modelling and applications in mathematics education*. Springer, Boston, MA, 45-56.

[4] Kulina, H., Gocheva-Ilieva, S., Voynikova, D., Atanasova, P., Ivanov, A., Iliev, A. (2018). Integrating of Competences in Mathematics through Software - Case Study, In *Proceedings of the 2018 IEEE Global Engineering Education Conference (EDUCON)*, 17-20 April 2018, Tenerife, Spain, IEEE, pp. 1586 – 1590. https://doi.org/10.1109/EDUCON.2018.8363424

Guide for a Problem

Linear Algebra

4.4. Matrices and determinants

LA3, LA4

Ascensión Hernández Encinas, Araceli Queiruga-Dios and Víctor Gayoso Martínez

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 37, to be achieved with the development of this activity guide are the following:

1. To understand what is meant by a matrix
2. To recall the basic terms associated with matrices (for example, diagonal, trace, square, triangular, identity)
3. To obtain the transpose of a matrix.
4. To determine any scalar multiple of a matrix.
5. To recognise when two matrices can be added and find, where possible, their sum.
6. To recognise when two matrices can be multiplied and find, where possible, their product.
7. To calculate the determinant of 2 x 2 and 3 x 3 matrices.
8. To understand the geometric interpretation of 2 x 2 and 3 x 3 determinants.
9. To use the elementary properties of determinants in their evaluation.
10. To state the criterion for a square matrix to have an inverse
11. To write down the inverse of a 2 x 2 matrix when it exists.
12. To determine the inverse of a matrix, when it exists, using row operations.
13. To calculate the rank of a matrix.
14. To use appropriate software to determine inverse matrices.
15. To represent a system of linear equations in matrix form
16. To understand how the general solution of a nonhomogeneous linear system of m equations in unknowns is obtained from the solution of the homogeneous system and a particular solution.
17. To recognize the different possibilities for the solution of a system of linear equations.
18. To give a geometrical interpretation of the solution of a system of linear equations.
19. To understand how and why the rank of the coefficient matrix and the augmented matrix of a linear system can be used to analyze its solution.
20. To use the inverse matrix to find the solution of 3 simultaneous linear equations when possible.
21. To understand the term “iff-conditioned”.
22. To apply the Gauss elimination method and recognize when it fails.
23. To understand the Gauss-Jordan variation.
24. To use appropriate software to solve simultaneous linear equations.

Table 37 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

= very important

= medium important

= less important

Table 37: Competencies that we measure with the global test model that is proposed.

		C1	C2	C3	C4	C5	C6	C7	C8
Linear Algebra									
LA3	3. Matrices and determinants								
LA31	Understand what is meant by a matrix								
LA32	Recall the basic terms associated with matrices (for example, diagonal, trace, square, triangular, identity)								
LA33	Obtain the transpose of a matrix								
LA34	Determine any scalar multiple of a matrix								
LA35	Recognise when two matrices can be added and find, where possible, their sum								
LA36	Recognise when two matrices can be multiplied and find, where possible, their product								
LA37	Calculate the determinant of 2 x 2 and 3 x 3 matrices								
LA38	Understand the geometric interpretation of 2 x 2 and 3 x 3 determinants								
LA39	Use the elementary properties of determinants in their evaluation								
LA40	State the criterion for a square matrix to have an inverse								
LA41	Write down the inverse of a 2 x 2 matrix when it exists								
LA42	Determine the inverse of a matrix, when it exists, using row operations								
LA43	Calculate the rank of a matrix								
LA44	Use appropriate software to determine inverse matrices								
LA4	4. Solution of simultaneous linear equations								
LA41	Represent a system of linear equations in matrix form								

LA42	Understand how the general solution of an inhomogeneous linear system of m equations in n unknowns is obtained from the solution of the homogeneous system and a particular solution								
LA43	Recognise the different possibilities for the solution of a system of linear equations								
LA44	Give a geometrical interpretation of the solution of a system of linear equations								
LA45	Understand how and why the rank of the coefficient matrix and the augmented matrix of a linear system can be used to analyse its solution								
LA46	Use the inverse matrix to find the solution of 3 simultaneous linear equations when possible								
LA47	Understand the term 'ill-conditioned'								
LA48	Apply the Gauss elimination method and recognise when it fails								
LA49	Understand the Gauss-Jordan variation								
LA50	Use appropriate software to solve simultaneous linear equations								

2 CONTENTS

1. Matrices of dimension 2 and 3.
2. Computation of transpose, inverse, rank and determinats of a matrix.
3. Linear equations and systems.
4. Recognition of ill-conditioned systems.
5. Gauss and Gauss-Jordan methods.
6. Software to calculate inverse of a matrix.
7. Software to solve simultaneous linear equations.

3 TEST

The time to solve this exam will be 3 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 38.

Table 38: Learning outcomes involved in this assessment activity.

Linear Algebra	
LA3	3. Matrices and determinants
LA31	Understand what is meant by a matrix
LA32	Recall the basic terms associated with matrices (for example, diagonal, trace, square, triangular, identity)
LA33	Obtain the transpose of a matrix
LA34	Determine any scalar multiple of a matrix
LA35	Recognize when two matrices can be added and find, where possible, their sum
LA36	Recognize when two matrices can be multiplied and find, where possible, their product
LA37	Calculate the determinant of 2 x 2 and 3 x 3 matrices
LA38	Understand the geometric interpretation of 2 x 2 and 3 x 3 determinants
LA39	Use the elementary properties of determinants in their evaluation
LA40	State the criterion for a square matrix to have an inverse
LA41	Write down the inverse of a 2 x 2 matrix when it exists
LA42	Determine the inverse of a matrix, when it exists, using row operations
LA43	Calculate the rank of a matrix
LA44	Use appropriate software to determine inverse matrices
LA4	4. Solution of simultaneous linear equations
LA41	Represent a system of linear equations in matrix form
LA42	Understand how the general solution of an inhomogeneous linear system of m equations in n unknowns is obtained from the solution of the homogeneous system and a particular solution
LA43	Recognize the different possibilities for the solution of a system of linear equations

LA44	Give a geometrical interpretation of the solution of a system of linear equations
LA45	Understand how and why the rank of the coefficient matrix and the augmented matrix of a linear system can be used to analyze its solution
LA46	Use the inverse matrix to find the solution of 3 simultaneous linear equations when possible
LA47	Understand the term 'ill-conditioned'
LA48	Apply the Gauss elimination method and recognize when it fails
LA49	Understand the Gauss-Jordan variation
LA50	Use appropriate software to solve simultaneous linear equations

The next subsections include the content of the proposed exam

3.1 MULTIPLE-CHOICE QUESTIONS

Determine the solution associated to the following statements:

1. Given the following matrix, detemine from wich of them you can calculate its determinant:

   i) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

   ii) $\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 4 \end{pmatrix}$

   iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   iv) $\begin{pmatrix} -1 & 3 & 0 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$

   v) $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$

2. The determinant of de matrix $\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$ is

   i) -13

   ii) 26

   iii) 0

   iv) It is not possible to calculate it

   v) None of the above

3. The range of de matrix $\begin{pmatrix} -1 & 3 & 0 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$ is

   i) 1

   ii) 2

   iii) 3

   iv) 4

   v) It is not possible to calculate it

4. The range of de matrix $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$ is

   i) 2

   ii) 3

   iii) 4

   iv) It is not possible to calculate it

   v) None of the above

5. Let A X = B be a system of equations with three unknowns. If |A|= 0, then

   i) A X = B is a dependent system

   ii) A X = B is an inconsistent system

   iii) A X = B is an independent system

   iv) It cannot be known the nature of A X = B

   v) None of the above

3.2 QUESTIONS 1

1. Given the matrix A= $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$ and the matrix B= $\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$, detemine:

   i) Transpose A

   ii) Adjoint of B

   iii) A+B, and 2 B

   iv) A.B and B.A

   v) A^{-1} and B^{-1}

2. Solve the system: $\begin{cases} x + y = -3 \\ x + 1.016y = 5 \end{cases}$. Solve it again but with only 2 decimals in the statement.

3. Solve the system $\begin{cases} x + y + z = 2 \\ 2x - y - 2z = 5 \\ -x + 3y + z = 1 \end{cases}$ using Gauss and Gauss-Jordan methods if it possible. Write it in matricial form and sove it using the inverse matrix. You can use the computer to solve it.

4. Calculate the determinant of $\begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & -3 & 1 & 3 \\ 2 & 0 & 3 & -1 \\ -1 & -2 & 0 & 1 \end{pmatrix}$ using the elementary properties of determinants.

5. Study the following system accordint to the real values of 'a', 'b' and 'c': $\begin{cases} a\,x + by + \ z = c \\ x + a\,y + bz = 0 \\ ax + b\,y + a\,z = c \end{cases}$.

   Solve it forthe part cular case when a=1.

3.3 QUESTIONS 2

1. The toll window of a motorway admits a traffic of 800 cars/hour. The distribution of entrances and links of this higway is:

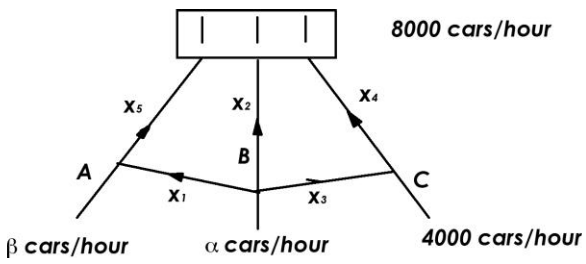

Figure 109

α and β are the control variables to the traffic density and the toll window V will be constant:

i) Find the relationship between α and β so that the above conditions are met

ii) If there is work on the AV section and therefore its traffic must be minimal, find the distribution of traffic by the different branches

iii) If there is work on the AV and BC sections and therefore their traffic must be minimal, find the distribution of traffic in all branches

2. From May 21-23, 2019, in Madrid (Spain) will celebrate the Digital Enterprise Show. And expert buyer intends to acquired movil phones, laptop and workstations. He would like to buy 100 units in total and. Phones cost 50€ each, laptop 1000€ each and workstations 5000€ each. He would like to spend 100000€ in total. How many items should he buy of each type?

3. We must make the following pipeline system works properly. This pipeline system has the scheme of inputs and outputs shown in the figure bellow. It must continue to function, even if it is not complete. We have set some small microcontrolers to get the data from the different zones. We need to get the amount of microcontrolers that go out in node D.

   Our first step is to close the node E and force that nothing circulates between nodes A and B. Between nodes B and D circulates 200 litres circulate and between nodes A and C circulates 100 litres circulate.

   The possible, quickest and easiest option consists in blocking node E. Obtain the solution in the other branches.

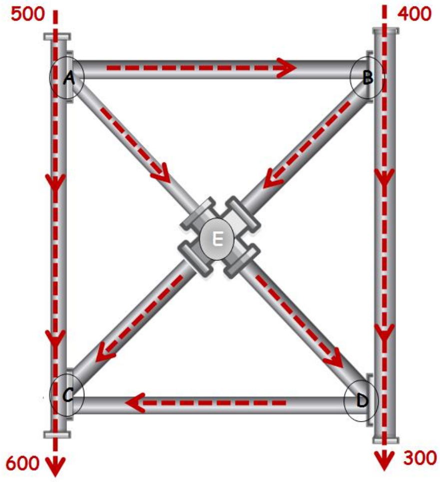

Figure 110

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. Given the following matrix, determine from which of them you can calculate its determinant:

   i) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

   ii) $\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 4 \end{pmatrix}$

   iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

   iv) $\begin{pmatrix} -1 & 3 & 0 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$

   v) $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$

   Solution:

   Correct answers are i), iii) and iv). Answer ii) is not a square matrix

   LO: LA32, LA37.

2. The determinant of de matrix $\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$ is

   i) -13

   ii) 26

   iii) 0

   iv) It is not possible to calculate it

   v) None of the above

   Solution:

   The solution is ii).

   LO: LA32, LA37.

3. The range of de matrix $\begin{pmatrix} -1 & 3 & 0 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$ is

   i) 1

   ii) 2

   iii) 3

   iv) It is not possible to calculate it

   v) None of the above

Solution:

The solution is ii).

LO: LA32, LA37, LA313.

4. The range of de matrix $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$ is

   i) 2

   ii) 3

   iii) 4

   iv) It is not possible to calculate it

   v) None of the above

Solution:

The correct option is ii).

LO: LA32, LA37, LA313.

5. Let A X = B be a system of equations with three unknowns. If |A|= 0, then

   i) A X = B is a dependent System

   ii) A X = B is an inconsistent System

   iii) A X = B is an independent System

   iv) It cannot be known the nature of A X = B

   v) None of the above

Solution:

The correct option is iii. If |A| = 0, rank (A) < 3, then smaller than the number of unknowns. According to de Rouché-Fröbenius theorem the system is inconsistent

LO: LA32, LA37, LA313, LA41, LA43, LA45.

4.2 QUESTIONS 1

1. Given the matrix A= $\begin{pmatrix} 1 & 2 & 0 & -3 \\ 5 & -3 & 4 & 1 \\ 3 & 0 & -2 & 6 \end{pmatrix}$ and the matrix B= $\begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$. Detemine:

   i) Transpose of A

   ii) Adjoint of B

   iii) A+B and 2 B

   iv) A.B and B.A

   v) A^{-1} and B^{-1}

Solution:

i) Transpose of A = $A^T = \begin{pmatrix} 1 & 5 & 3 \\ 2 & -3 & 0 \\ 0 & 4 & -2 \\ -3 & 1 & 6 \end{pmatrix}$

ii) Adjoint of B= Adj(B) = $\begin{pmatrix} -4 & -2 & 18 \\ 7 & -3 & 1 \\ 1 & 7 & -11 \end{pmatrix}$

iii) A+B cannot be calculated because they have not the same dimension.

2 B = $\begin{pmatrix} 2 & 8 & 4 \\ 6 & 2 & 10 \\ 4 & 2 & 2 \end{pmatrix}$

iv) A.B cannot be calculated because they have not the right dimension

B.A= $\begin{pmatrix} 27 & -10 & 12 & 13 \\ 23 & 3 & -6 & 22 \\ 10 & 1 & 2 & 1 \end{pmatrix}$

v) A^{-1} cannot be calculated because A is not a square matrix

B^{-1}= 1/26 $\begin{pmatrix} -4 & -2 & 18 \\ 7 & -3 & 1 \\ 1 & 7 & -11 \end{pmatrix}$

LO: LA32, LA33, LA34, LA35, LA36, LA37,LA310, LA312, LA313, LA314.

2. Solve the system: $\begin{cases} x + y = -3 \\ x + 1.016y = 5 \end{cases}$. Solve it again but with only 2 decimals in the statement.

Solution:

For the first system x = -503 and y = 500. For the second one, x = -403 and y = 400. This is because the system is "ill-conditioned"

LO: LA32, LA47.

3. Solve the system $\begin{cases} x + y + z = 2 \\ 2x - y - 2z = 5 \\ -x + 3y + z = 1 \end{cases}$ using Gauss and Gauss-Jordan methods if it possible. Write it in matricial form and solve it using the inverse matrix. You can use the computer to solve it

Solution:

The solution is: $x = \frac{19}{10}, y = \frac{7}{5}, z = -\frac{13}{10}$

i) Gauss form $\begin{cases} x + y + z = 2 \\ -3y - 4z = 1 \\ -\frac{10}{13}z = \frac{13}{3} \end{cases}$

ii) Gauss-Jordan form $\begin{cases} x + 0y + 0z = \frac{19}{10} \\ 0x + y + 0z = \frac{7}{5} \\ 0x + 0y + z = -\frac{13}{10} \end{cases}$

iii) Matricial form: AX = B → $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$

iv) Solved using the inverse matrix: X = A^{-1} B → $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} & -\frac{1}{10} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & -\frac{2}{5} & -\frac{3}{10} \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$

LO: LA31, LA33, LA37, LA310, LA312, LA313, LA314, LA41, LA42, LA43, LA45, LA46, LA48, LA49, LA50.

4. Calculate the determinant of $\begin{pmatrix} 1 & 1 & 3 & -2 \\ 0 & -3 & 1 & 3 \\ 2 & 0 & 3 & -1 \\ -1 & -2 & 0 & 1 \end{pmatrix}$ using the elementary properties of determinants.

Solution:

The solution is -14.

LO: LA39.

5. Study the following system according to the values of “a”, “b” and “c”: $\begin{cases} a\,x + by + \ z = c \\ \ x + a\,y + bz = 0 \\ ax + b\,y + a\,z = c \end{cases}$

Solve it for the particular case when a=1.

i) The asociated matrix: A = $\begin{pmatrix} a & b & 1 \\ 1 & a & b \\ a & b & a \end{pmatrix}$ and A* = $\begin{pmatrix} a & b & 1 & c \\ 1 & a & b & 0 \\ a & b & a & c \end{pmatrix} \sim \begin{pmatrix} a & b & 1 & c \\ 1 & a & b & 0 \\ 0 & 0 & a-1 & 0 \end{pmatrix}$.

Let see the rangk for each case.

|A| = $(a-1)(a^2-b)$

(1) If $a \neq 1$ and $a^2 \neq b$ then rank (A) = 3 = rank (A*) = number of unknows. Then system is an Independent system

(2) If $a^2 = b$, then A*= $\begin{pmatrix} a & a^2 & 1 & c \\ 1 & a & a^2 & 0 \\ 0 & 0 & a-1 & 0 \end{pmatrix}$

(a) If $a = 1$ then A*= $\begin{pmatrix} 1 & 1 & 1 & c \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so rank (A)= 1

(i) If $c \neq 0$, rank (A*) = 2 ≠ rank (A). The system is Inconsistent.

(ii) If $c = 0$, the system is **homogenous**. rank (A*) = 1 = rank (A), and the system is independent.

(b) $a \neq 1$ then rank (A) = 2.

A*= $\begin{pmatrix} a & a^2 & 1 & c \\ 1 & a & a^2 & 0 \\ 0 & 0 & a-1 & 0 \end{pmatrix}$, as $\begin{vmatrix} a & 1 & c \\ 1 & a^2 & 0 \\ 0 & a-1 & 0 \end{vmatrix} = c\,(a-1)$

(i) If $c \neq 0$ rank (A*) = 3≠ rank (A)=2. The system is inconsistent.

(ii) If $c = 0$ the system is **homogenous**. rank (A*) = 2 = rank (A), and the system is dependent.

(3) If $a = 1$ and $a^2 \neq b$ then rank (A) = 2

$$A^* = \begin{pmatrix} 1 & b & 1 & c \\ 1 & 1 & b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It can be started that rank (A*) = 2 = rank (A)), and the system is dependent for any value c.

Solve the particular case when a=1 (case i 2 a).

(i) If c ≠ 0, rank (A*) = 2 ≠ rank (A), and the system is inconsistent

(ii) c = 0, the system is homogenous. rank (A*) = 1 = rank (A).), and the system is dependent. The system is:

$x + y + z = 0 \rightarrow x = y - z$

LO: LA33, LA37, LA312, LA313, LA41, LA42, LA43, LA45, LA48, LA49, LA50.

4.3 QUESTIONS 2

In all the problems we must solve a system, so the **learning outcomes** in all of them are: LA31, LA33, LA37, LA310, LA312, LA313, LA314, LA41, LA42, LA43, LA45, LA46, LA48, LA49, LA50

1. The toll window of a motorway admits a traffic of 800 cars/hour. The distribution of entrances and links of this highway is:

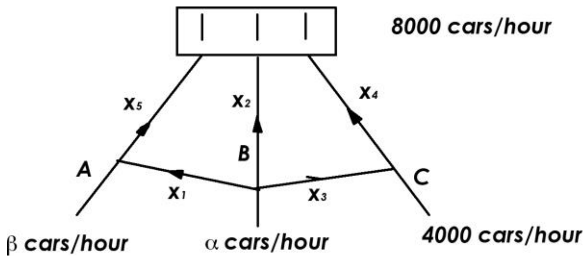

Figure 111

α and β are the control variables to the traffic density and the toll window V will be constant:

i) Find the relationship between α and β so that the above conditions are met

ii) If there is work on the AV section and therefore its traffic must be minimal, find the distribution of traffic by the different branches

iii) If there is work on the AV and BC sections and therefore their traffic must be minimal, find the distribution of traffic in all branches

Solution:

i) Find the relationship between α and β so that the above conditions are met.

In A, B, C and in V the traffic is the same to in and out, then:

$$A: x_1 - x_5 = -\beta$$
$$B: x_1 + x_2 + x_3 = \alpha$$
$$C: x_4 - x_3 = 4000$$
$$V: x_2 + x_4 + x_5 = 8000$$

This system is equivalent to:

$$x_1 - x_5 = -\beta$$
$$x_2 + x_3 + x_5 = \alpha + \beta$$
$$-x_3 + x_4 = 8000 - \alpha - \beta$$
$$-x_3 + x_4 = 4000$$

To be consistent it is necessary that the third and fourth equations are the same, so that:
$$8000 - \alpha - \beta = 4000$$

Then $\boxed{\alpha + \beta = 4000}$.

ii) If there is work on the AV section and therefore its traffic must be minimal, find the distribution of traffic by the different branches

The system that we must solve is:

$$x_1 - x_5 = -\beta$$
$$x_2 + x_3 + x_5 = 4000$$
$$-x_3 + x_4 = 4000$$

Making $x_3 = \lambda$ and $x_5 = \mu$, we obtain the following:

$$x_3 = \lambda \rightarrow x_4 = \lambda + 4000$$
$$x_5 = \mu \rightarrow \begin{cases} x_2 = 4000 - \mu - \lambda \\ x_1 = \mu - \beta \end{cases}$$

We need that result on the AV section so its traffic must is minimal, so we need to minimize x5.

If $x_5 = 0 \rightarrow x_1 = -\beta$. Making $\beta = 0 \rightarrow \alpha = 4000$, then:

$$x_1 = 0 \qquad x_2 = 4000 - \lambda \qquad x_3 = \lambda \qquad x_4 = 4000 + \lambda \qquad x_5 = 0$$
$\lambda \in \mathrm{N}, \lambda \in [0, 4000]$ to do x_2, x_3 and $x_4 \geq 0$

iii) If there is work on the AV and BC sections and therefore their traffic must be minimal, find the distribution of traffic in all branches.

Now we want minimize x5 and x3.

If $x_5 = 0 \rightarrow x_1 = -\beta$.

Making $\beta = 0 \rightarrow \alpha = 4000$

As $x_3 = \lambda$ making $\lambda = 0$ then:

$$x_1 = 0 \qquad x_2 = 4000 \qquad x_3 = 0 \qquad x_4 = 4000 \qquad x_5 = 0$$

2. From May 21 to May 23, 2019, Madrid (Spain) will celebrate the Digital Enterprise Show takes place. An expert buyer intends to acquire movil phones, laptops and workstations. He would like to buy 100 units in total. Phones cost 50€ each, laptops 1.000€ each and workstations

5000€ each. He would like to spend 100000€ in total. How many items should he buy of each type?

Solution:

x = phones

y = laptops

z = workstations

The system that we must solve is:

$$x + y + z = 100$$

$$50\,x + 1000\,y + 5000\,z = 100000$$

Simplifying the second equation: $x + 20\,y + 100\,z = 2000$

The system is equivalent to:

$$\begin{cases} x + y + z = 100 \\ 19\,y + 99z = 1900 \end{cases}$$

Making $z = \lambda \rightarrow \begin{cases} y = 100 - \frac{99\lambda}{19} \\ x = 100 - \lambda - \left(100 - \frac{99\lambda}{19}\right) = \frac{80}{19}\lambda \end{cases}$

It seems dependent but the unknowns must be natural numbers, so λ must be multiple of 19.

$\lambda = 19\,n$

If $\lambda = 0$, then z = 0. The buyer did not buy workstations (but he would like to buy at least one workstation).

If $\lambda = n\,19$, with n > 1 then y < 0, there is no solution.

Thus, $\lambda = 19$ and the buyer could buy 80 phones, 1 laptop and 19 workstations.

3. We must make the following pipeline system works properly. The pipeline system had the scheme of inputs and outputs shown in the figure bellow. It must continue to function, even if it is not complete. We have set some small microcontrolers to get the data from the different zones. We need to get the amount of microcontrolers that go out in node D.

   Our first step is to close the node E and force that nothing circulates between nodes A and B. Between nodes B and D circulates 200 litres circulate and between nodes A and C circulates 100 litres circulate.

   The possible, quickest and easiest option consists in blocking node E. Obtain the solution in the other branches.

   Solution:

   We can enumerate the amount of microcontrolers per pipe from x_1 to x_8.

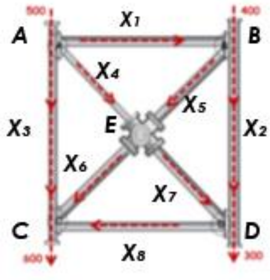

Figure 112

Solution:

The system that we obtain from the graphic was is the following:

$$A: x_1 + x_3 + x_4 = 500$$
$$B: -x_1 + x_2 + x_5 = 400$$
$$C: x_3 + x_6 + x_8 = 600$$
$$D: x_2 + x_7 - x_8 = 300$$
$$E: -x_4 - x_5 + x_6 + x_7 = 0$$

If nothing circulates between nodes A and B, then between nodes B an D 200 litres circulate and between nodes A and C circulates 100 litres circulate. Then, x1 =0, x2 =200 and x3 =100.

Then, the system to be solved is:

$$100 + x_4 = 500$$
$$200 + x_5 = 400$$
$$100 + x_6 + x_8 = 600$$
$$200 + x_7 - x_8 = 300$$
$$-x_4 - x_5 + x_6 + x_7 = 0$$

The solution is: x4 = 400, x5 = 200, x6 = 500, x7 = 100

5 BIBLIOGRAPHY

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Guide for a Problem

Linear Algebra

4.5. Least squares curve fitting

LA5

Michael Carr, Paul Robinson and

Ciarán O'Sullivan

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 39, to be achieved with the development of this activity guide are the following: As a result of learning this material you should be able to:

(Least squares curve fitting)

- Define the least squares criterion for fitting a straight line to a set of data points
- Understand how and why the criterion is satisfied by the solution of $A^T A \mathbf{x} = A^T \mathbf{b}$
- Understand the effect of outliers
- Modify the method to deal with polynomial models
- Use appropriate software to fit a straight line to a set of data points

Table 39 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 39: Competencies that we measure with the global test model that is proposed.

		C1	C2	C3	C4	C5	C6	C7	C8
Linear Algebra									
LA5	5. Least squares curve fitting								
LA51	Define the least squares criterion for fitting a straight line to a set of data points	green	yellow	green	yellow	green	green	yellow	yellow
LA52	Understand how and why the criterion is satisfied by the solution of $A^T A \mathbf{x} = A^T \mathbf{b}$	yellow	yellow	yellow	red	green	green	yellow	red
LA53	Understand the effect of outliers	green	red	green	red	red	red	yellow	yellow
LA54	Modify the method to deal with polynomial models	green	red	green	green	yellow	green	yellow	green
LA55	Use appropriate software to fit a straight line to a set of data points	red	red	red	yellow	red	yellow	green	green

2 CONTENTS

1. Least square curve fitting.
2. Least square curve fitting using GeoGebra and Matlab.

3 EVALUATION

The learning outcomes to be achieved with this procedure are detailed in Table 40.

Table 40: Learning outcomes (LO) involved in this assessment activity.

Linear Algebra	
LA5	5. Least squares curve fitting
LA51	Define the least squares criterion for fitting a straight line to a set of data points
LA52	Understand how and why the criterion is satisfied by the solution of $A^T A \mathbf{x} = A^T \mathbf{b}$
LA53	Understand the effect of outliers
LA54	Modify the method to deal with polynomial models
LA55	Use appropriate software to fit a straight line to a set of data points

3.1 MULTIPLE CHOICE PROBLEMS

1. For the Normal Equation $A^T A\underline{x} = A^T\underline{b}$ which one of the following is true:

   (a) The columns of A are orthogonal to each other

   (b) The columns of $\underline{b} - A\underline{x}$ are orthogonal to the rows of A^T

   (c) The columns of A^T are orthogonal to each other

   (d) The columns of $\underline{b} - A\underline{x}$ are orthogonal to the rows of A

2. The values of a and b so that the linear combination $a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}2\\0\\1\end{pmatrix}$ is closest in Euclidean norm to $\begin{pmatrix}3\\1\\1\end{pmatrix}$ are the solution of the equation

   (a) $\begin{pmatrix}1 & 2\\2 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$

   (b) $\left\| a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}2\\0\\1\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix}\right\| = 0$ where $\| \quad \|$ denotes Euclidean norm

   (c) $\begin{pmatrix}5 & 2\\2 & 5\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}5\\7\end{pmatrix}$

   (d) $\begin{pmatrix}1 & 2 & 0\\2 & 0 & 1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

3. If $\underline{x}$ is the solution of the Normal Equation $A^T A\underline{x} = A^T\underline{b}$ then the best approximation to $\underline{b}$ by the subspace spanned by the columns of A is

(a) $\underline{b} - A\underline{x}$

(b) $A^T A\underline{x}$

(c) $A\underline{x}$

(d) $A^T\underline{b}$

4. If $y = ax^2 + bx + c$ is the best least squares quadratic through the points (1, 1), (0, 0), (-2, 1) and (-2, 2) then the Normal Equation for a, b and c is

(a) $\begin{pmatrix} 9 & -3 & 4 \\ -3 & 9 & -3 \\ 4 & -3 & 9 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 13 \end{pmatrix}$

(b) $\begin{pmatrix} 33 & 9 & -15 \\ -15 & 9 & -3 \\ -15 & -3 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 13 \\ -5 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 33 & -15 & 9 \\ -15 & 9 & -3 \\ 9 & -3 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 13 \\ -5 \\ 4 \end{pmatrix}$

(d) None of the above three equations

5. The perpendicular distance from the point (1, 1, 3) to the space spanned by (1, -1, 0) and (1, 1, 1) is

(a) $\frac{\sqrt{24}}{3}$

(b) $\frac{\sqrt{75}}{3}$

(c) $\frac{4}{9}$

(d) $\frac{5}{9}$

3.2 PROBLEM (SAMPLE PROBLEMS ON LEAST SQUARES)

1. Use the Normal equation to find the point on the subspace spanned by the vectors b_1 = (1, -1, 3, 2), b_2 = (2, -1, 2, 0) which is closest in Euclidean norm to x = (3, 0, 1, -3). Write down the equation of the line through x which is orthogonal to the subspace and find the perpendicular distance from x to the subspace.

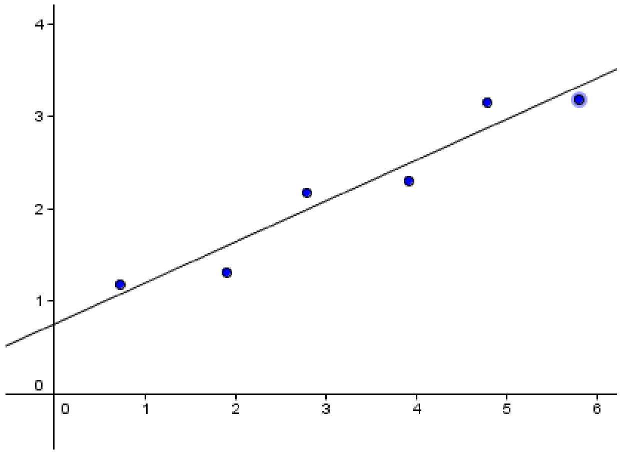


Figure 113

2. Points shown are (0.72, 1.19), (1.90, 1.31), (2.78, 2.19), (3.92, 2.31), (4.78, 3.15) and (5.79, 3.20)

   The line shown is of the form

   $y = a_1 x + a_2$

   (1) Show that you cannot write the vector b = (1.19, 1.31, 2.19, 2.31, 3.15, 3.20) as a linear combination of b_1 = (0.72, 1.90, 2.78, 3.92, 4.78, 5.79) and

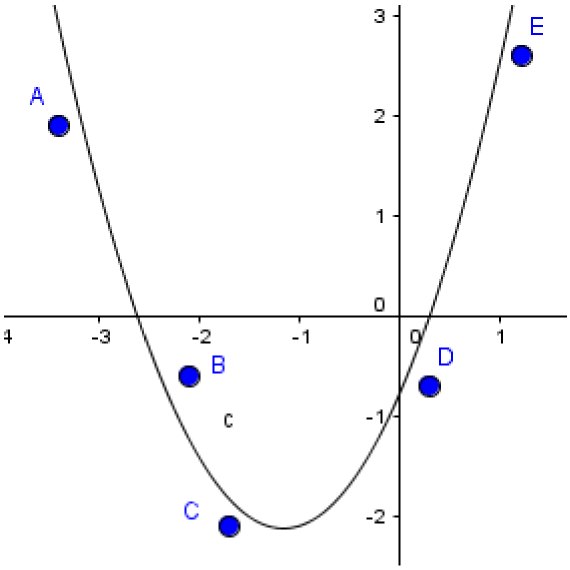
   b_2 = (1, 1, 1, 1, 1, 1).

   (2) Describe how the Least Squares Method relates the three vectors in part (1)

   (3) Write down and solve the Normal Equation $A^T A\hat{a} = A^T b$ for the least squares solution $\hat{a} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$ for this problem, and hence write down the best line through the data.

   (4) Calculate $b - A\hat{a}$ and the Euclidean norm $|b - A\hat{a}|$. What does this vector and norm represent? How do you know from part (1) that this norm cannot be 0?

3.



Five points are shown which lie approximately on a quadratic curve of the form $y = x^2 + ax + b$. Substituting in the five x and y coordinates of points A to E leads to the system of equations

$$-3.4a + b = -9.7$$
$$-2.1a + b = -3.8$$
$$-1.7a + b = -5.0$$

$$0.3a + b = -0.8$$
$$1.2a + b = 1.2$$

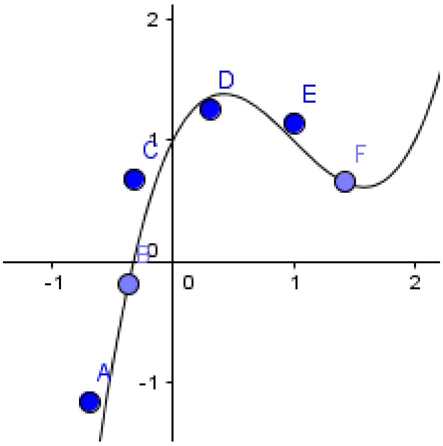
Figure 114

(1) Show that this system inconsistent.

(2) Use the Normal Equation to find the least squares solution $\hat{a}$ to the system. Work to 2 decimal places throughout.

(3) Write down the equation of the best quadratic through the given points. If x = 0.5 calculate the corresponding y value.

4.



(1) The points shown lie approximately on a cubic of the form

$y = a_1x^3 + a_2x^2 + a_3x + a_4$

Estimate the points A to F from the graph (to one decimal point) and write down the Normal Equation $A^T A\hat{a} = A^T b$ to find the coefficients a_1, a_2, a_3, a_4 which give the best least squared cubic approximation to the points (Do not solve it, see below for computer solution).

Figure 115

(2) In Matlab the matrix $C = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$ is written as C = [1 3;2 2;1 0], each row of C separated by semicolons. This system in (1) can be solved with

>>A= [rows of A]

>>b= [rows of y values]

>>a= (A'*A)\(A'*b)   where A' is the transpose of A and * is multiplication.

and a is the vector of coefficients required.

Solve the system and plot the cubic.

3.3 PROJECT (SIMPLE PROJECT COVERING THE LEARNING OUTCOMES)

Project

Helpful Websites

Geogebra Homepage: http://www.geogebra.org

Download GeoGebra Classic (you can change language in Options) and run it.

Geogebra Resources: https://www.geogebra.org/materials

Collection of re-usable resources.

Construction

We will construct the 'best' cubic $y = a_1x^3 + a_2x^2 + a_3x + a_4$ through 6 points in GeoGebra.

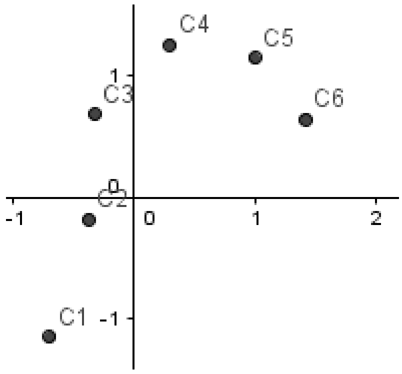


Figure 116

In GeoGebra go to View, Spreadsheet and type the numbers in columns A, B below. In C1 type = (A1,B1) and copy the formula down to C6.

A	B	C
-0.69	-1.15	(-0.69, -1.15)
-0.37	-0.18	(-0.37, -0.18)
-0.32	0.69	(-0.32, 0.69)
0.3	1.26	(0.3, 1.26)
1	1.15	(1, 1.15)
1.42	0.65	(1.42, 0.65)

In cell A8 type =A1^3, cell B8 type = A1^2, cell C8 type =A1, cell D8 type =1 and cell E8 type =B1. Select cells A8 to E8 and copy the contents down to row 13 to get

8	-0.33	0.48	-0.69	1	-1.15
9	-0.05	0.14	-0.37	1	-0.18
10	-0.03	0.1	-0.32	1	0.69
11	0.03	0.09	0.3	1	1.26
12	1	1	1	1	1.15
13	2.86	2.02	1.42	1	0.65

Select the cells A8:D13 with the mouse, right click and choose Create, Matrix. Right click the matrix in the Algebra window and rename it A.

Select the cells E8:E13 with the mouse, right click and choose Create, Matrix. Right click the matrix in the Algebra window and rename it b.

In the input line at the bottom of the screen type A_T = Transpose[A] followed by C = A_T*A and c = A_T*b pressing return after each command. The 4 least squares coefficients we require is the solution a of Ca = c, which in the input line is the following

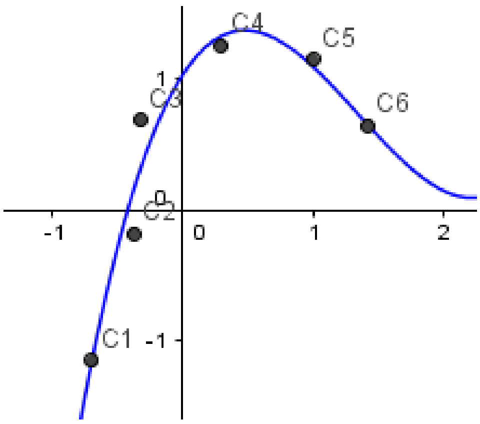


Figure 117

a = Invert[C]*c.

You should see

$$a = \begin{pmatrix} 0.49 \\ -1.98 \\ 1.56 \\ 1.03 \end{pmatrix}$$

Finally, type your best least squares cubic into the input line

$$y = a(1,1) * x\hat{}3 + a(2,1) * x\hat{}2 + a(3,1) * x + a(4,1)$$

What is the effect of changing the position of one of the points away from the curve?

Problem

Adapt the construction of the best cubic to find the 'best' plane through the six points

In a new GeoGebra file click View, 3D graphics (with spreadsheet view) and create the points (1.0, 2.2, 5.6), (0.3, 2.8, 10.1), (-1.2, 1.8, 8.2), (2.8, 3.6, 9.4), (-2.2, 3.5, 14.1), (3.0, 4.2, 9.6).

You need to find the best plane $a_1x + a_2y + a_3z - a_4 = 0$ through the points. Note that you may do a division in this equation and make one of the coefficients 1 so that there are really only 3 coefficients to find.

Plot the resulting plane.

4 SOLUTION

4.1 PROBLEMS

Multichoice Answers: 1 (b), 2 (c), 3 (c), 4 (c), 5 (a)

1.

$$a\begin{pmatrix}1\\-1\\3\\2\end{pmatrix}+b\begin{pmatrix}2\\-1\\2\\0\end{pmatrix}=\begin{pmatrix}1&2\\-1&-1\\3&2\\2&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}\text{ i.e.}A\begin{pmatrix}a\\b\end{pmatrix}$$

Need to find *a, b* so that this is nearest to $\underset{\sim}{x}=\begin{pmatrix}3\\0\\1\\2\end{pmatrix}$

Solve $$A^TA\begin{pmatrix}a_1\\b_1\end{pmatrix}=A^T\underset{\sim}{x}$$

$$\Rightarrow\begin{pmatrix}1&-1&3&2\\2&-1&2&0\end{pmatrix}\begin{pmatrix}1&2\\-1&-1\\3&2\\2&0\end{pmatrix}\begin{pmatrix}a_1\\b_1\end{pmatrix}=\begin{pmatrix}1&-1&3&2\\2&-1&2&0\end{pmatrix}\begin{pmatrix}3\\0\\1\\2\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}15&9\\9&9\end{pmatrix}\begin{pmatrix}a_1\\b_1\end{pmatrix}=\begin{pmatrix}10\\8\end{pmatrix}$$

Solving gives $a_1=\frac{3}{9}, b_1=\frac{5}{9}$ => Point nearest to $\underset{\sim}{x}$ is $\underset{\sim}{e}=\frac{3}{9}\begin{pmatrix}1\\-1\\3\\2\end{pmatrix}+\frac{5}{9}\begin{pmatrix}2\\-1\\2\\0\end{pmatrix}=\frac{1}{9}\begin{pmatrix}13\\-8\\19\\6\end{pmatrix}$

Vector $\underset{\sim}{x}-\underset{\sim}{e}$ is orthogonal to the subspace and

$$\underset{\sim}{x}-\underset{\sim}{e}=\begin{pmatrix}3\\0\\1\\2\end{pmatrix}-\frac{1}{9}\begin{pmatrix}13\\-8\\19\\6\end{pmatrix}=\frac{1}{9}\begin{pmatrix}27\\0\\9\\18\end{pmatrix}-\frac{1}{9}\begin{pmatrix}13\\-8\\19\\6\end{pmatrix}=\frac{1}{9}\begin{pmatrix}14\\8\\-10\\12\end{pmatrix}$$

Therefore line is $\underset{\sim}{r}=\underset{\sim}{x}+t(\underset{\sim}{x}-\underset{\sim}{e})=\begin{pmatrix}3\\0\\1\\2\end{pmatrix}+\frac{t}{9}\begin{pmatrix}14\\8\\-10\\12\end{pmatrix}, t\in\mathbb{R}$

Perpendicular distance $=|\underset{\sim}{x}-\underset{\sim}{e}|=\frac{1}{9}\sqrt{14^2+8^2+10^2+12^2}=2.494$

2.

(1) For the first 3 points assume,

$$0.72a_1+a_2=1.19(i)$$
$$1.90a_1+a_2=1.31(ii)$$
$$2.78a_1+a_2=2.19(iii)$$

(ii) – (i) => 1.18a1 = 0.12 => a1 =0.102

So (i) => a2 =1.19 – 0.72(0.102) = 1.117

But subbed into (iii), 2.78a1+a2 = 1.399 ≠ 2.19

Therefore the equations are inconsistent and b cannot be a combination of b1 and b2.

(2) The span of {b1, b2} defines a 2D subspace of$\mathbb{R}^6$. The point b is not in this subspace (see part (1)). The least squares method finds the closest point in the subspace to b with respect to the Euclidean norm.

(3)

$$\begin{pmatrix} 0.721 \\ 1.901 \\ 2.781 \\ 3.921 \\ 4.781 \\ 5.791 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \text{ is nearest to } \begin{pmatrix} 1.19 \\ 1.31 \\ 2.19 \\ 2.31 \\ 3.15 \\ 3.20 \end{pmatrix} \text{ and normal eqn } A^T A \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = A^T \underset{\sim}{b}$$

$$=> \begin{pmatrix} 83.60 & 19.89 \\ 91.89 & 6 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} 52.07 \\ 13.35 \end{pmatrix}$$

$$\text{Solving, } \hat{a}_2 = 0.758 and \hat{a}_1 = 0.443$$

Best line fit through the points is $y = 0.443x + 0.758$

(4)

$$\underset{\sim}{b} - A\underset{\sim}{\hat{a}} = \begin{pmatrix} 1.19 \\ 1.31 \\ 2.19 \\ 2.31 \\ 3.15 \\ 3.20 \end{pmatrix} - \begin{pmatrix} 0.721 \\ 1.901 \\ 2.781 \\ 3.921 \\ 4.781 \\ 5.791 \end{pmatrix} \begin{pmatrix} 0.443 \\ 0.758 \end{pmatrix} = \begin{pmatrix} 1.19 \\ 1.31 \\ 2.19 \\ 2.31 \\ 3.15 \\ 3.20 \end{pmatrix} - \begin{pmatrix} 1.08 \\ 1.60 \\ 1.99 \\ 2.49 \\ 2.88 \\ 3.32 \end{pmatrix} = \begin{pmatrix} 0.11 \\ -0.29 \\ 0.20 \\ -0.18 \\ 0.27 \\ -0.12 \end{pmatrix}$$

This is the perpendicular vector from the subspace to b.

So $|\underset{\sim}{b} - A\underset{\sim}{\hat{a}}| = \sqrt{0.11^2 + 0.29^2 + 0.20^2 + 0.18^2 + 0.27^2 + 0.12^2} = 0.51$

which is the perpendicular distance from b to the subspace. It cannot be 0 as we know from part(1) that b is not in the subspace.

3.

(1) $-3.4a + b = -9.7(1)$

$$-2.1a + b = -3.8(2)$$

Solving we get a = 4.54 and b = 5.74. Substitution into the third equation gives $-1.7(4.54) + 5.74 = -3.79 \neq -5.0$ and the equations are inconsistent.

(2) In matrix form

$$\begin{pmatrix} -3.4 & 1 \\ -2.1 & 1 \\ -1.7 & 1 \\ 0.3 & 1 \\ 1.2 & 1 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \textit{ is nearest to } \begin{pmatrix} -9.7 \\ -3.8 \\ -5.0 \\ -0.8 \\ 1.2 \end{pmatrix} \text{ and } A^T A\underset{\sim}{\hat{a}} = A^T \underset{\sim}{b} \text{ is}$$

$$\begin{pmatrix} 20.39 & -5.70 \\ -5.70 & 5 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 50.66 \\ -18.1 \end{pmatrix}$$

Solving we get $\hat{a} = 2.16$ and $\hat{b} = -1.16$.

(3) $y = x^2 + 2.16x - 1.16$ and y = 0.17 when x = 0.5.

4. Approximately, A = (-0.7, -1.2), B = (-0.4, 0.2), C = (-0.3, 0.7), D = (0.3, 1.2),

E = (1.0, 1.1) and F = (1.4, 0.6). Then six values of $a_1x^3 + a_2x^2 + a_3x + a_4$ are

$$-0.342a_1 + 0.49a_2 - 0.7a_3 + a_4$$
$$-0.064a_1 + 0.16a_2 - 0.4a_3 + a_4$$
$$-0.027a_1 + 0.09a_2 - 0.3a_3 + a_4$$
$$0.027a_1 + 0.09a_2 + 0.3a_3 + a_4$$
$$a_1 + a_2 + a_3 + a_4$$
$$2.744a_1 + 1.96a_2 + 1.4a_3 + a_4 \quad \text{or} \quad \begin{pmatrix} -0.342 & 0.49 & -0.7 & 1 \\ -0.064 & 0.16 & -0.4 & 1 \\ -0.027 & 0.09 & -0.3 & 1 \\ 0.27 & 0.09 & 0.3 & 1 \\ 1 & 1 & 1 & 1 \\ 2.744 & 1.96 & 1.4 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Then b is the column of 6 y values and the normal equation is

$$\begin{pmatrix} -0.342 & -0.064 & -0.027 & 0.027 & 1 & 2.744 \\ 0.49 & 0.16 & 0.09 & 0.09 & 1 & 1.96 \\ -0.7 & -0.4 & -0.3 & 0.3 & 1 & 1.4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -0.342 & 0.49 & -0.7 & 1 \\ -0.064 & 0.16 & -0.4 & 1 \\ -0.027 & 0.09 & -0.3 & 1 \\ 0.027 & 0.09 & 0.3 & 1 \\ 1 & 1 & 1 & 1 \\ 2.744 & 1.96 & 1.4 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} =$$
$$\begin{pmatrix} -0.342 & -0.064 & -0.027 & 0.027 & 1 & 2.744 \\ 0.49 & 0.16 & 0.09 & 0.09 & 1 & 1.96 \\ -0.7 & -0.4 & -0.3 & 0.3 & 1 & 1.4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1.2 \\ 0.2 \\ 0.7 \\ 1.2 \\ 1.1 \\ 0.6 \end{pmatrix}$$

(2) Type the three lines into the Matlab Command window.

4.2 Project Solution

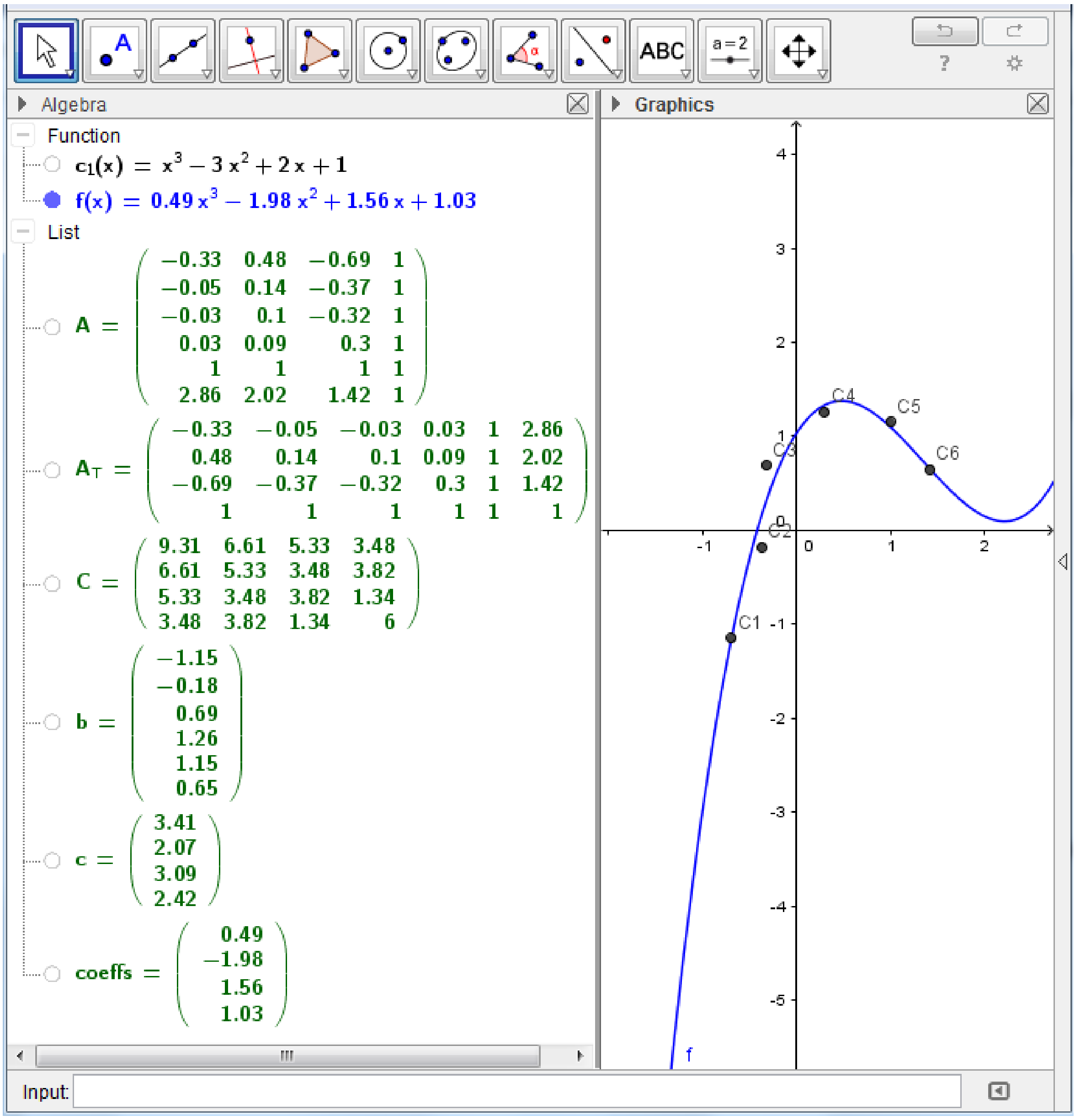
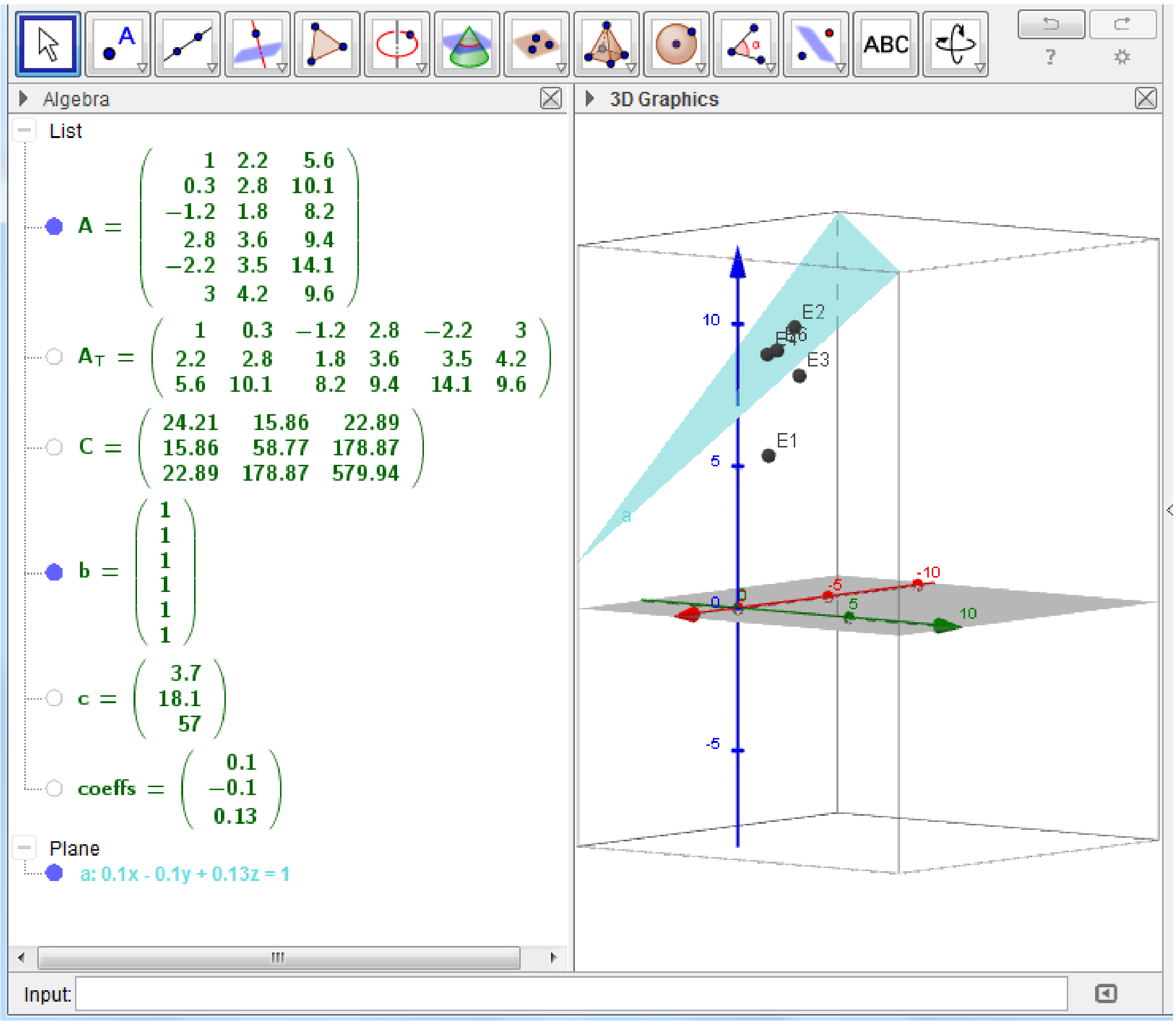


Figure 118

Problem Solution

A	B	C	D	E
1	2.2	5.6	1	(1, 2.2, 5.6)
0.3	2.8	10.1	1	(0.3, 2.8, 10.1)
-1.2	1.8	8.2	1	(-1.2, 1.8, 8.2)
2.8	3.6	9.4	1	(2.8, 3.6, 9.4)
-2.2	3.5	14.1	1	(-2.2, 3.5, 14.1)
3	4.2	9.6	1	(3, 4.2, 9.6)

Coefficient a_4 was taken to be 1 (but any of the 4 will do as the plane is clearly not going through the origin and is not parallel to any axis). So we are left with finding the best approximation to column D in the subspace spanned by columns A, B and C.


Algebra
List
A =
1 2.2 5.6
0.3 2.8 10.1
−1.2 1.8 8.2
2.8 3.6 9.4
−2.2 3.5 14.1
3 4.2 9.6
A_T =
1 0.3 −1.2 2.8 −2.2 3
2.2 2.8 1.8 3.6 3.5 4.2
5.6 10.1 8.2 9.4 14.1 9.6
C =
24.21 15.86 22.89
15.86 58.77 178.87
22.89 178.87 579.94
b =
1
1
1
1
1
1
c =
3.7
18.1
57
coeffs =
0.1
−0.1
0.13
Plane
a: 0.1x - 0.1y + 0.13z = 1
3D Graphics
E1
E2
E3
Input:
ABC
Figure 119

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Guide for a Problem

Linear Algebra

4.6. Linear spaces and transformations

LA6

Michael Carr, Paul Robinson and Ciarán O'Sullivan

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 41, to be achieved with the development of this activity guide are the following: As a result of learning this material you should be able to:

Linear spaces and transformations:

- Define a linear space.
- Define and recognise linear independence.
- Define and obtain a basis for a linear space.
- Define a subspace of a linear space and find a basis for it.
- Define scalar product in a linear space.
- Understand the concept of norm.
- Define the Euclidean norm.
- Define a linear transformation between two spaces; define the image space and the null space for the transformation.
- Derive the matrix representation of a linear transformations.
- Understand how to carry out a change of basis.
- Define an orthogonal transformation.
- Apply the above matrices of linear transformations in the Euclidean plane and Euclidean space.
- Recognise matrices of Euclidean and affine transformations: identity, translation, symmetry, rotation and scaling.

Table 41 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

= very important

= medium important

= less important

Table 41: Competencies that we measure with the global test model that is proposed.

		C1	C2	C3	C4	C5	C6	C7	C8
Linear Algebra									
LA6	6. Linear spaces and transformations								
LA6/1	Define a linear space								
LA6/2	Define and recognise linear independence								

LA6/3	Define and obtain a basis for a linear space								
LA6/4	Define a subspace of a linear space and find a basis for it								
LA6/5	Define scalar product in a linear space								
LA6/6	Understand the concept of norm								
LA6/7	Define the Euclidean norm								
LA6/8	Define a linear transformation between two spaces; define the image space and the null space for the transformation								
LA6/9	Derive the matrix representation of a linear transformations								
LA6/10	Understand how to carry out a change of basis								
LA69/11	Define an orthogonal transformation								
LA69/12	Apply the above matrices of linear transformations in the Euclidean plane and Euclidean space								
LA69/13	Recognise matrices of Euclidean and affine transformations: identity, translation, symmetry, rotation and scaling								

2 CONTENTS

1) Define a linear space.
2) Define and recognise linear independence.
3) Define and obtain a basis for a linear space.
4) Define a subspace of a linear space and find a basis for it.
5) Define scalar product in a linear space.
6) Understand the concept of norm.
7) Define the Euclidean norm.
8) Define a linear transformation between two spaces; define the image space and the null space for the transformation.
9) Derive the matrix representation of a linear transformations.
10) Understand how to carry out a change of basis.
11) Define an orthogonal transformation.
12) Apply the above matrices of linear transformations in the Euclidean plane and Euclidean space.
13) Recognise matrices of Euclidean and affine transformations: identity, translation, symmetry, rotation and scaling.

3 EVALUATION

The time to solve this exam will be 100 minutes (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 42.

Table 42: Learning outcomes (LO) involved in this assessment activity.

LA6	6. Linear spaces and transformations
LA61	Define a linear space
LA62	Define and recognise linear independence
LA63	Define and obtain a basis for a linear space
LA64	Define a subspace of a linear space and find a basis for it
LA65	Define scalar product in a linear space
LA66	Understand the concept of norm
LA67	Define the Euclidean norm
LA68	Define a linear transformation between two spaces; define the image space and the null space for the transformation
LA69	Derive the matrix representation of a linear transformations
LA691	Understand how to carry out a change of basis
LA692	Define an orthogonal transformation
LA693	Apply the above matrices of linear transformations in the Euclidean plane and Euclidean space
LA694	Recognise matrices of Euclidean and affine transformations: identity, translation, symmetry, rotation and scaling

3.1 MULTIPLE CHOICE QUESTIONS

Q1. If T<u>**x**</u> = <u>**0**</u> then

(a) <u>**x**</u> is in the range of T.

(b) <u>**x**</u> is in the kernel of T.

(c) <u>**x**</u> must be the zero vector.

(d) None of these answers.

Q2. How many of the four vectors <u>**u**</u> = (3, 15 , 9, 6), <u>**v**</u> = (1, 6 , 4, -3), <u>**w**</u> = (-2, 1 , 8, 1) and <u>**p**</u> = (7, 13, 7 ,4) are linearly independent?

(a) None

(b) 2

(c) 3

(d) 4

Q3. Which of the following matrices is the matrix for an anti-clockwise rotation about the origin through 45°:

(a) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Q4. If <u>**u**</u> = (1, -2, 3) and <u>**v**</u> = (5, 6, -8) then $\|u - v\|$, correct to two decimal places, is:

(a) 14.18

(b) -7.44

(c) 7.44

(d) 10.25

(e) 6.40

Q5. How many of the following six transformations are linear transformations?

$$T_1: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 3x - 2y \\ -x + y \end{pmatrix}, T_2: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + 6y \\ 3xy - 2y \end{pmatrix}, T_3: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 3x - 2y \\ x + 2 \end{pmatrix},$$

$$T_4: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -4y + x \\ y - 2x \end{pmatrix}, T_5: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x^2 - y \\ 3x + y \end{pmatrix}, T_6: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 5x \\ 3x - y \end{pmatrix}$$

(a) 4

(b) 5

(c) 2

(d) 3

3.2 Problems

Q1. Which of the following sets of vectors are linearly independent, and which are linearly dependent?

(a) [0, 1, 1] , [1, 0, 1] , [1, 1, 0]

(b) [1, -1, 1] , [1, 1, -1] , [3, 1, -1]

(c) [1, -2, -3] , [2, -3, 5] , [-5, 7, -18]

(d) [1, -3, 7] , [2, 5, 3] , [4, 5, 6]

(e) [0, 0, 0, 0] , [13, 7, 9, 2] , [3, -2, 5, 8]

In each case write down a spanning set of vectors and give the dimension of the space spanned by spanning set.

Q2. Use row reduction to write each the vectors,

$$\underline{a} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad \underline{c} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \quad \underline{d} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

as linear combinations of the vectors,

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \underline{x}_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \quad \underline{x}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Q3. Show that $W = \{(x_1, x_2, x_3, x_4)$ such that$3x_1 - x_2 + 2x_3 + 2x_4 = 0\}$ is a subspace of R^4.

Q4. Are the vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} and \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix}$ linearly independent?

If they are not, write the first vector as a linear combination of the other two.

Q5. Show that the matrices $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ are linearly independent.

Write the matrix $\begin{pmatrix} 3 & -2 \\ 2 & 4 \end{pmatrix}$ as a linear combination of the matrices.

Q6. Consider the set S of vectors in R^3

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 7 \\ 12 \end{pmatrix} \right\}$$

(i) Show that the vectors in S are not linearly independent.

(ii) Define a spanning set and a basis of a vector space. Is *S* a spanning set for R^3? Is *S* a basis for R^3?

(iii) Show that the vectors $\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$ are orthogonal. Find a non-zero vector which is orthogonal to both.

Q7. Show that the plane x – 2y + 4z = 0 can be written in vector form as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = sb_1 + tb_2$$

(a) Find $\mathbf{b}_1.\mathbf{b}_2$ and $\|b_1\|^2$. Show that $\mathbf{b}_1$ and $\mathbf{b}_2$ are not orthogonal with respect to the usual dot product in R^3.

(b) Calculate $u = b_2 - \left(\frac{(b_1.b_2)}{\|b_1\|^2}\right)b_1$ and show that $\mathbf{b}_1$ and **u** are orthogonal.

(c) $\mathbf{b}_1$ and **u** form an orthogonal basis for the subspace defined by the plane. Calculate $\|u\|$ and write down an orthonormal basis $B = \{a_1, a_2\}$for the plane.

(d) The point $\underset{\sim}{p} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ is not on the plane. The closest point **P** *on the plane* to **p** is the orthogonal projection

$$P = (p.a_1)a_1 + (p.a_2)a_2$$

Calculate P and show that it is a point on the plane. Find the distance from **P** to **p**.

(e) Visualization: In GeoGebra 3D (https://www.geogebra.org/3d) type points

A = (0,0,0), B = (2, 1, 0), C = (-4, 0,1) and also the vectors b_1 = (2,1,0), b_2=(-4,0,1). Use the graphics commands to create the plane through A, B and C. Type in vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, check that they lie on the plane and are orthogonal. Type in the point **P** and the vector **p**, and check that **P** is on the plane. Use the graphics commands to create a line segment between **P** and **p**, check that it looks orthogonal to the plane and that its length agrees with your calculation.

Q8. Transformations. Find the linear transformation T from the information given and write down its matrix

(i) $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(ii) $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(iii) $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

(iv) By considering the action on the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, find the matrix for an anti-clockwise rotation through an angle α.

Q9. The transformation T from R^3 to $R^3$is given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - y + z \\ x + 2y + 3z \end{pmatrix}$

(i) Write down the matrix for T

(ii) Define the kernel of a matrix T from $R^n$to R^m. Describe Ker(T) geometrically for T above. What is the dimension of Ker(T)?

(iii) Find a basis for the Ran(T). Describe Ran(T) geometrically. What is the dimension of Ran(T)?

Q10. Determine whether the following mappings are linear transformations. If it is linear then

(i) Write down the matrix of T if T is linear

(ii) Describe Ker(T) geometrically. What is the dimension of Ker(T)?

(iii) Find a basis for the Ran(T). Describe Ran(T) geometrically. What is the dimension of Ran(T)?

(a) $T_1 : R^2 \to R^3$ where $T_1(x_1, x_2) = (3x_1, x_1 + 2x_2, 7x_2)$

(b) $T_2 : R^2 \to R^2$ where $T_1(x_1, x_2) = (x_1^2, x_2^2)$

(c) $T_3 : R^3 \to R$ where $T_1(x_1, x_2, x_3) = 4x_2 - 3x_3$

(d) $T_4 : R^2 \to R^2$ where $T_1(x_1, x_2) = (x_1 - 3, x_2 + 4x_1)$

(e) $T_5 : R^4 \to R^4$ where $T_5\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1+2x_2+4x_3+x_4 \\ 2x_1-x_2+3x_3+7x_4 \\ -x_2-x_3+x_4 \\ -x_1-2x_3-3x_4 \end{pmatrix}$

3.3 Projects

Helpful Websites

Geogebra Homepage: http://www.geogebra.org

Download GeoGebra Classic (you can change language in Options) and run it.

Geogebra Resources: https://www.geogebra.org/materials

Collection of re-usable resources.

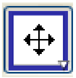
Project A: 2D Transformations.

Part(1)

In this project we will draw a square then apply some matrix transformations to it.

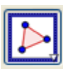
1. Make a square.

In the command line at the bottom of the screen type A = (0,0) and press return. Repeat for the points B = (1,0), C = (1, 1) and D = (0, 1) in that order. If you make a mistake, double click the point definition in the left hand window and edit.

Zoom in and centre the square so it has a reasonable size – use the *Panning Tool* .

In the *Options* menu click on *Labeling* and *No New Objects*. This will keep our screen uncluttered.

Note: You can always add or remove an objects label individually. Right click one of your objects, choose Properties and then the *Basic* Tab. Select each object in turn and click on/off its *Label*.

2. Click on the *Polygon Tool*. Click on your four points A, B, C and D in turn then back to A. Right click on the polygon (or its object in the left hand window) and change its colour and style (filling changes the colour density).

3. In the command line type **T = {{1, 1}, {2, -1}}**

This is how you create the matrix $T = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ in GeoGebra.

4. Apply this matrix to the 4 corners of our square and draw the new shape. To do this type

Polygon[T*A, T*B, T*C, T*D]

in the command line. Here, GeoGebra knows that T is a matrix and A is a vector and has done $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and the same for points B, C and D.

Note that we could have created the 4 points T*A, T*B, T*C and T*D then used the *Polygon Tool* as in step 2. Try that out sometime!

Problems

Change the definition of T so that you are applying

(1) $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (2) $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

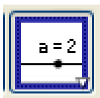
(3) $T = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ (4) $T = \begin{pmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{pmatrix}$

You may need to adjust the panning and axis scales.

Part(2) Interpolating Transformations

Our transformations took a square and immediately gave a new shape. It would be nice to change the square to the new shape slowly so that we can see where each corner goes. Here is one way to do this:

First of all put T back to {{1, 1}, {2, -1}}

1. Put a slider  on the screen with Name t, Min of 0 and Max of 1.

2. In the command line type **I = {{1, 0}, {0, 1}}**, which is the 2 by 2 identity matrix. This is the matrix which leaves the original square as it is.

3. In the command line type **M = t*T + (1-t)*I**. This is a new matrix which depends on t. When t = 0 we have M = I and the square is unchanged. When t = 1 we have M = T and the square will reach its final transformed shape.

4. type **Polygon[M*A, M*B, M*C, M*D]** in the command line.

As you move the slider t from 0 to 1 you will see the square change.

Problems

Change the definition of T so that you are applying

(1) $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (2) $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(3) $T = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ (4) $T = \begin{pmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{pmatrix}$

Note that there is no need to change the definition of M.

Slide t to see what happens. You may need to adjust the panning and axis scales. What might be good names for each transformation?

Extended Problem for Project A: Rotation Matrices (Start with a New File each question)

(i). The anticlockwise rotation by angle α is given by the matrix $R = \begin{pmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{pmatrix}$

(a) Put a slider on the screen which is an Angle rather than a Number, give it the Name α with Min of 0^o and Max of 360^o (These are the Angle defaults).

(b) Type in the matrix R on the command line.

(c) Create Polygon[R*A, R*B, R*C, R*D]

(d) Move the α slider to rotate the square.

(e) Change R so that you have a clockwise rotation.

(ii). Folding down a box

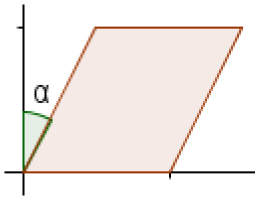


Figure 120

Start with a square box of side length 1 and corners A, B, C, D. To fold down the box by an angle α requires that (1, 0) transforms to (1, 0) i.e. the base does not move and (0, 1) transforms to (sin(α), cos(α)). What is the matrix of the transformation? Animate the transformation of the square for α from 0^0 to 90^0.

(iii). Bring up View, 3D Graphics. Repeat exercise 2 for a cube of side length 1 and one corner fixed at (0, 0, 0) with its sides along the positive x, y and z axes.

Your box should fold down along the x-axis. Find the matrix of the transformation first by considering its action on (1, 0, 0), (0, 1, 0) and (0, 0, 1).

<u>Project B: Inner products and linear independence.</u>

The vector space C[-1, 1] of continuous functions defined on the interval [-1, 1] has a *scalar product*

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx \text{ and induced norm } \|p\|^2 = \langle p, p\rangle = \int_{-1}^{1} p(x)^2 dx$$

This scalar product shares many of the same properties of the dot product in n dimensions. Two functions are defined to be *orthogonal* if this scalar product is 0.

The space $P_3(x)$ of polynomials of degree three or less is a *subspace* of C[-1, 1].

Type the functions below into GeoGebra (press return after each)

$$p_0(x) = 1p_1(x) = xp_2(x) = 3x^2 - 1p_3(x) = 5x^3 - 3x$$

Note: The command p_1(x) = Function[x, -1, 1] will restrict p_1 to the range -1 to 1.

[Note you may hide/unhide a drawing object by clicking the ball to its left in the Algebra Window]

(a) Use the Integral[] command to show that the set $B = \{p_0, p_1, p_2, p_3\}$is an orthogonal set with respect to the scalar product (Calculate $\langle p\_0, p\_1\rangle$ etc..). Can you see from the graphic why each integral is 0?

(b) Use the Integral[] command to calculate $b_0 = \|p_0\|, b_1 = \|p_1\|, b_2 = \|p_2\|, b_3 = \|p_3\|$and input an orthonormal basis $c_0(x), c_1(x), c_2(x), c_3(x)$ for $P_3(x)$ into GeoGebra. Hide $p_0(x), p_1(x), p_2(x), p_3(x)$.

(c) The 'closest' function to $h(x) = x^4$on the interval [-1, 1] by a function in $P_3(x)$ is the orthogonal projection

$$H(x) = \langle h, c_0\rangle c_0(x) + \langle h, c_1\rangle c_1(x) + \langle h, c_2\rangle c_2(x) + \langle h, c_3\rangle c_3(x)$$

Use the Integra[] command to compute H(x) and plot h(x) and H(x) in the interval [-1, 1] to compare the two. Compute $\|H - h\|$ as a measure of how close the functions are.

Edit h(x) to find an approximation to $x^4 + x^5$ and to $cos(\pi x)$

(d) The space spanned by

$$p_0(x) = 1 p_1(x) = cos(\pi x) p_2(x) = cos(2\pi x)$$
$$p_3(x) = sin(\pi x) p_4(x) = sin(2\pi x)$$

is also a subspace of C[-1, 1].

By simply editing your existing commands a little, show that these functions are also orthogonal and construct an orthonormal set. Find the 'closest' function in this subspace to $h(x) = x^4$on the interval [-1, 1] and plot it, and h(x), in GeoGebra.

4 SOLUTIONS

4.1 MULTIPLE CHOICE QUESTIONS: SOLUTIONS

Q1. (b) **x** is in the kernel of T.

Q2. (c) 3 vectors

Q3. (d) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\vartheta = 60°$ gives $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Q4. (a) 14.18

Q5. (d) Three of the transformations are linear.

4.2 PROBLEMS: SOLUTIONS

1.

(a)

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \overset{\text{Swap Row 1 and Row 2}}{\rightarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\underset{-Row1}{\rightarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\underset{-Row2}{\rightarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

No row of zeros $\Rightarrow$ linearly independent

The original set is spanning but also the final set:

$\{[1,0,1],[0,1,1],[0,0,-2]\}$

Dimsenion of the spanning set is 3.

Solutions (b) to (e) are similar.

2.

For $\underset{\sim}{a}$

$$c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

Augmented form,

$$\begin{pmatrix} 1 & 0 & 1\ 2 \\ 0 & -1 & 1\ 5 \\ -1 & -1 & -1 3 \end{pmatrix} \underset{+Row1}{\rightarrow} \begin{pmatrix} 1 & 0 & 12 \\ 0 & -1 & 15 \\ 0 & -1 & 05 \end{pmatrix}$$

$$-1 * Row2 \rightarrow \begin{pmatrix} 1 & 0 & 1\ 2 \\ 0 & 1 & -1 -5 \\ 0 & -1 & 0\ 5 \end{pmatrix}$$

$$\underset{+Row2}{\rightarrow} \begin{pmatrix} 1 & 0 & 1\ 2 \\ 0 & 1 & -1 -5 \\ 0 & 0 & -1\ 0 \end{pmatrix}$$

$$\begin{aligned}&Solving\\&-c_3 = 0 \Rightarrow c_3 = 0\\&\therefore c_2 - c_3 = -5 \Rightarrow c_2 = -5\\&\therefore c_1 + c_3 = 2 \Rightarrow c_1 = 2\\&\Rightarrow \underline{a} = 2\underline{x}_1 - 5\underline{x}_2 + 0\underline{x}_3\end{aligned}$$

Similar for b, c, d.

3.

If $\underline{x} = (x_1, x_2, x_3, x_4)$ and $\underline{y} = (y_1, y_2, y_3, y_4)$ are in W then,

$$\begin{aligned}&\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)\end{aligned}$$

and,

$$\begin{aligned}&3(x_1 + y_1) - (x_2 + y_2) + 2(x_3 + y_3) + 2(x_4 + y_4)\\&= [3x_1 - x_2 + 2x_3 + 2x_4] + [3y_1 - y_2 + 2y_3 + 2y_4]\\&= [0] + [0]\\&= 0\end{aligned}$$

$\therefore \underline{x} + \underline{y}$ is also in W.

Also for any t $\in R$

$$t\underline{x} = (tx_1, tx_2, tx_3, tx_4)$$

and,

$$\begin{aligned}&3(tx_1) - (tx_2) + 2(tx_3) + 2(tx_4)\\&= t[3x_1 - x_2 + 2x_3 + 2x_4]\\&= t[0]\\&= 0\end{aligned}$$

$\therefore t\underline{x}$ is also in W.

Since W is closed under addition and scalar multiplication it is a subspace of R^4

4. Write

$$\boldsymbol{c_1}\begin{pmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{-1}\end{pmatrix} + \boldsymbol{c_2}\begin{pmatrix}\mathbf{3}\\\mathbf{1}\\\mathbf{1}\end{pmatrix} + \boldsymbol{c_3}\begin{pmatrix}\mathbf{0}\\\mathbf{5}\\\mathbf{-3}\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c_1 + 3c_2\\2c_1 + c_2 + 5c_3\\-c_1 + c_2 - 3c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1 & 3 & 0\\2 & 1 & 5\\-1 & 1 & -3\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Augmented form,

$$\begin{pmatrix}1 & 3 & 0 & 0\\2 & 1 & 5 & 0\\-1 & 1 & -3 & 0\end{pmatrix}\begin{matrix}\\-2Row1\rightarrow\\+Row1\end{matrix}\begin{pmatrix}1 & 3 & 0 & 0\\0 & -5 & 5 & 0\\0 & 4 & -3 & 0\end{pmatrix}$$

$$Row2 \div 5 \rightarrow \begin{pmatrix}1 & 3 & 0 & 0\\0 & 1 & -1 & 0\\0 & 4 & -3 & 0\end{pmatrix}$$

$$\begin{matrix}\\\rightarrow\\-4 * Row2\end{matrix}\begin{pmatrix}1 & 3 & 0 & 0\\0 & 1 & -1 & 0\\0 & 0 & 1 & 0\end{pmatrix}$$

Reading from the bottom up we have the equations,

$$c_3 = 0$$

$$\begin{aligned}&\therefore c_2 - c_3 = -5 \Rightarrow c_2 = 0\\&\therefore c_1 + 3c_2 = 2 \Rightarrow c_1 = 0\end{aligned}$$

Since all 3 c values are zero => original vectors are independent.

Further we cannot write the first vector as a combination of the other two.

5.

$$\text{Write } c_1\begin{pmatrix}1&1\\1&0\end{pmatrix}+c_2\begin{pmatrix}1&0\\0&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\0&1\end{pmatrix}+c_4\begin{pmatrix}0&1\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Comparing coefficients of each matrix,

$$c_1+c_2=0(1,1\text{entry})$$
$$c_1+c_4=0(1,2\text{entry})$$
$$c_1=0(2,1\text{entry})$$
$$c_3+c_4=0(2,2\text{ entry})$$

Since $c_1 = 0$ => $c_4 = 0$ and $c_2 = 0$. Also $c_4 = 0$ => $c_3 = 0$.

Since all four *c* values are 0 => the four matrices are independent.

We can write

$$c_1\begin{pmatrix}1&1\\1&0\end{pmatrix}+c_2\begin{pmatrix}1&0\\0&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\0&1\end{pmatrix}+c_4\begin{pmatrix}0&1\\0&1\end{pmatrix}=\begin{pmatrix}3&-2\\2&4\end{pmatrix}$$

Comparing coefficients of each matrix,

$$c_1+c_2=3(1,1\text{entry})$$
$$c_1+c_4=-2(1,2\text{entry})$$
$$c_1=2(2,1\text{entry})$$
$$c_3+c_4=4(2,2\text{ entry})$$

Since $c_1 = 2$ => $c_2 = 1$ and $c_4 = -4$. Also $c_4 = -4$ => $c_3 = 8$.

$$\therefore \begin{pmatrix}3&-2\\2&4\end{pmatrix}=2\begin{pmatrix}1&1\\1&0\end{pmatrix}+\begin{pmatrix}1&0\\0&0\end{pmatrix}+8\begin{pmatrix}0&0\\0&1\end{pmatrix}-4\begin{pmatrix}0&1\\0&1\end{pmatrix}$$

6. (i) Writing down S^T and performing row reduction

$$\begin{pmatrix}1&1&4\\2&-2&0\\-1&7&12\end{pmatrix}\begin{matrix}\\-2*R1\\+R1\end{matrix}\rightarrow\begin{pmatrix}1&1&4\\0&-4&-8\\0&8&16\end{pmatrix}\begin{matrix}\\ \\+2*R1\end{matrix}\rightarrow\begin{pmatrix}1&1&4\\0&-4&-8\\0&0&0\end{pmatrix}$$

So that S only has 2 linearly independent vectors (the third is a combination of the other two)

(ii) A spanning set S for V is such that any vector in V can be written as a linear combination of vectors in S. A basis for V is a minimal spanning set. S is not a spanning set for R^3 as it has dimension 2 while R^3 has dimension 3. It cannot be a basis with dimension 2

(iii) Their dot product is 1(2)+1(-2)+4(0) = 0 , so orthogonal. Suppose that

$$\begin{pmatrix}1\\a\\b\end{pmatrix}\text{ is orthogonal to }\begin{pmatrix}1\\1\\4\end{pmatrix}\text{ and }\begin{pmatrix}2\\-2\\0\end{pmatrix}$$

taking dot products

$$1+a+4b=0$$

$$2-2a=0 \;\Rightarrow a=1 \text{ and } b=-\tfrac{1}{2} \text{ giving the vector } \begin{pmatrix}1\\1\\-\frac{1}{2}\end{pmatrix}$$

7. Put y = s and z = t then

$$x = 2y - 4z = 2s - 4t$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s - 4t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} s, t \in R$$

(a) $\underset{\sim}{b}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \underset{\sim}{b}_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underset{\sim}{b}_1 . \underset{\sim}{b}_2 = -8 + 0 + 0 = -8 \Rightarrow$ **not orthogonal.** Also,

$$\|\underset{\sim}{b}_1\|^2 = 2^2 + 1^2 + 0^2 = 5$$

(b) $\underset{\sim}{u} = \underset{\sim}{b}_2 - \left(\frac{(\underset{\sim}{b}_1 . \underset{\sim}{b}_2)}{\|\underset{\sim}{b}_1\|^2}\right) \underset{\sim}{b}_1 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} - \frac{-8}{5}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 \\ 8 \\ 5 \end{pmatrix}$

and

$$\underset{\sim}{u} . \underset{\sim}{b}_1 = \frac{1}{5}[(-4)(2) + (8)(1) + (5)(0)] = 0 \Rightarrow orthogonal$$

(c) $\|\underset{\sim}{u}\|^2 = \left(\frac{-4}{5}\right)^2 + \left(\frac{8}{5}\right)^2 + 1^2 = \frac{105}{25} \Rightarrow \|\underset{\sim}{u}\| = \sqrt{\frac{21}{5}}$ so that

$$\underset{\sim}{a}_1 = \frac{\underset{\sim}{b}_1}{\|\underset{\sim}{b}_1\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} and \underset{\sim}{a}_2 = \frac{\underset{\sim}{u}}{\|\underset{\sim}{u}\|} = \sqrt{\frac{5}{21}} . \frac{1}{5}\begin{pmatrix} -4 \\ 8 \\ 5 \end{pmatrix} = \sqrt{\frac{1}{105}}\begin{pmatrix} -4 \\ 8 \\ 5 \end{pmatrix}$$

(d) $\underset{\sim}{c} . \underset{\sim}{a}_1 = \frac{1}{\sqrt{5}}[(3)(2) + (4)(1) + (2)(0)] = \frac{10}{\sqrt{5}} and \underset{\sim}{c} . \underset{\sim}{a}_2 = \frac{1}{\sqrt{105}}[(3)(-4) + (4)(8) + (2)(5)] = \frac{30}{\sqrt{105}}$

$$\Rightarrow C = \left(\frac{10}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \left(\frac{30}{\sqrt{105}}\right)\left(\frac{1}{\sqrt{105}}\right)\begin{pmatrix} -4 \\ 8 \\ 5 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 20 \\ 30 \\ 10 \end{pmatrix}$$

To be on the plane C must satisfy $x - 2y + 4z = 0$ which is easily checked. Also $\|\underset{\sim}{C} - \underset{\sim}{c}\|^2 = \frac{3}{7}$

8. (i) $T\begin{pmatrix} x \\ y \end{pmatrix} = xT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix} \Rightarrow T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(ii) $T\begin{pmatrix} x \\ y \end{pmatrix} = xT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\begin{pmatrix} 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2x + y \\ -x - y \end{pmatrix} \Rightarrow T = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

(iii) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xT\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + zT\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + z\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x + y - 2z \\ y + z \\ -x - y - z \end{pmatrix} \Rightarrow$
$T = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$

(iv)

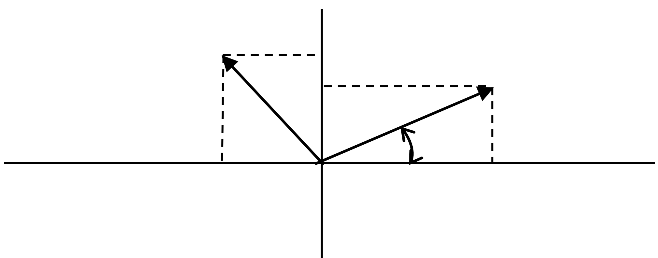


Figure 121

Applying trigonometry of right angled triangles we see

$$T\binom{x}{y} = xT\binom{1}{0} + yT\binom{0}{1} = x\binom{cos(\alpha)}{sin(\alpha)} + y\binom{-sin(\alpha)}{cos(\alpha)} = \binom{cos(\alpha)x - sin(\alpha)y}{sin(\alpha)x + cos(\alpha)y}$$
$$\Rightarrow T = \begin{pmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{pmatrix}$$

9. (i) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - y - z \\ x + 2y + 3z \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow T = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix}$

(ii) $Ket(T) = \{\underset{\sim}{x} \in R^n : T(\underset{\sim}{x}) = \underset{\sim}{0}\}$

$\Rightarrow \begin{pmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$ and in augmented matrix form

$$\begin{pmatrix} 3 & -1 & -1 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix} \updownarrow\rightarrow \begin{pmatrix} 1 & 2 & 3 & 0 \\ 3 & -1 & -1 & 0 \end{pmatrix} -3R_1 \rightarrow \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -7 & -8 & 0 \end{pmatrix}$$

$\Rightarrow z = tisfree, -7y - 37 = 0 \Rightarrow y = -\frac{8}{7}tandx + 2y + 3z = 0 \Rightarrow x = -\frac{5}{7}t$

$\therefore Ker(T) = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{t}{7}\begin{pmatrix} -5 \\ -8 \\ 7 \end{pmatrix}, t \in R\right\}$, a 1 Dim. line of points in R^3 through the origin.

(iii) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\binom{3}{1} + y\binom{-1}{2} + z\binom{1}{3}$ and Ran(*T*) is the space spanned by the three vectors

To find independent vectors do row reduction on the transpose of T

$$\begin{pmatrix} 3 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} \xrightarrow{swapRow1,Row3} \begin{pmatrix} 1 & 3 \\ -1 & 2 \\ 3 & 1 \end{pmatrix} \begin{matrix} +Row1 \\ -3Row1 \end{matrix} \rightarrow \begin{pmatrix} 3 & 1 \\ 0 & 5 \\ 0 & -8 \end{pmatrix} \div 5 \rightarrow \begin{pmatrix} 3 & 1 \\ 0 & 1 \\ 0 & -8 \end{pmatrix} +8Row2 \rightarrow \begin{pmatrix} 3 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

A basis of Ran(*T*) is $\left\{\binom{1}{3}.\binom{0}{1}\right\}$ and $Ran(T) = \left\{s\binom{1}{3} + t\binom{0}{1}, s, t \in R\right\}$ which is all of R^3 with dimension 2.

10.

(a) $T_1\binom{x_1}{x_2} = \begin{pmatrix} 3x_1 \\ x_1 + 2x_2 \\ 7x_2 \end{pmatrix}$

(i)

$$T_1\binom{x_1}{x_2} = \begin{pmatrix} 3x_1 \\ x_1 + 2x_2 \\ 7x_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 12 \\ 07 \end{pmatrix}\binom{x_1}{x_2}$$

$\therefore T_1$ is linear as it has a matrix $T_1 = \begin{pmatrix} 30 \\ 12 \\ 07 \end{pmatrix}$

(ii) $\boldsymbol{T_1}\begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{pmatrix} = \underline{\boldsymbol{0}} \Rightarrow \begin{pmatrix} \boldsymbol{30} \\ \boldsymbol{12} \\ \boldsymbol{07} \end{pmatrix}\begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}$ and in augmented matrix form is $\begin{pmatrix} 300 \\ 120 \\ 070 \end{pmatrix}$. So

$$\begin{pmatrix} 300 \\ 120 \\ 070 \end{pmatrix} \begin{matrix} \text{Swaprow1} \\ \text{androw2} \end{matrix} \rightarrow \begin{pmatrix} 120 \\ 300 \\ 070 \end{pmatrix} Row2 - 3Row_1$$

$$\rightarrow \begin{pmatrix} 1\ 2\ 0 \\ 0{-}60 \\ 0\ 7\ 0 \end{pmatrix} Row2 \div (-6) \rightarrow \begin{pmatrix} 120 \\ 010 \\ 070 \end{pmatrix} Row3 - 7Row2$$

$$\rightarrow \begin{pmatrix} 120 \\ 010 \\ 000 \end{pmatrix}$$

$$\Rightarrow x_2 = 0, x_1 + 2(0) = 0 \text{and} x_3 = t \text{isfree.}$$

$\therefore Ker(T_1) = \left\{ t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, t \in \mathbb{R} \right\}$, a vertical <u>line</u> through the <u>origin</u> R^3 through the origin. Also *Dim*(*Ker*(T_1)) = 1

(iii) $T_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix}, Ran(T_1) = \text{spacespannedbythe2vectors.}$

Clearly,

$$\Rightarrow Ran(T_1) \text{ space spanned by } \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix} \right\} \text{a plane through the origin}$$

$$\text{of the form } \underline{x} = s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \\ 7 \end{pmatrix}, s,t \in R \text{ and of dim 2.}$$

the vectors are linearly independent.

10. (b) $T_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} {x_1}^2 \\ {x_2}^2 \end{pmatrix}$So

$$2T_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\text{But} T_2\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2^2 \\ 2^2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \neq 2T_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore T_2 \text{is} not \text{linear}$$

10. (c) (i)$T_3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (4x_2 - 3x_3)$so $T_3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\therefore T_3$ is linear as it has a matrix.

(ii) $\boldsymbol{T_1\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\underset{\sim}{0}}$ $\Rightarrow x_1 = t \text{and} x_3 = s(or x_2\text{=s})\text{arefree.}$

$$\Rightarrow 4x_2 - 3x_3 = 0 \Rightarrow x_2 = \frac{3}{4}s$$

$$\therefore Ker(T_3) = \left\{\begin{pmatrix}t\\ \frac{3}{4}s\\ s\end{pmatrix} = t\begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}0\\ \frac{3}{4}\\ 1\end{pmatrix}, s,t \in R\right\}$$

$a$planethroughtheorigin in $R^3$of dim 2.

(iii) $T_3\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = (4x_2 - 3x_3)(1)$

$\therefore$ basis for $Ran(T_3) = \{(1)\} \Rightarrow Ran(T_3) = \{t(1), t \in R\}$ a <u>one</u> dimensional *<u>line</u>* of points.

(d) (i) $T_4\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1 - 3\\x_2 + 4x_1\end{pmatrix} \Rightarrow T\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}-3\\0\end{pmatrix}$ so that T_4 cannot be linear

(e) (i) $T_5\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}1&2&4&1\\2&-1&3&7\\0&-1&-1&1\\-1&0&-2&-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \Rightarrow T_5 = \begin{pmatrix}1&2&4&1\\2&-1&3&7\\0&-1&-1&1\\-1&0&-2&-3\end{pmatrix}$

T_5 must be linear as it has a matrix.

(ii) $Ket(T_5) = \{\underset{\sim}{x} \in R^n : T_5(\underset{\sim}{x}) = \underset{\sim}{0}\}$

in augmented matrix form

$$\begin{pmatrix}1&2&4&1&0\\2&-1&3&7&0\\0&-1&-1&1&0\\-1&0&-2&-3&0\end{pmatrix} \rightarrow \ldots rowreduction \ldots \rightarrow \begin{pmatrix}1&2&4&1&0\\0&1&1&-1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

$\Rightarrow x_3 = s and x_4 = t are free,$

$\Rightarrow x_2 + s - t = 0 \Rightarrow x_2 = -s + t and$

$x_1 + 2(-s+t) + 4s + t = 0 \Rightarrow x_1 + 2s + 3t = 0 \Rightarrow x_1 = -2s - 3t$

$\therefore Ker(T_5) = \left\{\underset{\sim}{x} = \begin{pmatrix}-2s-3t\\-s+t\\s\\t\end{pmatrix} = s\begin{pmatrix}-2\\-1\\1\\0\end{pmatrix} + t\begin{pmatrix}-3\\1\\0\\1\end{pmatrix}, s,t \in R\right\}$, a 2 Dim. plane of points in R^4 through the origin.

(iii) $T_5\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_1\begin{pmatrix}1\\2\\0\\-1\end{pmatrix} + x_2\begin{pmatrix}2\\-1\\-1\\0\end{pmatrix} + x_3\begin{pmatrix}4\\3\\-1\\-2\end{pmatrix} + x_4\begin{pmatrix}1\\7\\1\\-3\end{pmatrix}$ and Ran(T) is the space spanned by the four vectors

To find independent vectors do row reduction on the transpose of T

$$\begin{pmatrix}1 & 2 & 0 & -1\\ 2 & -1 & -1 & 0\\ 4 & 3 & -1 & -2\\ 1 & 7 & 1 & -3\end{pmatrix}\begin{matrix}\\ -2Row1\\ -4Row1\\ -Row1\end{matrix} \to \begin{pmatrix}1 & 2 & 0 & -1\\ 0 & -5 & -1 & 2\\ 0 & -5 & -1 & 2\\ 0 & 5 & 1 & -2\end{pmatrix}\begin{matrix}\\ \\ -R2\\ +R2\end{matrix} \to \begin{pmatrix}3 & 1\\ 0 & 1\\ 0 & -8\end{pmatrix}+8R2 \to$$

$$\begin{pmatrix}1 & 2 & 0 & -1\\ 0 & -5 & -1 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

∴a basis of $Ran(T_5) is \left\{\begin{pmatrix}1\\2\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\-5\\-1\\2\end{pmatrix}\right\}$ and $Ker(T_5) = \left\{ s\begin{pmatrix}1\\2\\0\\-1\end{pmatrix} + t\begin{pmatrix}0\\-5\\-1\\2\end{pmatrix}, s, t \in R\right\}$

which is another 2D plane through the origin in R^4.

4.3 Projects: Solutions

Project A: 2D Transformations.

Follow the constructions!

Project B: Inner products and linear independence.

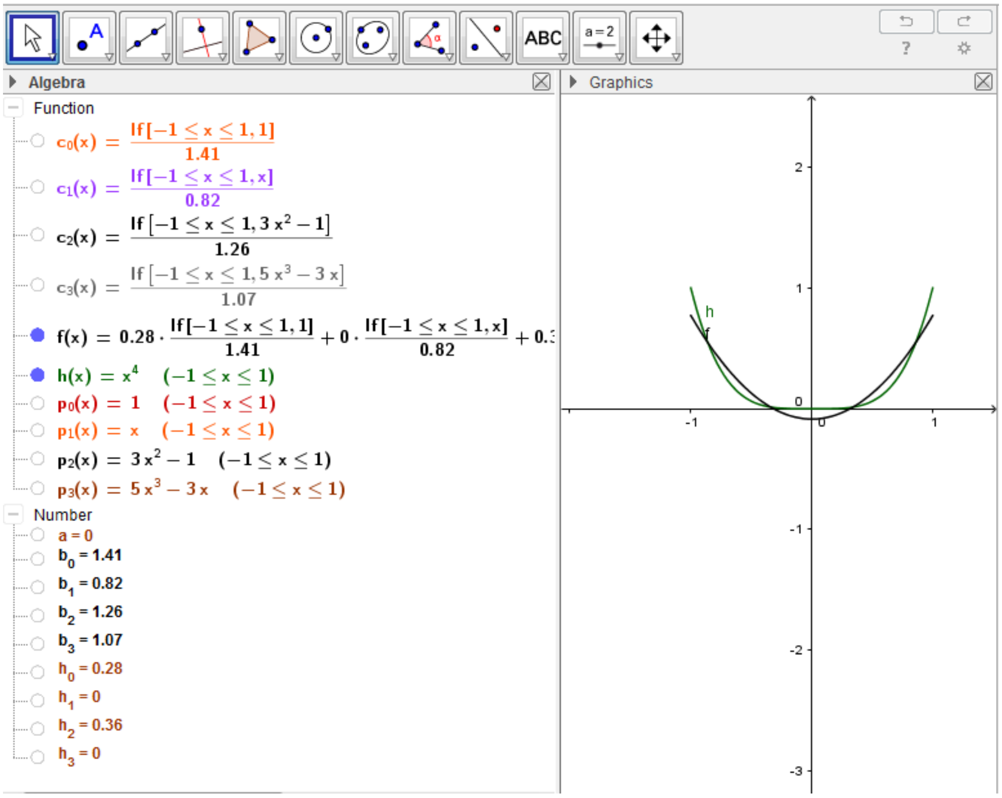


Figure 122

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CHAPTER 5

Statistics and Probability

Guide for a Problem

Statistics and Probability

5.1. Data handling

SP1

Jana Gabková and Daniela Richtáriková

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 43, to be achieved with the development of this activity guide are the following:

1. To recognize the difference between population and random sample.
2. To define and to distinguish between terms – random sample, statistic, value of the statistic, ordered random sample, ordered statistic and value of ordered statistic.
3. To explain why representative random sampling is important in research.
4. To explain the meaning and calculate the basic statistical characteristics of numeric methods in descriptive statistics.
5. To construct a stem-and-leaf diagram to visualize a set of data.
6. To explain the meaning and to construct a frequency table.
7. To explain and to construct a histogram and a polygon from the frequency table.
8. To explain the meaning, to construct and to use a box-and-whisker plot..
9. To understand the meaning and to use a normal probability plot.
10. To apply and to analyse numerical and graphical methods in descriptive statistics
11. To calculate measures of average and dispersion for a grouped set of data
12. To know normality tests, e.g. Shapiro-Wilk test
13. To be able to use statistical software and to analyse its results in statistical data processing.

The Table 43 shows the list of the competencies that we measure with the global test model that is proposed, where

- = very important
- = medium important
- = less important

Table 43: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
STATISTICS AND PROBABILITY									
SP1	Data handling								
SP11	Calculate the range, inter-quartile range, variance and standard deviation for a set of data items								
SP12	Distinguish between a population and a sample								

SP13	Know the difference between the characteristic values (moments) of a population and of a sample								
SP14	Construct a suitable frequency distribution from a data set								
SP15	Calculate relative frequencies								
SP16	Calculate measures of average and dispersion for a grouped set of data								

2 CONTENTS

1. The numeric methods of descriptive statistics.
2. The graphical methods of descriptive statistics.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 44.

Table 44: Learning outcomes involved in this assessment activity.

Statistics and Probability	
SP1	Data handling
SP11	Calculate the range, inter-quartile range, variance and standard deviation for a set of data items
SP12	Distinguish between a population and a sample
SP13	Know the difference between the characteristic values (moments) of a population and of a sample
SP14	Construct a suitable frequency distribution from a data set
SP15	Calculate relative frequencies
SP16	Calculate measures of average and dispersion for a grouped set of data

3.1 MULTIPLE CHOICE QUESTIONS

1) A population is not characterized by
   a) μ
   b) s^2
   c) σ
   d) σ^2
2) A sample median
   a) is average value of a sample
   b) is the most frequent item in a sample
   c) is the value which divides ordered sample into two parts with equal sizes
   d) is influenced by the value of an outlier
3) Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$, then with respect to graphs of distribution
   a) $\sigma_1 < \sigma_2$
   b) $\mu_1 > \mu_2$
   c) $\mu_1 < \mu_2$
   d) $\sigma_1 > \sigma_2$

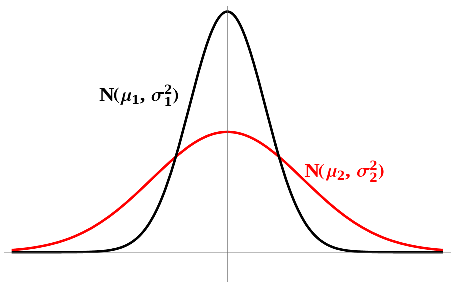


Figure 123. Graph of distributions

4) The distribution
   a) is skewed to the left
   b) Median < Mode
   c) The sign of skewness coefficient is positive
   d) Mean < Mode

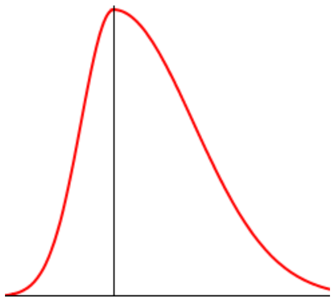


Figure 124. Graph of distributions

5) Interquartile range is calculated as
   a) $xmin_{max}$
   b) $x_{0,25} - x_{0,75}$
   c) $x_1 - x_0$
   d) $x_{0,75} - x_{0,25}$

6) The measured data set has an unimodal mode.
   a) It is from the interval $t_1,\ t_2$).
   b) It is from the interval $t_3,\ t_4$).
   c) It is from the interval $t_5,\ t_6$) or $t_6,\ t_7$) .
   d) We do not know to determine its position.

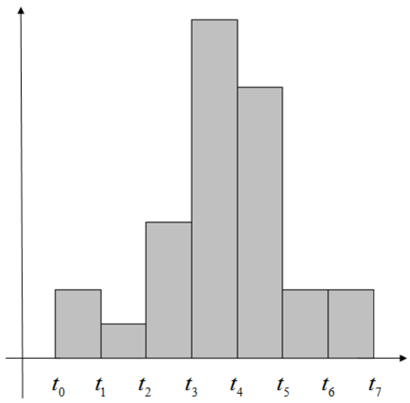


Figure 125. Histogram of frequencies

3.2 QUESTIONS 1

Problem 1

A certain engineering company specializes in the production of components with atypical sizes, the standard length of which is set by a norm to *l* = 70 mm. The customer, who agrees with the standard length, ordered a production of 15 components with atypical diameter ∅ (mm) and atypical thickness *h* (mm).

Measured sizes of components are shown in the following table. We assume that the component sizes are random variables with normal distribution.

i	∅ (*mm*)	*h* (*mm*)	*l* (*mm*)	*i*	∅ (*mm*)	*h* (*mm*)	*l* (*mm*)
1.	14,38	3,96	–	**9.**	14,22	4,26	70,02
2.	14,38	4,02	69,90	**10.**	14,30	4,12	70,02
3.	14,21	3,93	70,05	**11.**	14,27	3,92	–
4.	14,25	4,16	–	**12.**	14,23	4,26	69,85
5.	14,15	3,90	70,02	**13.**	14,16	3,93	69,95
6.	14,20	4,17	–	**14.**	14,30	4,17	–
7.	14,13	4,12	69,85	**15.**	14,01	4,28	70,03
8.	14,37	4,22	69,95				

Problem 1.1: Determine the basic statistics of the normal distribution of the component sizes.

Problem 1.2: Compare atypical dimensions of components in terms of their variability.

Problem 1.3: For the length *l*:

a) Calculate other measures of average.

b) Sketch a graph of normal data distribution based on the calculated measures of central tendency.

c) Sketch a box-and-whisker-plot with corresponding numerical characteristics.

Problem 2

All employees of certain computer company travel every day to work by public transport. The daily travel times (min) home-to-work are shown in the table below. (We assume that travel time to work is a random variable with a normal distribution). Compute the mean and the standard deviation of the daily times of traveling to the work.

Daily time	**Number of employees**
(1 – 10]	8
(10 – 20]	14
(20 – 30]	12
(30 – 40]	9
(40 – 50]	7

Problem 3

The computer company from previous problem participated in the International Sports Games of Companies. It was represented by all young employees aged 25-45. The scores of all participated employees are shown in the following table. (We assume that the score achieved is a random variable with a normal distribution). Find out the mean and the standard deviation of the score achieved in the sports game by the company.

score	The number of attendees
46 – 60	2
61 – 75	2
76 – 90	5
91 – 105	9
106 – 120	8
121 – 135	5
136 – 150	4

3.3 QUESTIONS 2

Problem 4

The Alfa machine cuts the component used in special safety lock inserts to the required width of 8,500 mm. Thirty components were selected at random and their widths were checked with the same measuring instrument (see the table below).

i	x_i	*i*	x_i	*i*	x_i	*i*	x_i	*i*	x_i	*i*	x_i
1	8,462	6	8,505	11	8,482	16	8,493	21	8,511	26	8,497
2	8,489	7	8,519	12	8,499	17	8,510	22	8,496	27	8,506
3	8,500	8	8,486	13	8,498	18	8,502	23	8,470	28	8,501
4	8,486	9	8,502	14	8,462	19	8,526	24	8,539	29	8,49
5	8,504	10	8,498	15	8,534	20	8,498	25	8,514	30	8,479

The following results were obtained through a statistical software by processing the data using numerical and graphical methods of descriptive statistics. Analyze the results.

Numerical methods

Summary Statistics for the_required_width

Count	30
Average	8,4986
Median	8,4985
Mode	8,498
Variance	0,000324524
Standard deviation	0,0180146
Coeff. of variation	0,211971%
Standard error	0,00328899
Minimum	8,462
Maximum	8,539
Range	0,077
Lower quartile	8,489
Upper quartile	8,506
Interquartile range	0,017
Stnd. skewness	0,113747
Stnd. kurtosis	0,601168

Graphical methods

Stem-and-Leaf Display for the_required_width: unit = 0,001   1|2 represents 0,012

```
        LO|8,462 8,462

  2    846|
  4    847|09
  8    848|2669
 (8)   849|03678889
 14    850|0122456
  7    851|0149
  3    852|6

        HI|8,534 8,539
```

Frequency Tabulation for the_required_width

	Lower	*Upper*			*Relative*	*Cumulative*	*Cum. Rel.*
Class	*Limit*	*Limit*	*Midpoint*	*Frequency*	*Frequency*	*Frequency*	*Frequency*
	at or below	8,45		0	0,0000	0	0,0000
1	8,45	8,46667	8,45833	2	0,0667	2	0,0667
2	8,46667	8,48333	8,475	3	0,1000	5	0,1667
3	8,48333	8,5	8,49167	12	0,4000	17	0,5667
4	8,5	8,51667	8,50833	9	0,3000	26	0,8667
5	8,51667	8,53333	8,525	2	0,0667	28	0,9333
6	8,53333	8,55	8,54167	2	0,0667	30	1,0000
	above	8,55		0	0,0000	30	1,0000

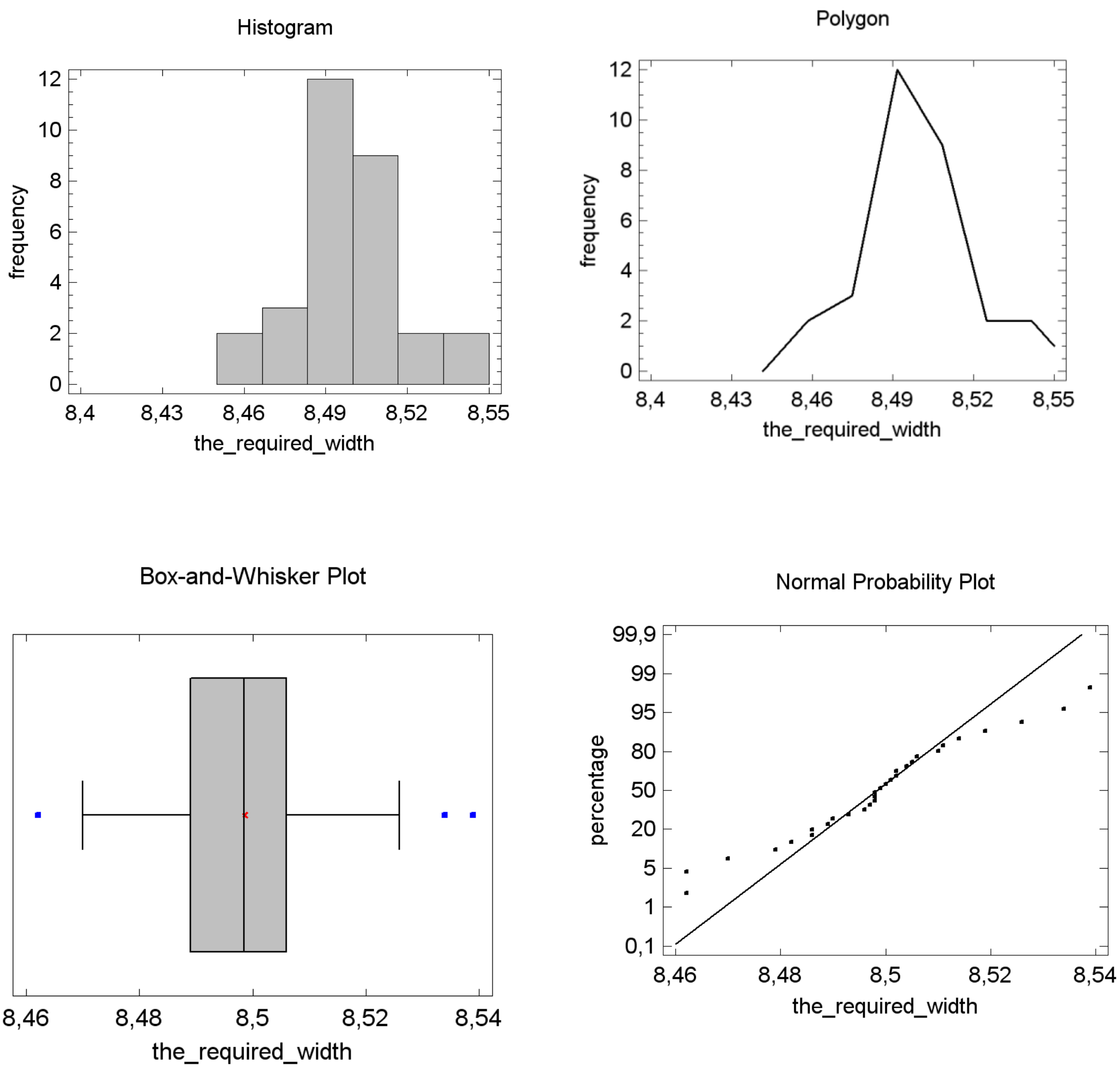

Histogram
frequency
12
10
8
6
4
2
0
8,4
8,43
8,46
8,49
8,52
8,55
the_required_width
Polygon
frequency
12
10
8
6
4
2
0
8,4
8,43
8,46
8,49
8,52
8,55
the_required_width
Box-and-Whisker Plot
8,46
8,48
8,5
8,52
8,54
the_required_width
Normal Probability Plot
percentage
99,9
99
95
80
50
20
5
1
0,1
8,46
8,48
8,5
8,52
8,54
the_required_width

Figure 126

4 SOLUTION

4.1 MULTIPLE CHOICE QUESTIONS

1) A population is not characterized by

   a) μ

   b) s^2

   c) σ

   d) σ^2

   Correct answer: b)

   LO: SP12, SP13.

2) A sample median

   a) is average value of a sample

   b) is the most frequent item in a sample

   c) is the value which divides ordered sample into two parts with equal sizes

   d) is influenced by the value of an outlier

   Correct answer: c)

   LO: SP12, SP13.

3) Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$, then with respect to graphs of distribution

   a) $\sigma_1 < \sigma_2$

   b) $\mu_1 > \mu_2$

   c) $\mu_1 < \mu_2$

   d) $\sigma_1 > \sigma_2$

   Correct answer: a)

   LO: SP11.

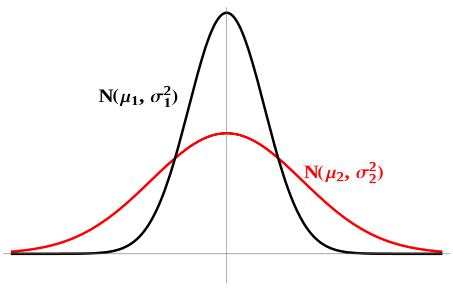


Figure 127. Graph of distributions

4) The distribution is

   a) skewed to the left

   b) Median<Mode

   c) The sign of skewness coefficient is positive

   d) Mean<Mode

   Correct answer: c)

   LO: SP1

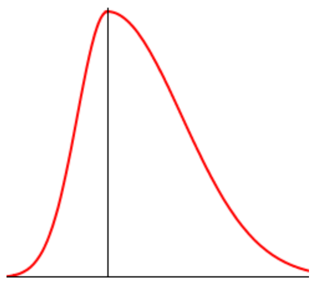


Figure 128. Graph of distribution

5) Interquartile range is calculated as

   a) $xmin_{max}$

   b) $x_{0,25} - x_{0,75}$

   c) $x_1 - x_0$

   d) $x_{0,75} - x_{0,25}$

   Correct answer: d)

   LO: SP11.

6) The measured data set has an unimodal mode.

   a) It is from the interval $t_1,\ t_2)$.

   b) It is from the interval $t_3,\ t_4)$.

   c) It is from the interval $t_5,\ t_6)$ or $t_6,\ t_7)$.

   d) We do not know to determine its position.

   LO: SP1.

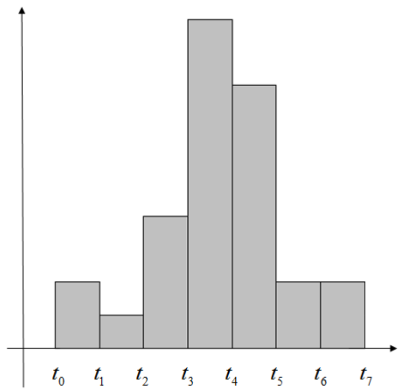


Figure 129. Histogram of frequencies

4.2 QUESTIONS 1

A certain engineering company specializes in the production of components with atypical sizes, the standard length of which is set by a norm to *l* = 70 mm. The customer, who agrees with the standard length, ordered a production of 15 components with atypical diameter ∅ (mm) and atypical thickness *h* (mm).

Measured sizes of components are shown in the following table. We assume that the component sizes are random variables with normal distribution.

i	*Ø (mm)*	*h (mm)*	*l (mm)*	*i*	*Ø (mm)*	*h (mm)*	*l (mm)*
1.	14,38	3,96	–	**9.**	14,22	4,26	70,02
2.	14,38	4,02	69,90	**10.**	14,30	4,12	70,02
3.	14,21	3,93	70,05	**11.**	14,27	3,92	–
4.	14,25	4,16	–	**12.**	14,23	4,26	69,85
5.	14,15	3,90	70,02	**13.**	14,16	3,93	69,95
6.	14,20	4,17	–	**14.**	14,30	4,17	–
7.	14,13	4,12	69,85	**15.**	14,01	4,28	70,03
8.	14,37	4,22	69,95				

Problem 1.1: Determine the basic statistics of the normal distribution of the component sizes.

Problem 1.2: Compare atypical dimensions of components in terms of their variability.

Problem 1.3: For the length *l*

a) calculate other measures of central tendency

b) sketch a graph of normal data distribution based on the calculated measures of central tendency

c) sketch a box-and-whisker-plot with corresponding numerical characteristics

Solution

Problem 1.1

statistic	∅ (mm)	*h* (mm)	*l* (mm)
n	15	15	10
$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$	14,2373	4,0947	69,9710
$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$	0,0098 = $0,0989^2$	0,0179 = $0,01337^2$	–
$\sigma/\bar{x}$	0,0069 = 0,69 %	0,0326 = 3,26 %	–
$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$	–	–	0,00574 = $0,07575^2$
$s/\bar{x}$	–	–	0,0010827 = 0,10827 %

$X_\emptyset$– diameter – a random variable $X_\emptyset \sim N(\mu; \sigma^2 = 0{,}0989^2); \hat{\mu} = \bar{x} = 14{,}2373$

X_h– thickness – a random variable$X_h \sim N(\mu; \sigma^2 = 0{,}0134^2); \hat{\mu} = \bar{x} = 4{,}0947$

X_l – required length – a random variable: $X_l \sim N(\mu; \sigma^2); \hat{\mu} = \bar{x} = 69{,}964; \widehat{\sigma^2} = s^2 = 0{,}07575^2$

$X_\emptyset$, X_h– the population – all data are considered

X_l– the sample – 10 values were randomly measured

Problem 1.2

The variability of thickness is more considerable, its variance is nearly 5 times greater than the variance of diameter.

Problem 1.3

For correct determination of required number characteristics, it is necessary to order the sample.

i	1	2	3	4	5	6	7	8	9	10
l_i **(mm)**	69,90	70,05	70,02	69,85	69,95	70,02	70,02	69,85	69,95	70,03
$l_{(i)}$ **(mm)**	69,85	69,85	69,90	69,95	69,95	70,02	70,02	70,02	70,03	70,05

a) Median: the size of sample is even (n = 10), the sample median is not an item of the sample, it will be the average of two middle values:

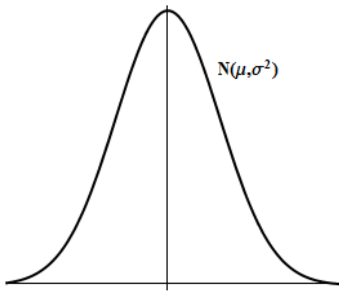
$$x_{MED} = \tilde{x} = x_{0,50} = q_2 = \frac{x_{(5)} + x_{(6)}}{2} = \frac{69{,}95 + 70{,}02}{2} = 69{,}985$$

Mode: $x_{MOD} = \hat{x} = 70{,}02$. The sample mode is unimodal.

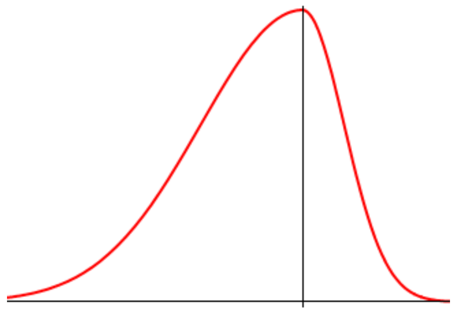
b) The measures of central tendency

$$\left.\begin{aligned} \bar{x} &= 69{,}964 \\ x_{MOD} &= 69{,}702 \\ x_{MED} &= 69{,}985 \end{aligned}\right\} \Rightarrow \bar{x} < x_{MED} < x_{MOD}$$

The distribution is skewed to the left, the skewness coefficient is negative.



$\bar{x} = x_{MOD} = x_{MED}$



$\bar{x} < x_{MED} < x_{MOD}$

Figure 130. Comparison of symmetric and skewed distribution

c) The size of sample n = 10, n/4 is not integer

The first (lower) quartile : $q_1 = x_{0,25} = x_{(3)} = 69{,}90$

The third (upper quartile: $q_3 = x_{0,75} = x_{(8)} = 70{,}02$

Interquartile range: $IQR = q_3 - q_1 = 70{,}02 - 69{,}90 = 0{,}12$

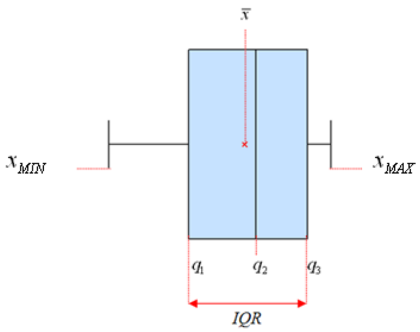


Figure 131. The box-plot

LO: SP11, SP12, SP13.

Problem 2

All employees of certain computer company travel every day to work by public transport. The daily travel times (min) home-to-work are shown in the table below. (We assume that travel time to work is a random variable with a normal distribution). Compute the mean and the standard deviation of the daily times of traveling to the work.

Daily time	**Number of employees**
(1 – 10]	8
(10 – 20]	14
(21 – 30]	12
(30 – 40]	9
(40 – 50]	7

Solution:

k	**travel time** $x_i; x_{i+1}$	**Number of emploees** n_i	**Class representative** $x_i^* = \frac{x_{i+1} + x_i}{2}$	$(x_i^*)^2$	$n_i \cdot x_i^*$	$n_i \cdot (x_i^*)^2$
1.	(1 ; 10]	8	5,5	30,25	44	242
2.	(10 ; 20]	12	15,5	240,25	217	3363,5
3.	(20 ; 30]	14	25,5	650,25	306	7803
4.	(30 ; 40]	9	35,5	1260,25	319,5	11 342,3
5.	(40 ; 50]	7	45,5	2070,25	318,5	14 491,8
	$\sum$:	**50**	–		**1205**	**37 242,6**

The mean of travel time: $\bar{x} = \frac{\sum_{i=1}^{k} n_i \cdot x_i^*}{\sum_{i=1}^{k} n_i} = \frac{1205}{50} = 24{,}1$ and the standard deviation of travel time:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{k} n_i \cdot (x_i^*)^2 - n \cdot \bar{x}^2}{\sum_{i=1}^{k} n_i}} = \sqrt{\frac{37242{,}6 - 50 \cdot 24{,}1^2}{50}} = \sqrt{164{,}04} \doteq 12{,}81.$$

LO: SP12, SP13, SP16.

Problem 3

The computer company from previous problem participated in the International Sports Games of Companies. It was represented by all young employees aged 25-45. The scores of all participated employees are shown in the following table. (We assume that the score achieved is a random variable with a normal distribution). Find out the mean and the standard deviation of the score achieved in the sports game by the company.

score	**The number of attendees**
46 – 60	2
61 – 75	2
76 – 90	5
91 – 105	9
106 – 120	8
121 – 135	5
136 – 150	4

Solution:

k	**score**		**The number of company attendees** n_i	**Class representative** $x_i^* = \frac{x_{i+1} + x_i}{2}$	$(x_i^*)^2$	$n_i \cdot x_i^*$	$n_i \cdot (x_i^*)^2$
	absolute	$[x_i; x_{i+1}]$					
1.	46 – 60	$[45{,}5; 60{,}5]$	2	53	2809	106	5618
2.	61 – 75	$[60{,}5; 75{,}5]$	2	68	4624	136	9248
3.	76 – 90	$[75{,}5; 90{,}5]$	5	83	6889	415	34 445
4.	91 – 105	$[90{,}5; 105{,}5]$	9	98	9604	882	86 436
5.	106 – 120	$[105{,}5; 120{,}5]$	8	113	12 769	904	102 152
6.	121 – 135	$[120{,}5; 135{,}5]$	5	128	16 384	640	81 920
7.	136 – 150	$[135{,}5; 150{,}5]$	4	143	20 449	572	81 796
		$\sum$:	35	–		**3655**	**401 615**

The mean of score: $\bar{x} = \frac{\sum_{i=1}^{k} n_i \cdot x_i^*}{\sum_{i=1}^{k} n_i} = \frac{3655}{35} = 104{,}43$

The sample standard deviation of score:

$$s = \sqrt{\frac{\sum_{i=1}^{k} n_i \cdot \left(x_i^*\right)^2 - n \cdot \bar{x}^2}{\sum_{i=1}^{k} n_i - 1}} = \sqrt{\frac{401615 - 35 \cdot 104{,}43^2}{35 - 1}} = \sqrt{585{,}827} \doteq 24{,}2$$

LO: SP12, SP13, SP16

4.3 QUESTIONS 2

Problem 4

The Alfa machine cuts the component used in special safety lock inserts to the required width of 8,500 mm. Thirty components were selected at random and their widths were checked with the same measuring instrument (see the table below).

i	x_i	*i*	x_i	*i*	x_i	*i*	x_i	*i*	x_i	*i*	x_i
1	8,462	6	8,505	11	8,482	16	8,493	21	8,511	26	8,497
2	8,489	7	8,519	12	8,499	17	8,510	22	8,496	27	8,506
3	8,500	8	8,486	13	8,498	18	8,502	23	8,470	28	8,501
4	8,486	9	8,502	14	8,462	19	8,526	24	8,539	29	8,49
5	8,504	10	8,498	15	8,534	20	8,498	25	8,514	30	8,479

The following results were obtained through a statistical software by processing the data using numerical and graphical methods of descriptive statistics. Analyze the results.

Numerical methods

Summary Statistics for the_required_width

Count	30
Average	8,4986
Median	8,4985
Mode	8,498
Variance	0,000324524
Standard deviation	0,0180146
Coeff. of variation	0,211971%
Standard error	0,00328899
Minimum	8,462
Maximum	8,539
Range	0,077
Lower quartile	8,489
Upper quartile	8,506
Interquartile range	0,017
Stnd. skewness	0,113747
Stnd. kurtosis	0,601168

Graphical methods

Stem-and-Leaf Display for the_required_width: unit = 0,001   1|2 represents 0,012

```
         LO|8,462 8,462

  2     846|
  4     847|09
  8     848|2669
 (8)    849|03678889
 14     850|0122456
  7     851|0149
  3     852|6

         HI|8,534 8,539
```

Frequency Tabulation for the_required_width

	Lower	*Upper*			*Relative*	*Cumulative*	*Cum. Rel.*
Class	*Limit*	*Limit*	*Midpoint*	*Frequency*	*Frequency*	*Frequency*	*Frequency*
	at or below	8,45		0	0,0000	0	0,0000
1	8,45	8,46667	8,45833	2	0,0667	2	0,0667
2	8,46667	8,48333	8,475	3	0,1000	5	0,1667
3	8,48333	8,5	8,49167	12	0,4000	17	0,5667
4	8,5	8,51667	8,50833	9	0,3000	26	0,8667
5	8,51667	8,53333	8,525	2	0,0667	28	0,9333
6	8,53333	8,55	8,54167	2	0,0667	30	1,0000
	above	8,55		0	0,0000	30	1,0000

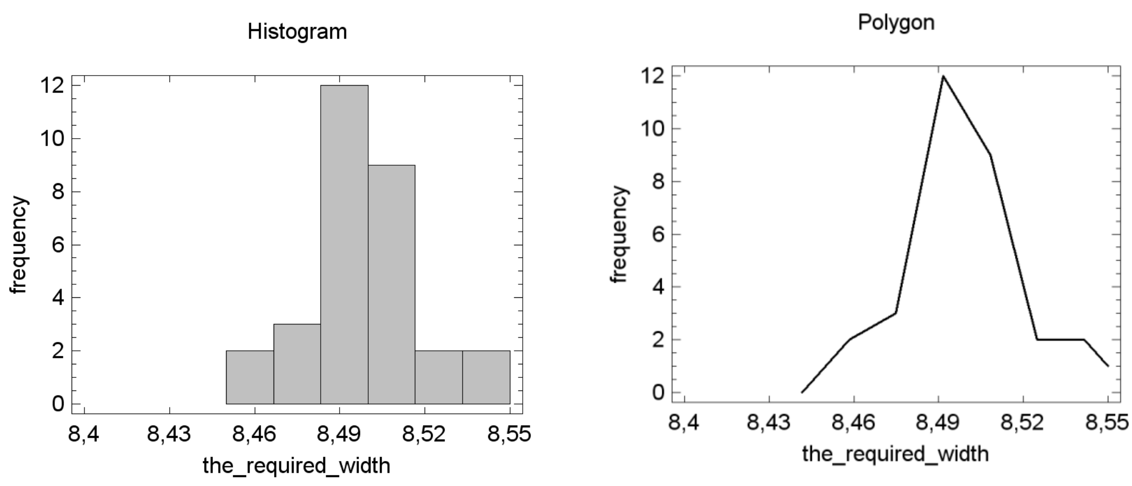


Figure 132

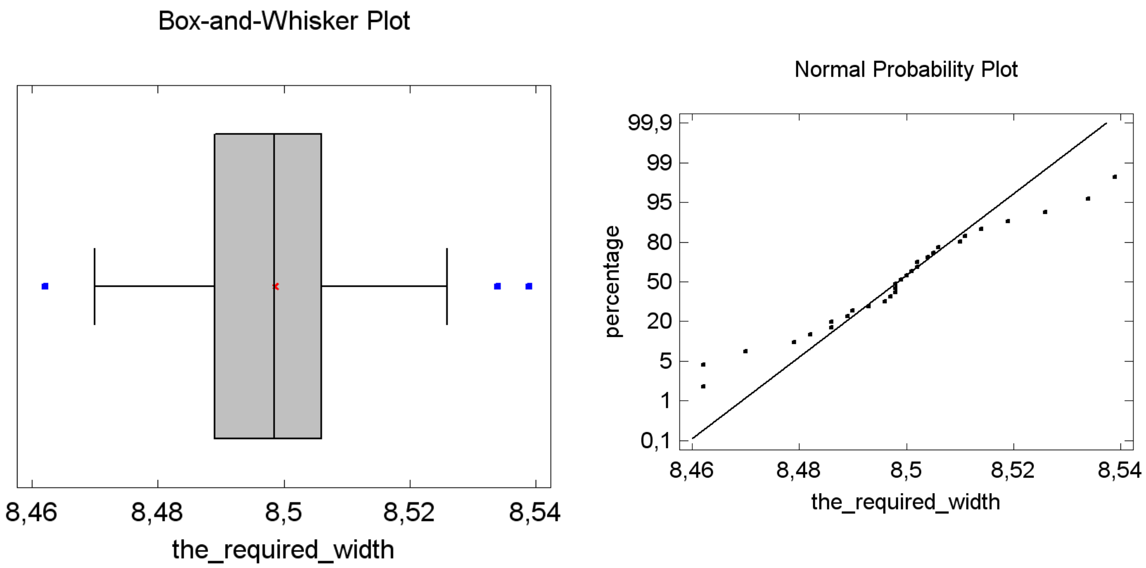


Figure 133

Solution:

Comment. The input data are not sorted. Not sorted data are usually sorted automatically by a software.

First we have to look at the Outliers. They

a) directly affect the value of
   - sample mean $\bar{x}$, where all the data are considered in the calculation
   - range $(R = x_{MAX} - x_{MIN})$
   - quartiles q_1, q_3, and the interquartile range IQR
   - possibly also mode $x_{MOD} = \hat{x}$
b) indirectly affect the values of those numerical characteristics that include sample mean (sample variance or sample standard deviation and consequently also sample standard error, sample standardized skewness coefficient, sample standardized kurtosis coefficient, sample variation coefficient)
c) do not affect the position of median $x_{MED} = \tilde{x}$

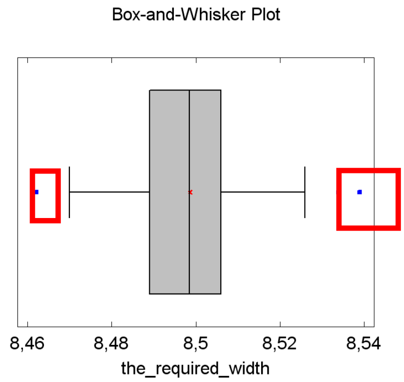
Outliers can be seen in the Stem and leaf Display, Box-and-Whisker Plot or in the Normal Probability Plot.

Stem-and-Leaf Display for the_required_width: unit = 0,001 1|2 represents 0,012

```
          LO|8,462 8,462

  2       846|
  4       847|09
  8       848|2669
 (8)      849|03678889
 14       850|0122456
  7       851|0149
  3       852|6

          HI|8,534 8,539
```



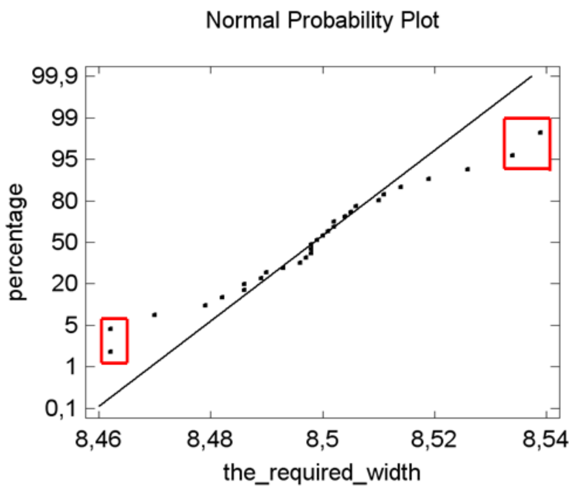


Figure 134

It is necessary to detect whether the values are only outliers or extreme outliers.

- Interval for lower outliers:

$$(x_{0,25} - 3 \times \text{IQR}; x_{0,25} - 1{,}5 \times \text{IQR}) = (8{,}489 - 3 \times 0{,}017; 8{,}489 - 1{,}5 \times 0{,}017) = (8{,}435;\ 8{,}4635)$$

- Interval for lower extreme outliers:

$$(-\infty, x_{0,25} - 3 \times \text{IQR}) = (-\infty; 8{,}489 - 3 \times 0{,}017) = (-\infty; 8{,}435)$$

- Interval for upper outliers:

$$(x_{0,75} + 1{,}5 \times \text{IQR};\ x_{0,75} + 3 \times \text{IQR}) = (8{,}506 + 1{,}5 \times 0{,}017;\ 8{,}506 + 3 \times 0{,}017) = (8{,}5315;\ 8{,}557)$$

- Interval for upper extreme outliers:

$$(x_{0,75} + 3 \times \text{IQR};\ \infty) = (8{,}506 + 3 \times 0{,}017;\ \infty) = (8{,}557;\ \infty)$$

Lower outliers: $x_{(1)} = x_{(2)} = 8{,}462$

Upper outliers: $x_{(29)} = 8{,}534$, $x_{(30)} = 8{,}539$

– With respect to the causes of their occurrence it is necessary to revise whether these values will stay in or they will be omitted from the sample.

Since the lower outliers are very close to the upper bound of the interval $(8{,}435;\ \ 8{,}4635)$, and the upper outliers are very close to the lower bound of the interval $(8{,}5315;\ \ 8{,}557)$, we decide to keep them in the random sample, and so the calculated characteristics via software can be considered and there in no need to revise them.

a) The size of the sample is even number *n* = 30

- the sample median $x_{MED} = \tilde{x} = x_{0{,}50} = q_2 = 8{,}4985$is not an item of the sample
- n/4 is not an integer, so the value of lower quartile is $q_1 = x_{0{,}25} = x_{(8)} = 8{,}489$ and the value of upper quartile is $q_3 = x_{0{,}75} = x_{(23)} = 8{,}506$

b) The sample mode is $x_{MOD} = \hat{x} = 8{,}498$. The software does not give the information on unimodality. We can come to it from the sorted values of the sample, i.e. $x_{MOD} = \hat{x} = 8{,}498$ is unimodal.

i	x_i	*i*	x_i	*i*	x_i	*i*	x_i	*i*	x_i	*i*	x_i
1	8,462	6	8,486	11	8,496	16	8,499	21	8,504	26	8,514
2	8,462	7	8,486	12	8,497	17	8,500	22	8,505	27	8,519
3	8,470	8	8,489	13	8,498	18	8,501	23	8,506	28	8,526
4	8,479	9	8,49	14	8,498	19	8,502	24	8,510	29	8,534
5	8,482	10	8,493	15	8,498	20	8,502	25	8,511	30	8,539

c) The value of the standardized skewness coefficient is 0,0306566. It is positive and very small, what means that the distribution is almost symmetric and only a little skewed to the right. (see plots)

d) The value of the standardized coefficient of the kurtosis is positive of the value 0,66439, what means that the distribution will be more acute than a normal distribution.

e) The value of the sample variation coefficient is $\frac{s}{\bar{x}} \cdot 100\% = 0{,}21114\%$, what means that the standard deviation makes 0,21114% of the sample mean. To claim whether the products are convenient or not it is necessary to confront the result with the norm. (generally we can say that the data vary a little).

f) The number of classes was set with respect to the formula $k = 1 + 3{,}322\ log(30) = 5{,}907 \doteq 6$, as well as the lower boarder of the first class and the upper boarder of the sixth class.

g) Normal distribution

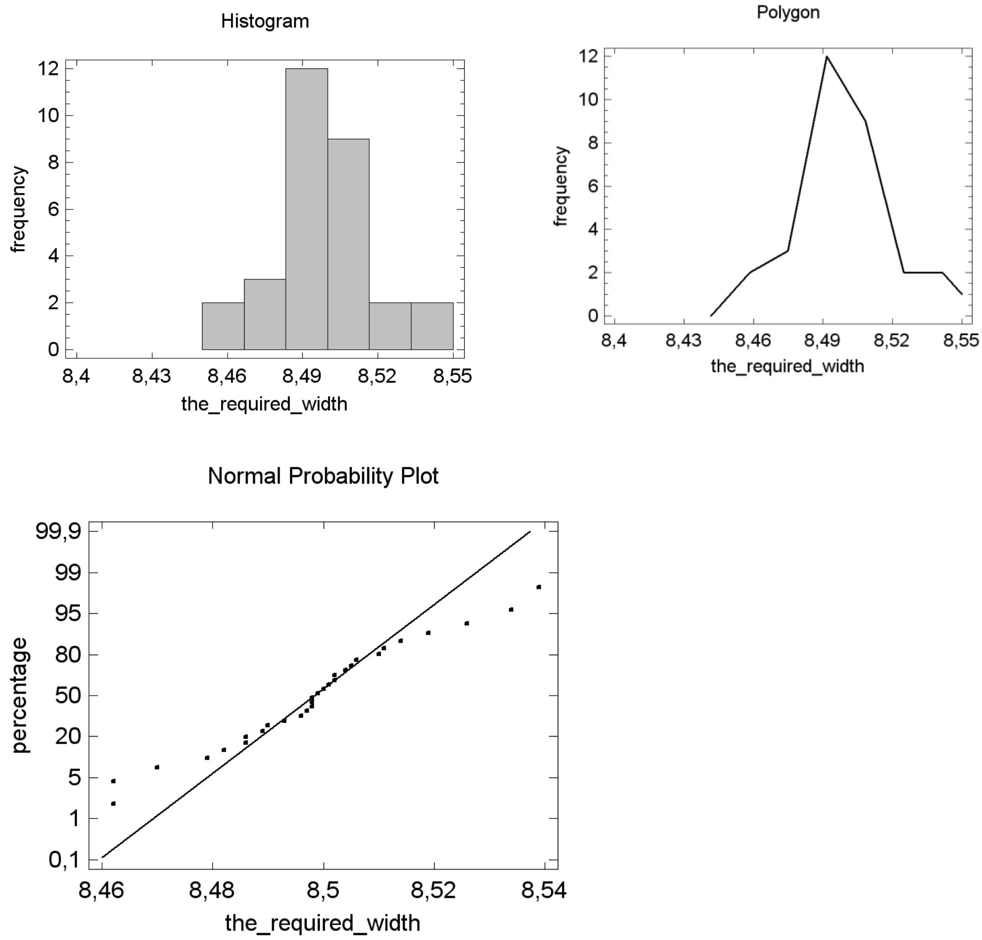


Figure 135

With respect to the shape of the histogram, polygon and normal probability plot, we can claim that the data can come from the normal distribution. The normality of the data can be checked with some normality test, e.g. Shapiro–Wilk test.

LO: SP1.

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Guide for a Problem

Statistics and Probability

5.2. Combinatorics

SP2

Snezhana Gocheva-Ilieva and

Hristina Kulina

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 45, to be achieved with the development of this activity guide are the following:

1. Determine the type of needed arrangements, the number of elements, division in subsets, like and unlike objects.
2. Applying the formula of permutations.
3. Applying the formula of variations.
4. Understanding the difference between permutation and combination.
5. Applying the formula of combination.
6. Evaluating the number of ways of arranging unlike objects in a line.
7. Evaluating the number of ways of arranging objects in a line, where some are alike.
8. Evaluating the number of ways of arranging unlike objects in a ring.
9. Evaluating the number of ways of permuting r objects from n unlike objects.
10. Evaluating the number of combinations of r objects from n unlike objects.
11. Understanding and applying the multiplication principle for combinations.
12. Achieving the symbolic representation of combinatorial formulas.

Table 45 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 45: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Statistics and Probability									
SP2	2. Combinatorics								
SP2/1	Evaluate the number of ways of arranging unlike objects in a line	green	green	green	yellow	yellow	yellow	red	green
SP2/2	Evaluate the number of ways of arranging objects in a line, where some are alike	green	green	green	yellow	yellow	yellow	red	green
SP2/3	Evaluate the number of ways of arranging unlike objects in a ring	green	green	green	yellow	yellow	yellow	red	green

SP2/4	Evaluate the number of ways of permuting r objects from n unlike objects								
SP2/5	Evaluate the number of combinations of r objects from n unlike objects								
SP2/6	Use the multiplication principle for combinations								

2 CONTENTS

The content of this guide is focused on the subject of elementary combinatorics. Combinatorics has a large number of applications and is an essential part of the engineering education. Some of the most important applications are in the areas of genetics and reading the DNA code, statistical physics, structural chemistry, computer network traffic, signal processing, optical communication systems, wireless communication systems, variability in electronic circuits and more.

The following notations are used:

P_n - permutations of n elements;

V_n^k - variations of n elements of class k (permutations of n elements taken k at a time);

C_n^k - combination of n elements of class k (combination of n elements taken k at a time);

$\binom{n}{k}$ - binomial coefficient, $\binom{n}{k} = C_n^k$.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 46.

Table 46: Learning outcomes involved in this assessment activity.

Statistics and Probability	
SP2	2. Combinatorics
SP2/1	Evaluate the number of ways of arranging unlike objects in a line
SP2/2	Evaluate the number of ways of arranging objects in a line, where some are alike
SP2/3	Evaluate the number of ways of arranging unlike objects in a ring
SP2/4	Evaluate the number of ways of permuting r objects from n unlike objects
SP2/5	Evaluate the number of combinations of r objects from n unlike objects
SP2/6	Use the multiplication principle for combinations

3.1 Multiple-Choice Questions

Solve the following test.

1. An engineer in technical sales living in Madrid must visit enterprises in Salamanca, Zaragoza and Malaga. How many different sequences of visiting these three enterprises exist?

   i) 3

   ii) 4

   iii) 6

   iv) 12

2. Students in one class study 9 different subjects - 6 primary and 3 elective. How many ways can be set up for one school day from 5 different school hours to study 5 different subjects, provided that the elective one is in the third hour.

   i) 120

   ii) 240

   iii) 760

   iv) 1080

3. A computer virus erases files from a disk drive in random order. If there are 1000 files on the disk, in how many different orders can 600 of files be erased from the drive?

   i) $\frac{1000!}{600}$

   ii) $\frac{1000!}{400!}$

   iii) $\frac{1000!}{600!}$

   iv) $\frac{1000!}{400!.600!}$

4. The security system code consists of 4 different odd numbers. What is the maximum number of attempts that must be made to find the system code?

   i) 360

   ii) 820

   iii) 240

   iv) 120

5. A new production line produces 12 different machine components per product per minute. The components undergo automatic testing, which shows that for each series of 12 components produced 2 are defective and the others are nondefective. In how many different orders can the 12 items be arranged if all the defective components are considered identical and all the nondefective components are identical of a different class?

   i) 66

   ii) 142

   iii) 256

   iv) 1020

6. A 12-person jury is to be selected from a group of 25 potential jurors of which 15 are men and 10 are women. How many 12-person juries are there with five men and seven women?

   i) 8220

   ii) 360360

   iii) $\frac{25!}{12!}$

   iv) $\frac{25!}{5!.7!}$

3.2 QUESTIONS 1

Six people are to be seated round the circular table.

a) How many ways are there of achieving this?

b) How many if A prefers to sit beside B?

c) How many if A refuses to sit beside B?

3.3 QUESTIONS 2

a) In how many different ways can 15 people be allocated to three wagons on a train, if there are 5 available seats in the first wagon, 4 in the second wagon and 6 in the third?

b) Choose the appropriate notations and write down the general formula for any number of passengers, allocated to three wagons.

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test:

1. An engineer in technical sales living in Madrid must visit enterprises in Salamanca, Zaragoza and Malaga. How many different sequences of visiting these three enterprises exist?

   i) 3

   ii) 4

   iii) 6

   iv) 12

   Solution:

   The number of different sequences is equal to $P_3 = 3! = 1.2.3 = 6$ different permutations.

   The solution is iii).

2. Students in one class study 9 different subjects - 6 primary and 3 elective. How many ways can be set up for one school day from 5 different school hours to study 5 different subjects, provided that the elective one is in the third hour.

   i) 120

   ii) 240

   iii) 760

   iv) 1080

   Solution:

   Denote by * the primary subject, and by E – the elective subject. We have the following distribution of hours **E**. Since different arrangements of the subjects result in different curricula, then the possible ones for the primary subjects are $V_6^4 = 6.5.4.3 = 360$. Then for 3 elective all arrangements are $3.V_6^4 = 1080$.

   The solution is iv).

3. A computer virus erases files from a disk drive in random order. If there are 1000 files on the disk, in how many different orders can 600 of files be erased from the drive?

   i) $\frac{1000!}{600}$

   ii) $\frac{1000!}{400!}$

   iii) $\frac{1000!}{600!}$

   iv) $\frac{1000!}{400!.600!}$

   Solution:

   Let have in total n files on the disk. Then there are n choices for the first file to be erased, $n-1$ for the second, and so on until the k^{th} file. Hence, there are $n(n-1)\ldots(n+1-k) = \frac{n!}{(n-k)!}$ different orders in which files can be erased from the disk. For $n = 1000$ and $k = 600$, the all ways are $\frac{1000!}{400!}$.

The solution is ii).

4. The security system code consists of 4 different odd numbers. What is the maximum number of attempts that must be made to find the system code?

   i) 360

   ii) 820

   iii) 240

   iv) 120

   Solution:

   All digits are ten: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Five digits are odd numbers, i.e. the digits used are 5. Since the order of the digits is important, it is the number of all attempts is equal to the number of variations of 5 elements of class 4, or: $V_5^4 = 5.4.3.2 = 120$, i.e. the maximum number of attempts we have to make to find the system code is 120.

   The solution is iv).

5. A new production line produces 12 different machine components per product per minute. The components undergo automatic testing, which shows that for each series of 12 components produced 2 are defective and the others are nondefective. In how many different orders can the 12 items be arranged if all the defective components are considered identical and all the nondefective components are identical of a different class?

   i) 66

   ii) 142

   iii) 256

   iv) 1020

   Solution:

   The number of ways of arranging 10 nondefective and 2 defective conponents is: $\frac{12!}{10!.2!} = \frac{11.12}{2} = 66.$

   The solution is i).

6. A 12-person jury is to be selected from a group of 25 potential jurors of which 15 are men and 10 are women. How many 12-person juries are there with five men and seven women?

   v) 8220

   vi) 360360

   vii) $\frac{25!}{12!}$

   viii) $\frac{25!}{5!.7!}$

Solution:

There are $\binom{15}{5}$ ways to choose the five men, and there are $\binom{10}{7}$ ways to choose the seven women. Hence, there are $\binom{15}{5}\binom{10}{7} = \frac{15!}{5!10!}\frac{10!}{7!3!} = 360360$ possible juries with five men and seven women.

The solution is ii).

LO: SP2.

4.2 Questions 1

Six people are to be seated round the circular table. Considering all places identical find:

a) How many ways are there of achieving this?

b) How many if A prefers to sit beside B?

c) How many if A refuses to sit beside B?

Solution:

a) One of them can sit where he wants and the other five are arranged in relation to him. This gives permutations of 5 elements, or $P_5 = 5! = 120$ways.

b) Person A can choose any place and B can sit beside A in two different ways (on the left or on the right). There are 4 people left who can sit in 4! different ways to them. That is, the number of different ways is $x = 2.4! = 48$ different ways.

c) This can be obtained by excluding from all possible arrangements in a) the arrangements in b). We get $5! - 2.4! = 72$ways.

LO: SP2.

4.3 Questions 2

a) In how many different ways can 15 people be allocated to three wagons on a train, if there are 5 available seats in the first wagon, 4 in the second wagon and 6 in the third?

b) Choose the appropriate notations and write down the general formula for any number of passengers, allocated to three wagons.

Solution:

a) In the first wagon of 15 people we can choose 5 in C_{15}^5 ways. Of the other 10 passengers we can choose 4 in C_{10}^4 different ways. There are 6 passengers left for the last wagon, i.e. we have the C_6^6 ways. Then by the multiplication principle for combinations the number of different ways is $C_{15}^5 . C_{10}^4 . C_6^6 = \frac{15!}{5!.\boxed{10!}} . \frac{\boxed{10!}}{4!.\boxed{6!}} . \frac{\boxed{6!}}{6!.0!} = \frac{15!}{5!.4!.6!} = 630630.$

b) Let the total number of passengers be n, for the first wagon $-i$ and for the second $-k$. Then the formula is: $C_n^i . C_{n-i}^k . C_{n-i-k}^{n-i-k} = \frac{n!}{i!.k!.(n-i-k)!}$.

LO: SP2.

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Guide for a Problem

Statistics and Probability

5.3. Simple probability

SP3, SP4, SP5, SP6

Deolinda D. Rasteiro

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 47, to be achieved with the development of this activity guide are the following:

1. To interpret probability of an event as a degree of belief.
2. To understand the distinction between a priori and posteriori probabilities.
3. To use a tree diagram to calculate probabilities.
4. To know what conditional probability is and be able to use it (Bayes' Theorem).
5. To calculate probabilities for series and parallel connections.
6. To define a random variable and a discrete probability distribution.
7. To state the criteria for a binomial model and define its parameters.
8. To calculate probabilities for a binomial model.
9. To state the criteria for a Poisson model and define its parameters.
10. To calculate probabilities for a Poisson model.
11. To state the expected value and variance for each of these models.
12. To understand when a random variable is continuous.
13. To explain the way in which probability calculations are carried out in the continuous case.
14. To handle probability statements involving continuous random variables.
15. To convert a problem involving a normal variable to the area under part of its density curve.
16. To relate the general normal distribution to the standardised normal distribution.
17. To use tables for the standardised normal variable.
18. To solve problems involving a normal variable using tables.
19. To define a random sample.
20. To know what a random distribution is.
21. To understand the term "mean square error" of an estimate.
22. To understand the term "unbiasedness" of an estimate.

The Table 47 shows the list of the competencies that we measure with the global test model that is proposed, where

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 47: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
Statistics and Probability									
SP3	3. Simple probability								
SP31	Interpret probability as a degree of belief								
SP32	Understand the distinction between a priori and a posteriori probabilities								
SP33	Use a tree diagram to calculate probabilities								
SP34	Know what conditional probability is and be able to use it (Bayes' Theorem)								
SP35	Calculate probabilities for series and parallel connections								
SP4	4. Probability models								
SP41	Define a random variable and a discrete probability distribution								
SP42	State the criteria for a binomial model and define its parameters								
SP43	Calculate probabilities for a binomial model								
SP44	State the criteria for a Poisson model and define its parameters								
SP45	Calculate probabilities for a Poisson model								
SP46	State the expected value and variance for each of these models								
SP47	Understand when a random variable is continuous								
SP48	Explain the way in which probability calculations are carried out in the continuous case								

SP5	5. Normal distribution								
SP51	Handle probability statements involving continuous random variables								
SP52	Convert a problem involving a normal variable to the area under part of its density curve								
SP53	Relate the general normal distribution to the standardized normal distribution								
SP54	Use tables for the standardized normal variable								
SP55	Solve problems involving a normal variable using tables								
SP6	6. Sampling								
SP61	Define a random sample								
SP62	Know what a sampling distribution is								
SP63	Understand the term “mean square error” of an estimate								
SP64	Understand the term “unbiasedness” of an estimate								

2 CONTENTS

1. Simple probability.
2. Probabilities for series and parallel connections.
3. Probability models: binomial and *Poisson* distributions.
4. Probability of events involving continuous random variables.
5. Normal distribution.
6. Sampling distributions.
7. Mean squared error of an estimate.
8. Unbiased estimate.

3 TEST

The time to solve this exam will be 3 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 48.

Table 48: Learning outcomes involved in this assessment activity.

Statistics and Probability	
SP3	3. Simple probability
SP31	Interpret probability as a degree of belief
SP32	Understand the distinction between a priori and a posteriori probabilities
SP33	Use a tree diagram to calculate probabilities
SP34	Know what conditional probability is and be able to use it (Bayes' Theorem)
SP35	Calculate probabilities for series and parallel connections
SP4	4. Probability models
SP41	Define a random variable and a discrete probability distribution
SP42	State the criteria for a binomial model and define its parameters
SP43	Calculate probabilities for a binomial model
SP44	State the criteria for a Poisson model and define its parameters
SP45	Calculate probabilities for a Poisson model
SP46	State the expected value and variance for each of these models
SP47	Understand when a random variable is continuous
SP48	Explain the way in which probability calculations are carried out in the continuous case
SP5	5. Normal distribution
SP51	Handle probability statements involving continuous random variables
SP52	Convert a problem involving a normal variable to the area under part of its density curve
SP53	Relate the general normal distribution to the standardized normal distribution
SP54	Use tables for the standardized normal variable
SP55	Solve problems involving a normal variable using tables
SP6	6. Sampling
SP61	Define a random sample
SP62	Know what a random distribution is
SP63	Understand the term "mean square error" of an estimate
SP64	Understand the term "unbiasedness" of an estimate

The statement of the exam is the next:

3.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test.

1. Let Ω be the space of results associated to a certain random experience. Let A and B be two random events from Ω. Knowing that $P(\bar{A}) = 0.7, P(\bar{B}) = 0.5$ and $P(A \cap B) = 0.2$ the value of $P(\bar{B}/\bar{A})$ is:

   i) $4/5$

   ii) $4/7$

   iii) $6/7$

   iv) $2/7$

2. The probability distribution function of a real random variable X is given by the following table

x	-2	-1	1	2	3
$P(X = x)$	a	b	0.25	b	a

   where a and b are real numbers. Knowing that $P(X = -2) = 2P(X = 2)$ the value of $(X = 4a + 8b)$ is

   i) $1/16$

   ii) $1/5$

   iii) $1/8$

   iv) $1/4$

3. In a production line the probability of producing a piece with defect is 0.05. Assuming that a large number of pieces are produced the probability that, in a batch of 20 pieces, there are at most 18 perfect pieces is

   i) 0.2642

   ii) 0.0755

   iii) 0.7358

   iv) 1

4. The number of cars that cross Rainha Santa Isabel's bridge per minute is a random variable following a *Poisson* distribution with mean 10. The probability that more than 620 cars cross that bridge in one hour is:

   i) 0.2123

   ii) 0.7993

   iii) 0.2007

   iv) 0.7877

3.2 QUESTIONS 1

1. Let X and Y be two independent real random variables such that $X \sim P(0.2)$ and $Y \sim B(4,0.4)$. Determine, indicating all the necessary calculus and justifications, the value of $P(2X + Y = 4)$.
2. Let $X_i, i = 1, \dots, 50$, be a sample of the random variable $X \sim P(0.5)$.
   a. Calculate the mean value and the variance of the random variable $\bar{X} = \frac{1}{50}\sum_{i=1}^{50} X_i$.
   b. Determine $P(\bar{X} \geq 0.5)$ and give an interpretation of the calculated value.

3.3 QUESTIONS 2

Suppose that you are playing a computer game where the objective is to destroy a section of a railway line piloting an airplane (Figure 1). You are an element of the airplane crew that receives the order to destroy and you are the one that has to do the calculations and give the instructions to your fellow colleagues.

Figure 136. Railway section to be destroyed.

1. To the crew of an airplane is assigned the task of destroying a section of railway line. The crew's aim statistics reveal that the variable X, distance, in meters, from the point of impact of a bomb to the targeted line, follows a Normal distribution with standard deviation $6m$. It is considered equally probable that the bomb falls to one side (for which it is agreed to take $X > 0$) or to the other (for which it is agreed to take $X < 0$) of the line. The mission is considered fulfilled if at least one bomb hits the target, that is, if it "falls to less than $1m$ of the line".
   (a) Justify that X has mean value equal to zero.
   (b) What is the probability of the bomb hits the target?
   (c) Characterize the distribution function of the random variable Y = "Number of bombs that hit the target, supposing that n bombs are launched ($n \in$ IN)".
   (d) Suppose that $n > 20$ and determine, using the approximate distribution of Y, the minimum number of bombs that is necessary to launch in order to have a successful mission with probability greater than 0.95.

2. Suppose the weight, W, of each bomb is a random variable with Normal distribution with mean 65 kg and standard deviation $3kg$ and that the maximum load supported by the mission plane is $3\ tones$.

   (a) Deduce, justifying, the distribution of the random variable $\sum_{i=1}^{n} W_i$, where W_i is the weight of the i^{th} bomb.

   (b) Justify if the following statement is true or false:

   "For $n = 32$, the total weight of the bombs is almost certain supported by the plane!"

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

Solve the following test

1. Let Ω be the space of results associated to a certain random experience. Let A and B be two random events from Ω. Knowing that $P(\bar{A}) = 0.7, P(\bar{B}) = 0.5$ and $P(\cap B) = 0.2$ the value of $P(\bar{B}/\bar{A})$ is:

   i) $4/5$

   ii) $4/7$

   iii) $6/7$

   iv) $2/7$

   Solution:

   Since $P(\bar{A}) = 0.7, P(\bar{B}) = 0.5$ and $P(A \cap B) = 0.2$ we have

$$P(\bar{B}/\bar{A}) = \frac{P(\bar{B} \cap \bar{A})}{P(\bar{A})} = \frac{1 - P(A) - P(B) + P(A \cap B)}{0.5} = \frac{0.7 - 0.5 + 0.2}{0.7} = \frac{4}{7}$$

   The solution is ii).

2. The probability distribution function of a real random variable X is given by the following table

x	-2	-1	1	2	3
$P(X = x)$	a	b	0.25	b	a

   where a and b are real numbers. Knowing that $P(X = -2) = 2P(X = 2)$ the value of $P(X = 4a + 8b)$ is

   ii) $1/16$

   iii) $1/5$

   iv) $1/8$

   v) $1/4$

   Solution:

   Solving the system with two unknowns (a and b), we will have

$$\begin{cases} P(X = -2)2P(X = 2) \\ 2a + 2b + 0.25 = 1 \end{cases} \Leftrightarrow \begin{cases} a = 2b \\ 3a = 0.75 \end{cases} \Leftrightarrow \begin{cases} b = 0.125 \\ a = 0.25 \end{cases}.$$

   Thus, $P(X = 4a + 8b) = P(X = 2) = 0.125 = \frac{1}{8}$.

   The solution is iii).

3. In a production line the probability of producing a piece with defect is 0.05. Assuming that a large number of pieces are produced the probability that, in a batch of 20 pieces, there are at most 18 perfect pieces is

i) 0.2642

ii) 0.0755

iii) 0.7358

iv) 1

Solution:

$D =$ "A piece is produced with defect"

$P(D) = 0.05$; n is large. We have a batch of 20 pieces.

Let $E =$ "Extract randomly one piece from the production". Repeat E 20 times not in the same conditions (at each extraction the number of pieces is the previous minus 1).

Assume $X =$ "number of defect pieces of the 20 extracted". Since n is large we may assume that $X \dot{\sim} B(20, 0.05)$. Thus we have

$P(20 - X \ \leq 18) = P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1) = 1 - 0.73584 \cong 0.2642.$

The solution is i).

4. The number of cars that cross Rainha Santa Isabel's bridge per minute is a random variable following a *Poisson* distribution with mean 10. The probability that more than 620 cars cross that bridge in one hour is:

i) 0.2123

ii) 0.7993

iii) 0.2007

iv) 0.2129

Solution:

Let $X =$ "Number of cars that cross Rainha Santa Isabel's bridge per minute"

Since $\mathrm{E}(X) = 10$ and X follows a Poisson distribution we may conclude that $X \sim P(10)$.

Considering $Y =$ "Number of cars that cross Rainha Santa Isabel's bridge per hour$= \sum_{i=1}^{60} X_i$

where $X_1, X_2, \dots, X_{50}$ are independent random variables following the same distribution as X. Thus,

$Y \sim P\left(\sum_{i=1}^{60} 10\right) = P(600)$ and $P(Y > 620) = 1 - P(Y \leq 620) = 1 - 0.7993 = 0.2007.$

The solution is iii).

Observe that since the Poisson parameter is large, the student may approximate the distribution of Y to a normal distribution with mean 600 and variance also equal to 600 and calculate the asked probablitity, using continuity correction, as

$$P(Y > 620) = P\left(\frac{Y - 600}{\sqrt{600}} \geq \frac{619.5 - 600}{\sqrt{600}}\right) = 1 - P\left(Z \leq \frac{19.5}{24.49}\right) = 1 - 0.7871 = 0.2129.$$

In this case the solution is iv).

LO: SP3, SP4, SP5.

4.2 QUESTIONS 1

1. Let X and Y be two independent real random variables such that $X \sim P(0.2)$ and $Y \sim B(4{,}0.4)$. Determine, indicating all the necessary calculus and justifications, the value of $P(2X+Y=4)$.

Solution:

X and Y be two independent real random variables such that $X \sim P(0.2)$ and $Y \sim B(4{,}0.4)$ thus, $S_X = \mathrm{IN}_0$ and $S_Y = \{0,1,2,3,4\}$.

If we construct a double entry table with possible values for $X$and Y we have

X \ Y	0	1	2	3	4
0	0	1	2	3	4
1	2	3	4	5	6
2	4	5	6	7	8
⋮	⋮	⋮	⋮	⋮	⋮

and it follows that,

$$P(2X+Y=4) = P[(X=0 \cap Y=4) \cup (X=1 \cap Y=2) \cup (X=2 \cap Y=0)]$$

since the events are disjoint and X and Y are independent

$$= P(X=0) \times P(Y=4) + P(X=1) \times P(Y=2) + P(X=2) \times P(Y=0)$$

$$= 0.81873 \times 0.0256 + 0.16375 \times 0.3456 + 0.01638 \times 0.1296 = 0.079674.$$

2. Let $X_i, i = 1, \dots, 50$, be a sample of the random variable $X \sim P(0.5)$.

a) Calculate the mean value and the variance of the random variable $\bar{X} = \frac{1}{50}\sum_{i=1}^{50} X_i$.

b) Determine $P(\bar{X} \geq 0.5)$ and give an interpretation of the calculated value.

Solution:

a) Since $X \sim P(0.5)$ we know that $E(X) = V(X) = 0.5$. The mean value of $\bar{X}$ is

$$E(\bar{X}) = E\left(\frac{1}{50}\sum_{i=1}^{50} X_i\right) = \frac{1}{50}\sum_{i=1}^{50} E(X_i) = E(X) = 0.5.$$

The variance of $\bar{X}$ is

$$V(\bar{X}) = V\left(\frac{1}{50}\sum_{i=1}^{50} X_i\right) = \frac{1}{50^2}\sum_{i=1}^{50} V(X_i) = \frac{V(X)}{50} = \frac{0.5^2}{50} = 0.01.$$

b) Since $X_1, X_2, \dots, X_{50}$ is a sample of $X, X_i, i = 1, \dots, 50$ are independent and identically distributed with X. Thus, using the Central Limit Theorem, we have

$$\frac{\bar{X} - E(\bar{X})}{\sqrt{V(\bar{X})}} \dot{\sim} N(0,1)$$

From a) we know that $E(\bar{X}) = 0.5$ and $V(\bar{X}) = 0.01$, once

$$\frac{\bar{X} - 0.5}{\sqrt{0.01}} \dot{\sim} N(0,1) \Leftrightarrow \frac{\bar{X} - 0.5}{0.1} \dot{\sim} N(0,1) \Leftrightarrow \bar{X} \dot{\sim} N(0.5, 0.1)$$

and

$P(\bar{X} \geq 0.5) \cong 0.5$, that is 0.5 is the median of the $\bar{X}$ sample distribution.

LO: SP4, SP5, SP6.

4.3 QUESTIONS 2

Suppose that you are playing a computer game where the objective is to destroy a section of a railway line piloting an airplane (Figure 1). You are an element of the airplane crew that receives the order to destroy and you are the one that has to do the calculations and give the instructions to your fellow colleagues.

Figure 137. Railway section to be destroyed

1. To the crew of an airplane is assigned the task of destroying a section of railway line. The crew's aim statistics reveal that the variable X, distance, in meters, from the point of impact of a bomb to the targeted line, follows a Normal distribution with standard deviation $6m$. It is considered equally probable that the bomb falls to one side (for which it is agreed to take $X > 0$) or to the other (for which it is agreed to take $X < 0$) of the line. The mission is considered fulfilled if at least one bomb hits the target, that is, if it "falls to less than $1m$ of the line".

   (a) Justify that X has mean value equal to zero.

If $X =$ "Distance, in meters, from the point of impact of a bomb to the target line" and $X \sim N(\mu, 6)$, X is symmetric with respect to μ. Since $P(X > 0) = P(X < 0)$ and X is continuous we may conclude that $\mu = 0$.

(b) What is the probability of the bomb hits the target?

$$P(\text{"Bomb hits the target"}) = P(-1 < X < 1) =$$
$$P\left(\frac{-1-0}{6} < \frac{X-0}{6} < \frac{1-0}{6}\right) = P\left(\frac{-1}{6} < Z < \frac{1}{6}\right) = 0.132368,$$

where $Z = \frac{X}{6} \sim N(0,1)$.

(c) Characterize the distribution function of the random variable Y = "Number of bombs that hit the target, supposing that n bombs are launched ($n \in$ IN)".

Let $E =$ "A bomb is launched". E is repeated n times in the same conditions. Then

$$S_Y = \{0,1,\dots,n\}; Y \sim B\big(n, P(-1 < X < 1)\big) = B(n, 0.132368)$$

(d) Suppose that $n > 20$ and determine, using the approximate distribution of Y, the minimum number of bombs that is necessary to launch in order to have a successful mission with probability greater than 0.95.

Since $n > 20,\ Y \dot{\sim} N\left(0.132368 \times n, \sqrt{0.132368(1-0.132368) \times n}\right) = N(0.132368 \times n, 0.114847\sqrt{n})$.

The objective is to find the minimum number of bombs that should be launched in order to have a successful mission with probability greater than 0.95, that is,

$$P(Y \geq 1) > 0.95 \iff P\left(\frac{Y - 0.132368n}{0.114847\sqrt{n}} \geq \frac{1-0.566n}{0.114847\sqrt{n}}\right) > 0.95$$
$$\iff 1 - P\left(Z \geq \frac{1-\ 0.132368n}{0.114847\sqrt{n}}\right) > 0.95$$
$$\iff \frac{1-0.132368n}{0.114847\sqrt{n}} < -1.645$$

Solving the nonlinear inequality towards n we will obtain

$$\sqrt{n} > 4.5925 \iff n > 21.091$$

Then the minimum number of bombs that have to be launched is 22.

2. Suppose the weight, W, of each bomb is a random variable with Normal distribution with mean 65 kg and standard deviation $3kg$ and that the maximum load supported by the mission plane is 3 $tones$.

(a) Deduce, justifying, the distribution of the random variable $\sum_{i=1}^{n} W_i$, where W_i is the weight of the i^{th} bomb.

The random variables $W_i, i = 1, \dots, n$ are independent and identically distributed with W that has normal distribution with mean 65 and variance 9.

Using the stability of the normal distribution we conclude

$$\sum_{i=1}^{n} W_i \sim N\left(\sum_{i=1}^{n} 65\,, \sqrt{\sum_{i=1}^{n} 3^2}\right) = N\left(65 \times n, 3 \times \sqrt{n}\right).$$

(b) Justify if the following statement is true or false:

"For $n = 32$, the total weight of the bombs is almost certain supported by the plane!"

If $n = 32$ then

$$\sum_{i=1}^{32} W_i \sim N\left(\sum_{i=1}^{32} 65\,, \sqrt{\sum_{i=1}^{32} 3^2}\right) = N\left(65 \times 32, 3 \times \sqrt{32}\right) = N(2080{,}12\sqrt{2}).$$

and thus,

$$P\left(\sum_{i=1}^{32} W_i \leq 3000\right) \cong 1,$$

Which means that the statement is true.

LO: SP4, SP5.

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Guide for a Problem

Statistics and Probability

5.4. Probability models

SP4

Ion Mierlus-Mazilu

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 49, to be achieved with the development of this activity guide are the following:

1. Define a random variable and a discrete probability distribution.
2. State the criteria for a binomial model and define its parameters.
3. Calculate probabilities for a binomial model.
4. State the criteria for a Poisson model and define its parameters.
5. Calculate probabilities for a Poisson model.
6. State the expected value and variance for each of these models.
7. Understand when a random variable is continuous.
8. Explain the way in which probability calculations are carried out in the continuous case.

Table 49 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- = very important
- = medium important
- = less important

Table 49: Competencies that we measure with the global test model that is proposed

		C1	C2	C3	C4	C5	C6	C7	C8
Statistics and Probability									
SP4	4. Probability models								
SP41	Define a random variable and a discrete probability distribution								
SP42	State the criteria for a binomial model and define its parameters								
SP43	Calculate probabilities for a binomial model								
SP44	State the criteria for a Poisson model and define its parameters								
SP45	Calculate probabilities for a Poisson model								

SP46	State the expected value and variance for each of these models								
SP47	Understand when a random variable is continuous								
SP48	Explain the way in which probability calculations are carried out in the continuous case								

2 CONTENTS

1. Define a random variable and a discrete probability distribution.
2. State the criteria for a binomial model and define its parameters.
3. Calculate probabilities for a binomial model.
4. State the criteria for a Poisson model and define its parameters.
5. Calculate probabilities for a Poisson model.
6. State the expected value and variance for each of these models.
7. Understand when a random variable is continuous.
8. Explain the way in which probability calculations are carried out in the continuous case.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 50.

Table 50: Learning outcomes (LO) involved in this assessment activity.

Statistics and Probability	
SP4	4. Probability models
SP41	Define a random variable and a discrete probability distribution
SP42	State the criteria for a binomial model and define its parameters
SP43	Calculate probabilities for a binomial model
SP44	State the criteria for a Poisson model and define its parameters
SP45	Calculate probabilities for a Poisson model
SP46	State the expected value and variance for each of these models
SP47	Understand when a random variable is continuous
SP48	Explain the way in which probability calculations are carried out in the continuous case

The statement of the exam is the next:

3.1 Multiple-Choice Questions

1. In a research report by Richard H. Weindruch of the UCLA Medical School, it is claimed that mice with an average life span of 32 months will live to be about 40 months old when 40% of the calories in their food are replaced by vitamins and protein. Is there any reason to believe that μ < 40 if 64 mice that are placed on this diet have an average life of 38 months with a standard deviation of 5.8 months? Use a P-value in your conclusion.

   i) Reject H_0

   ii) Don't reject H_0

   iii) It does not matter

2. If the distribution of life spans in exercise #4 is approximately normal, how large a sample: is required in order that the probability of committing a type 11 error be 0.1 when the true mean is 35.9 months? Assume that a = 5.8 months.

   i) 18

   ii) 20

   iii) 17

3. In the American Heart Association journal Hypertension, researchers report that individuals who practice Transcendental Meditation (TM) lower their blood pressure significantly. If a random sample of 225 male TM practitioners meditate for 8.5 hours per week with a standard deviation of 2.25 hours, does that suggest that, on average, men who use TM meditate more than 8 hours per week?

   i) Yes

   ii) No

   iii) Sometimes

4. How large a sample is required in exercise #7 if the power of our test is to be 0.8 when the true mean meditation time exceeds the hypothesized value by 1.2σ ? Use α = 0.05.

   i) 4

   ii) 5

   iii) 6

5. A fuel oil company claims that one-fifth of the homes in a certain city are heated by oil. Do we have reason to believe that less than 1/5 are heated by oil if, in a random sample of 1000 homes in this city, it is found that 136 are heated by oil? Use a P-value in **y**our conclusion.

   i) Yes

   ii) No

   iii) Not always

3.2 QUESTIONS 1

In a factory there were 10 accidents during 2 weeks. After this equipment were renewed and during the following 3 weeks there were 5 accidents. Did the measures have a significant effect on the rate of accidents?

3.3 QUESTIONS 2

A manufacturer has developed a new fishing line, which he claims has a mean breaking strength of 15 kilograms with a standard deviation of 0.5 kilogram. To test the hypothesis that, μ = 15 kilograms against the alternative that p < 15 kilograms, a random sample of 50 lines will be tested. The critical region is defined to be $\bar{x} < 14.9$.

(a) Find the probability of committing a type I error when H_0 is true.

(b) Evaluate β for the alternatives p =14.8 and μ = 14.9 kilograms

3.4 QUESTIONS 3

An electrical firm manufactures light bulbs that have a lifetime that is approximately normally distributed with a mean of 800 hours and a standard deviation of 40 hours. Test tbe hypothesis that *p* = 800 hours against the alternative $p \neq 800$ hours if a random sample of 30 bulbs has an average life of 788 hours. Use a P-value in your answers.

3.5 QUESTIONS 4

Amstat News (December 2004) lists median salaries for associate professors of statistics at research institutions and at liberal arts and other institutions in the United States. Assume a sample of 200 associate professors from research institutions having an average salary of $70,750 per year with a standard deviation of S6000. Assume also a sample of 200 associate professors from other types of institutions having an average salary of $65,200 with a standard deviation of $5000. Test the hypothesis that the mean salary for associate professors in research institutions is $2000 higher than for those in other institutions. Use a 0.01 level of significance. Assume equal variances.

3.6 QUESTIONS 5

A new radar device is being considered for a certain defense missile system. The system is checked by experimenting with actual aircraft in which a kill or a no kill is simulated. If in 300 trials, 250 kills occur, accept or reject, at the 0.04 level of significance, the claim that the probability of a kill with the new system does not exceed the 0.8 probability of the existing device.

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. The hypotheses are

   $H_0: \mu = 40$ months

   $H_0: \mu < 40$ months

   Now, $z = \frac{38-40}{5.8/\sqrt{64}} = -2.76$, and P-value $= P(Z < -2.76) = 0.0029$

   Decision: reject H_0

2. $\beta = 0.1, \sigma = 5.8, \delta = 35.9 - 40 = -4.1$. Assume $\alpha = 0.05$ then, $z_{0.05} = 1.645$, $z_{0.10} = 1.28$.

   Therefore,

$$n = \frac{(1.645 + 1.28)^2(5.8)^2}{(-4.1)^2} = 17.12 \approx 18$$

   due to round up.

3. The hypotheses are

$$H_0: \mu = 8$$
$$H_1: \mu > 8$$

   Now, $z = \frac{8.5-8}{2.25/\sqrt{225}} = 3.33$, and P-value $= P(Z > 3.33) = 0.0004$.

   Decision: Reject H_0 and conclude that men who use *TM*, on average, mediate more than 8 hours per week.

4. $n = \frac{(1.645+0.842)^2(2.25)^2}{[(1.2)(2.25)]^2} = 4.29$.

   The sample size would be 5.

5. The hypotheses are:

$$H_0: p = 0.2$$
$$H_1: p < 0.2$$

   Then

$$P_{-value} \approx P\left(Z < \frac{136 - (1000)(0.2)}{\sqrt{(1000)(0.2)(0.8)}}\right) = P(Z < -5.06) \approx 0$$

   Decision: Reject H_0; less than 1/5 of the homes in the city are heated by oil.

4.2 QUESTIONS 1

$X(2)$ = Number of accidents during 2 weeks before the renewal, with intensity λ_X.

$Y(3) = 3$ after λ_Y

Then $(Y(3)|X(2) + Y(3) = n = 15) \sim$ Binomial $(n = 15, p = \frac{3\lambda_Y}{2\lambda_X + 3\lambda_Y})$.

We want to test $H_0: \lambda_X = \lambda_Y \Leftrightarrow p = \frac{3}{5}$. The observed number of accidents is 5 and therefore a one-sided ρ -value is obtained by

$$P(Y(3) \leq 5) = \binom{15}{0}(3/5)^0(2/5)^{15} + \cdots + \binom{15}{5}(3/5)^5(2/5)^{10} = 0.0338.$$

The computation of the last expression is simplified by using tables over the Binomial distribution, e.g. Table 1, p. 839 in Wackerly et al 2007.

For a two-tailed test we also have to compute the probability of extremely large values. Since the expected number of accidents is $n \cdot p = 15 \cdot 3/5 = 9$, we compute

$$P(Y(3) \geq 13) = \binom{15}{13}(3/5)^{13}(2/5)^2 + \binom{15}{14}(3/5)^{14}(2/5)^1 + \binom{15}{15}(3/5)^{15}(2/5)^0 = 0.0271.$$

The ρ -value for a two-tailed test is 0.0338 + 0.0271 > 0.05 and the conclusion is that the effect of renewed equipment is not significant.

4.3 QUESTIONS 2

(a) $n = 50, \mu = 0.5, \sigma = 0.5$, and $\sigma_{\bar{X}} = \frac{0.5}{\sqrt{50}} = 0.071$, with $z = \frac{14.9-15}{0.071} = -1.41$. Hence $\alpha = P(Z < -1.141) = 0.0793$.

(b) If $\mu = 14.8, z = \frac{14.9-14.8}{0.071} = 1.141$. So, $\beta = P(Z > 1.41) = 0.0793$. If $\mu = 14.9$, then $z = 0$ and $\beta = P(Z > 0) = 0.5$.

4.4 QUESTIONS 3

The hypotheses are

$$H_0: \mu = 800$$
$$H_1: \mu \neq 800$$

Now, $z = \frac{788-800}{40/\sqrt{30}} = -1.64$, and P-value $= 2P(Z < -1.64) = (2)(0.0505) = 0.1010$

Hence, the mean is not significantly different from 800 for α < 0.101.

4.5 QUESTIONS 4

The hypotheses are:

$$H_0: \mu_1 - \mu_2 = \$2{,}000$$
$$H_1: \mu_1 - \mu_2 > \$2{,}000$$

α = 0.01. Critical region: z > 2.33.

$z = \frac{(70750-65200)-2000}{\sqrt{(6000)^2/200+(5000)^2/200}} = 6.43$, with a P-value $= P(Z > 6.43) \approx 0$

Reject H_0 and conclude that the mean salary for associate professors in research institutions is $2000 higher than for those in other institutions.

4.6 QUESTIONS 5

The hypotheses are

$$H_0: p = 0.8$$
$$H_1: p > 0.8$$

α = 0.04

Critical region: $z > 1.75$

Computation:

$$z = \frac{250 - (300)(0.8)}{\sqrt{(300)(0.8)(0.2)}} = 1.44$$

Decision: Fail to reject H_0; it cannot conclude that the new missile system is more accurate.

5 PROJECTS

5.1 PROJECT 1

1. Let σ = 3.1 be the true standard deviation of the population from which a random sample is chosen. How large should the sample size be for testing H_0 : μ = 5 versus Ha : μ = 5.5, in order that α = 0.01 and β = 0.05?

2. A machine is considered to be unsatisfactory if it produces more than 6% defectives. It is suspected that the machine is unsatisfactory. A random sample of 130 items produced by the machine contains 16 defectives. Does the sample evidence support the claim that the machine is unsatisfactory? Use $\alpha = 0.01$.

3. In attempting to control the strength of the wastes discharged into a nearby river, an industrial firm has taken a number of restorative measures. The firm believes that they have lowered the oxygen consuming power of their wastes from a previous mean of 400 manganate in parts per million. To test this belief, readings are taken on $n = 20$ successive days. A sample mean of 315.5 and the sample standard deviation 105.23 are obtained. Assume that these 20 values can be treated as a random sample from a normal population. Test the appropriate hypothesis. Use α = 0.05.

5.2 PROJECT 2

In a frequently traveled stretch of the I-75 expressway, where the posted speed is 70 kmph, it is thought that people travel on the average of at least 75 km/h. To check this claim, the following radar measurements of the speeds (in km/h) is obtained for 10 vehicles traveling on this stretch of the interstate highway.

66 74 79 80 69 77 78 65 79 81

Do the data provide sufficient evidence to indicate that the mean speed at which people travel on this stretch of highway is at most 75 km/h? Test the appropriate hypothesis using $\alpha = 0.01$. Draw a box plot and normal plot for this data, and comment.

6 SOLUTIONS OF THE PROJECTS

6.1 PROJECT 1

1. Let σ = 3.2 be the true standard deviation of the population from which a random sample is chosen. How large should the sample size be for testing H_0: μ = 7 versus Ha : μ = 4.5, in order that α = 0.01 and β = 0.05?

   We are given μ_0 = 5 and μ_a = 5.5. Also, $z_\alpha = z_{0.01}$ = 2.33 and $z_\beta = z_{0.05}$ = 1.645. Hence, the sample size

   $$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_a - \mu_0)^2} = \frac{(2.33+1.645)^2 (3.2)^2}{(2.5)^2} = 25.88.$$

   So, *n* = 26 will provide the desired levels. That is, in order for us to test the foregoing hypothesis, we must randomly select 608 observations from the given population.

2. A machine is considered to be unsatisfactory if it produces more than 6% defectives. It is suspected that the machine is unsatisfactory. A random sample of 130 items produced by the machine contains 16 defectives. Does the sample evidence support the claim that the machine is unsatisfactory? Use $\alpha = 0.01$.

   Let *Y* be the number of observed defectives. This follows a binomial distribution. However, because np_0 and nq_0 are greater than 5, we can use a normal approximation to the binomial to test the hypothesis. So we need to test $H_0 : p = 0.06$ versus $H_a : p > 0.06$. Let the point estimate of p be $\hat{p} = \left(\frac{Y}{n}\right) = 0.119$, the sample proportion. Then the value of the TS is

   $$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} = \frac{0.119 - 0.06}{\sqrt{\frac{(0.06)(0.92)}{130}}} = 2.92\ .$$

   For α = 0.01, $z_{0.01}$ = 2.33. Hence, the rejection region is {z > 2.33}.

   Decision: Because 2.92 is greater than 2.33, we reject H_0. We conclude that the evidence supports the claim that the machine is unsatisfactory.

3. In attempting to control the strength of the wastes discharged into a nearby river, an industrial firm has taken a number of restorative measures. The firm believes that they have lowered the oxygen consuming power of their wastes from a previous mean of 400 manganate in parts per million. To test this belief, readings are taken on $n = 20$ successive days. A sample mean of 315.5 and the sample standard deviation 105.23 are obtained. Assume that these 20 values can be treated as a random sample from a normal population. Test the appropriate hypothesis. Use α = 0.05.

   Here we need to test the following hypothesis: H0 : μ = 450 vs. Ha : μ < 450 Given $n = 20$, $\bar{x} = 312.5$, and $s = 106.23$. The observed test statistic is

   $$t = \frac{315.5 - 400}{105.23/\sqrt{20}} = -0.17\ .$$

   The rejection region for α = 0.05 and with 19 degrees of freedom is the set of t-values such that

   $$\{t < -t0.05.19\} = \{t < -1.729\}.$$

   Decision: Because t = −0.17 is more than −1.729, do not reject H_0. There is sufficient evidence to confirm the firm’s belief. For large random samples, the following procedure is used to perform tests of hypotheses about the population proportion, *p*.

6.2 PROJECT 2

In a frequently traveled stretch of the I-75 expressway, where the posted speed is 70 kmph, it is thought that people travel on the average of at least 75 km/h. To check this claim, the following radar measurements of the speeds (in km/h) is obtained for 10 vehicles traveling on this stretch of the interstate highway.

66 74 79 80 69 77 78 65 79 81

Do the data provide sufficient evidence to indicate that the mean speed at which people travel on this stretch of highway is at most 75 km/h? Test the appropriate hypothesis using $\alpha = 0.01$. Draw a box plot and normal plot for this data, and comment.

We need to test $H_0 : \mu = 75$ vs. $Ha : \mu > 75$

For this sample, the sample mean is x = 74.8 km/h and the standard deviation is σ = 5.9963 km/h. Hence, the observed test statistic is

$$t = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{74.8 - 75}{5.9963/\sqrt{10}} = -0.10547$$

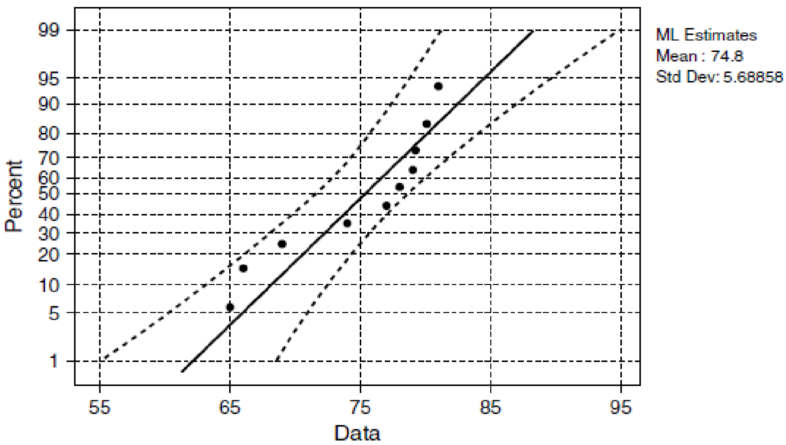

Figure 138

From the t-table, t0.019 = 2.821. Hence, the rejection region is {t > 2.821}. Because, $t = -0.10547$ does not fall in the rejection region, we do not reject the null hypothesis at $\alpha = 0.01$.

Note that we assumed that the vehicles were randomly selected and that collected data follow the normal distribution, because of the small sample size, n < 30, we use the *t*-test.

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Guide for a Problem

Statistics and Probability

5.5. Statistical inference

SP7

Jana Gabková and Daniela Richtáriková

1 OBJECTIVES

The objectives, related to the learning outcomes detailed in Table 51, to be achieved with the development of this activity guide are the following:

1. To identify two major areas of statistical inference.
2. To distinguish between point estimation and interval estimation.
3. To distinguish between confidence, prediction and statistical tolerance intervals.
4. To interpret a 100(1–α)% confidence interval.
5. To explain the relationship between the length of a confidence interval and precision of estimation.
6. To determinate the error (E) when estimating a parameter.
7. For the mean μ of a normal distribution with known variance σ^2:
   - to determine: the point estimator, 100(1–α)% confidence interval on μ, a sample size n to satisfy a predetermined level of error (E) in estimation of μ;
   - to explain the meaning "the critical value of normal distribution" and to be able to find it in tables.
8. For the mean μ of a normal distribution with unknown variance σ^2:
   - to determine: the point estimator, 100(1–α)% confidence interval on μ,
   - to explain the meaning and to be able to find the critical value of t–distribution in tables.
9. For variance σ^2:
   - to determine the point estimator and 100(1–α)% confidence interval on σ2,
   - to explain the meaning and be able to find the critical value of χ^2 -distribution in tables.
10. To explain the meaning and be able to calculate 100(1–α)% prediction interval for a new observation.
11. To set a p-tolerance interval with 100(1–α)% confidence for normal population with unknown parameters μ and σ^2.
12. To explain terms: null hypothesis, alternative hypothesis, test statistic, acceptance region, rejection region, critical value, type I error probability (α), type II error probability (β) and power of a test ($1-\beta$).
13. To explain the meaning and determinate the values of the type I and II error probability, power of a test and explain the relationships between α and β.
14. To determine the procedure of hypothesis testing and compare the α–value approach with the *P-value* approach in evaluating hypothesis test results.
15. To test hypotheses on the mean μ of a normal distribution, variance σ^2 known using z–test, 100(1–α)% confidence interval for μ, and *P–value*.
16. To determine the sample size n of a z-test for statistical inference on μ by applying an appropriate sample size formula and operating characteristic (OC) curve.
17. To explain the effect of the sample size n on the statistical significance and power of the test.

18. To test hypotheses on mean μ of a normal distribution, variance σ^2 unknown using: *t*–test, 100(1–α)% confidence interval for μ, and *P*–value.
19. To determine the sample size *n* of a *t*-test for statistical inference on μ by using an appropriate operating characteristic (OC) curve.
20. To test hypotheses on the σ^2 of a normal population using: χ^2– test, 100(1–α)% confidence interval for σ^2, and *P*–value.
21. To determine the sample size *n* of a χ^2-test for statistical inference on σ^2 by using an appropriate operating characteristic (OC) curve.
22. To test hypotheses on the mean difference $\mu_1 - \mu_2$of a normal distribution, when σ_1^2 and σ_2^2 are known, using: *z*–test, 100(1–*α*)% confidence interval for $\mu_1 - \mu_2$ and *P–value.*
23. To determine the sample size *n* of a z-test for the statistical inference on $\mu_1 - \mu_2$, which satisfies a preselected level of error E, using an appropriate sample size formula and operating characteristic (OC) curve.
24. To test hypotheses on the mean difference $\mu_1 - \mu_2$of a normal distribution, when σ_1^2 and σ_2^2 are unknown, using: *t*–test, 100(1–*α*)% confidence interval for $\mu_1 - \mu_2$ and *P–value.*
25. To determine the sample size *n* of a *t*-test for statistical inference on $\mu_1 - \mu_2$ by using an appropriate operating characteristic (OC) curve.
26. To test hypotheses on the ratio of two variances σ_1^2/σ_2^2 using: *F*-test, $100(1-\alpha)\%$ confidence interval for σ_1^2/σ_2^2 and *P–value.*
27. To determine the sample size of an *F*-test for statistical inference on σ_1^2/σ_2^2 using an appropriate operating characteristic (OC) curve.
28. To apply the normality test on measured data, for example by using Shapiro-Wilk test.
29. To be able to use statistical software and to analyse its results in statistical data processing.

Table 51 shows the list of the competencies that are measured with the global test model that is proposed, where the different colors used have the following meaning:

- (green) = very important
- (yellow) = medium important
- (red) = less important

Table 51: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

		C1	C2	C3	C4	C5	C6	C7	C8
STATISTICS AND PROBABILITY									
SP7	7. Statistical inference								
SP71	Apply confidence intervals to sample estimates	yellow	green	red	red	yellow	green	red	yellow
SP72	Follow the main steps in a test of hypothesis.	green	green	green	yellow	green	yellow	yellow	green

SP73	Explain the difference between a test of hypothesis and a significance test (p–value)								
SP74	Define the level of a test (error of the first kind)								
SP75	Define the power of a test (error of the second kind)								
SP76	State the link between the distribution of a normal variable and that of means of samples								
SP77	Place confidence intervals around the sample estimate of a population mean								
SP78	Test claims about the population mean using results from sampling								
SP79	Recognise whether an alternative hypothesis leads to a one-tail or a two-tail test								
SP80	Compare the approaches of using confidence intervals and hypothesis tests								

2 CONTENTS

1. Point estimation.
2. Interval estimation.
3. Two sided and one sided confidence interval for the mean μ of a random sample from normal population $N(\mu, \sigma^2)$ with a variance σ^2 known.
4. Two sided and one side confidence interval for the mean μ of a random sample from normal population $N(\mu, \sigma^2)$ with a variance σ^2 unknown.
5. Prediction interval.
6. Statistical p-tolerance interval.
7. Tests of Hypotheses for a Single Sample.
8. Tests on the mean of a normal distribution, variance known.
9. Tests on the mean of a normal distribution, variance unknown.
10. Hypothesis tests on the variance of a normal population.
11. Statistical Inference for Two Samples.
12. Inference for a difference in means of two normal distributions, variances known.
13. Inference for a difference in means of two normal distributions, variances unknown.
14. Inference on the variances of two normal populations.

3 TEST

The time to solve this exam will be 2 hours (approximately).

The learning outcomes to be achieved with this procedure are detailed in Table 52.

Table 52: Learning outcomes (LO) involved in this assessment activity.

Statistics and Probability	
SP7	7. Statistical Inference
SP71	Apply confidence intervals to sample estimates
SP72	Follow the main steps in a test of hypothesis.
SP73	Explain the difference between a test of hypothesis and a significance test (p–value)
SP74	Define the level of a test (error of the first kind)
SP75	Define the power of a test (error of the second kind)
SP77	Place confidence intervals around the sample estimate of a population mean
SP78	Test claims about the population mean using results from sampling
SP79	Recognise whether an alternative hypothesis leads to a one-tail or a two-tail test
SP80	Compare the approaches of using confidence intervals and hypothesis tests

3.1 MULTIPLE-CHOICE QUESTIONS

1. Suppose that $\left[23{,}7 - 1{,}96\frac{20}{\sqrt{50}}; 23{,}7 + 1{,}96\frac{20}{\sqrt{50}}\right]$is the 100 $(1-\alpha)$ % confidence interval for the mean μ of a random sample from normal population $N(\mu, \sigma^2)$ with a variance σ^2 known.
   a) the value 23,7 represents
      i) sample mean
      ii) population mean
      iii) sample variance
      iv) sample median
   b) the variance of a population is
      i) 20
      ii) 400
      iii) 50
      iv) $\sqrt{50}$

c) the significance level is

i) 90%

ii) 5%

iii) 95%

iv) 99%

d) What size of a sample *n* is necessary if the length of the confidence interval is 40?

i) 3

ii) 4

iii) 5

iv) 8

2. How is the length of confidence interval influenced by a level of test significance α?

a) The smaller α is, the wider confidence interval is.

b) The bigger α is, the wider confidence interval is.

c) The bigger α is, the narrower confidence interval is.

d) The smaller α is, the narrower confidence interval is.

3. Which of the below relations defines the probability of type II error?

a) $\alpha = P(\text{reject} H_0 | H_0 \text{is false})$

b) $\alpha = P(\text{failtoreject} H_0 | H_0 \text{is true})$

c) $\beta = P(\text{reject} H_0 | H_0 \text{is true})$

d) $\beta = P(\text{failtoreject} H_0 | H_0 \text{is false})$

4. What relation is between *P–value* and the level of significance α, if hypothesis $H_0$is rejected at the level of significance α?

a) $P - value \geq \alpha$

b) $P - value \geq 2\alpha$

c) $P - value \geq \alpha/2$

d) $P - value \leq \alpha$

e) $P - value \leq 2\alpha$

f) $P - value \leq \alpha/2$

5. *z*–test is a test of hypothesis on

a) mean, variance unknown

b) variance, mean known

c) variance, mean unknown

d) mean, variance known

6. χ^2– test is used for hypothesis tests on
   a) mean of single normal population
   b) the variance of two normal populations
   c) the variance of single normal population
   d) mean of two normal populations

7. The figure presents the result of testing
   a) $H_0: \mu_1 = \mu_2 \rightarrow H_1: \mu_1 > \mu_2$
   b) $H_0: \mu_1 = \mu_2 \rightarrow H_1: \mu_1 < \mu_2$
   c) $H_0: \mu_1 < \mu_2 \rightarrow H_1: \mu_1 = \mu_2$
   d) $H_0: \sigma_1^2 = \sigma_2^2 \rightarrow H_1: \mu_1 = \mu_2$

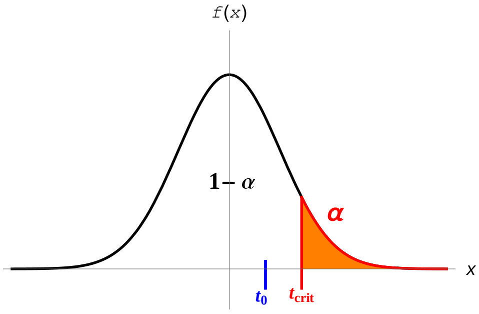

Figure 139

8. The figure presents the result
   a) $H_0: \mu_1 = \mu_2$ fail to reject
   b) $H_0: \mu_1 = \mu_2$ reject
   c) $H_0: \mu_1 < \mu_2$ reject
   d) $H_0: \sigma_1^2 = \sigma_2^2$ fail to reject

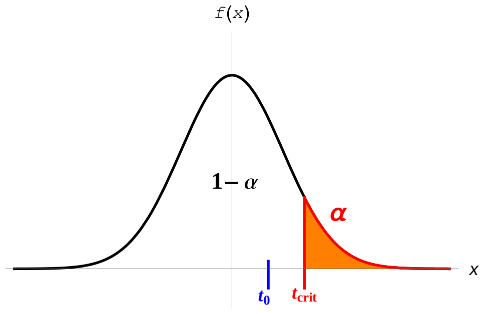

Figure 140

3.2 QUESTIONS 1

Task A

The joint seal thickness *X* (mm) of the gas pipe must satisfy the standard $\sigma_N^2 = 0{,}06$. The following table presents a random sample from a normal population *X*~ *N*(μ, σ^2) where seven pipe connections were randomly checked.~

i	1.	2.	3.	4.	5.	6.	7.
x_i	1,3	1,8	1,4	1,2	0,9	1,5	1,7

1.
   a) Estimate parameters of joint seal thickness normal population
   b) Does the thickness in the inspected section satisfy the standard? Justify your claim via the appropriate confidence interval for α = 0,05.
   c) With a probability of 95%, estimate the endpoints of the interval that covers the unknown mean value of the pipe joint seal thickness.

2. Suppose the seal thickness selection is a random sample from a normal distribution $X \sim N(\mu; 0{,}09)$
   a) Estimate the upper bound of an interval which covers the unknown mean value of the pipe joint seal thickness with a probability of 99 %.
   b) How many randomly selected seals must be checked to avoid the error greater than 0,02 mm when calculating the 95% two sided confidence interval for an unknown mean of the seal thickness?
   c) Estimate the value of seal thickness for next observation with probability 97 %.
   d) Calculate two sided intervals that covers at least 95% seal thickness values from the population with probability 90 %.
   e) Product inspection of joint seal thicknesses found out that 24 from 300 randomly inspected pieces were defective. Find the 95% confidence interval with the lower limit for the proportion of defective pipe joint seal thicknesses.
   f) How many randomly selected pipe joints must be checked to ensure that an error greater than 0,02 mm does not occur when calculating the 95% confidence interval for the proportion of defective pipe joint seal thicknesses?
      i) Solve the task for an unknown proportion of defective pipe joint seal thicknesses.
      ii) Solve the task for given proportion of defective pipe joint seal thicknesses $p = 5\%$.

Task B

Instructions

Express hypotheses in symbolic and also in verbal notation.

Find necessary data via statistical software.

Test hypotheses using

a) a statistic

b) a confidence interval

c) a *P–value*

d) the graph of a probability density function (mark the critical value, α, and *P–value*).

A company tested the life length of Infinity Light Bulbs (X; unit: hour). The following table presents a random sample of 30 bulb life lengths x_i.

i	x_i	*i*	x_i	*i*	x_i
1	727	11	831	21	725
2	755	12	742	22	735
3	714	13	784	23	770
4	840	14	807	24	792

5	772	15	820	25	765
6	750	16	812	26	749
7	814	17	804	27	829
8	820	18	754	28	821
9	753	19	715	29	816
10	796	20	845	30	743

Problem 1A

Suppose that the life length of an Infinity light bulb (X; unit: hour) follows the normal distribution with $X \sim N(\mu, \sigma^2)$. The value of variance of an Infinity light bulb specified by the norm is 40^2 hours.

Verify at the level of significance α = 0,05, if the standard deviation of the life length satisfies the norm.

Problem 1B

Determine the random sample size n for the hypothesis test in problem 1A, which is necessary to detect whether the true value of the standard deviation of the life length is 50 hours with a test strength of 0.8.

Problem 2A

Suppose that the life length of Infinity Light Bulbs is a random variable (X; unit: hour) which follows the normal distribution $X \sim N(\mu, 1600)$. The production guarantees 765 hours to be the mean of the life length of Infinity Light Bulbs.

Examine at the level of significance α = 0,01 if it is possible to claim with respect to results of hypothesis test that the mean of the life length exceeds the value garanteed by the production.

Problem 2B

Determine the random sample size n required for test hypothesis in the problem 2A to detect the difference between the sample mean 785 hours and the hypothetical life length mean 765 hours with power of the test 0,9 (Use the formula or the OC–curve.)

Task C

Analyse the results obtained from a statistical software:

Hypothesis Tests
Sample mean = 775,0
Sample standard deviation = 39,855
Sample size = 25

95,0% confidence interval for mean: 775,0 +/- 16,4514 [758,549;791,451]

Null Hypothesis: mean = 758,0
Alternative: not equal
Computed t statistic = 2,13273
P-Value = 0,043377

3.3 Questions 2

A company tested the life length of an Infinity Light Bulb (X_1; unit: hour) and the life length of a Forever Light Bulb (X_2; unit: hour). The life lengths were tested on two random samples: The sample of randomly selected n_1 = 30 pieces of Infinity Light Bulbs and n_2 = 25 pieces of Forever bulbs. The life lengths of both samples are seen in the following table.

Table: Two samples of Life length: life length of an Infinity Light Bulb (X_1; unit: hour) and the life length of a Forever Light Bulb (X_2; unit: hour)

i	Life lengths		*i*	Life lengths	
	X_1	X_2		X_1	X_2
1	727	789	16	812	756
2	755	835	17	804	787
3	714	765	18	754	788
4	840	796	19	715	794
5	772	797	20	845	822
6	750	776	21	725	837
7	814	769	22	735	798
8	820	836	23	770	837
9	753	847	24	792	841
10	796	769	25	765	766
11	831	755	26	749	

12	742	813	27	829	
13	784	828	28	821	
14	807	771	29	816	
15	820	829	30	743	

Suppose that X_1 and X_2 are normally distributed. Test at the level of significance $\alpha = 0{,}05$ in terms of precision and accuracy. Show the results using all indicators (statistic, a confidence interval, a *P–value,* the graph of a probability density function (mark the critical value, α, and *P–value*).) We would like to know

a) whether Infinity and Forever light bulbs differ in their life lengths,

b) the necessary sample sizes to detect the difference 20 hours between life length means, and

c) the necessary sample sizes to detect the true ratio of brand life lengths variances $1{,}5^2$ hours.

d) In case the Infinity variance is 30^2, Forever variance is 40^2, and the level of significance $\alpha = 0{,}01$, is the mean of Infinity bulbs less than the mean of Forever bulbs lifetime? How many bulbs do we need to detect the true difference 30 hours between means of bulb life lengths with power of test 0,9?

4 SOLUTION

4.1 MULTIPLE-CHOICE QUESTIONS

1. Suppose that $\left[23{,}7 - 1{,}96\frac{20}{\sqrt{50}}; 23{,}7 + 1{,}96\frac{20}{\sqrt{50}}\right]$is the 100 (1 – α) % confidence interval for the mean μ of a random sample from normal population $N(\mu, \sigma^2)$ with a variance σ^2known.

   a) the value 23,7 represents

      i) sample mean

      ii) population mean

      iii) sample variance

      iv) sample median

   Solution: The correct answer is i).

   b) the variance of a population is

      i) 20

      ii) 400

      iii) 50

      iv) $\sqrt{50}$

   Solution: The correct answer is ii).

   c) the significance level is

      i) 90%

      ii) 5%

      iii) 95%

      iv) 99%

   Solution: The correct answer is iii).

   d) What size of a sample n is necessary if the length of the confidence interval is 40?

      i) 3

      ii) 4

      iii) 5

      iv) 8

   Solution: The correct answer is ii).

$$\bar{x} - k_\alpha \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + k_\alpha \frac{\sigma}{\sqrt{n}} \quad \Leftrightarrow \quad 23{,}7 - 1{,}96\frac{20}{\sqrt{50}} \leq \mu \leq 23{,}7 + 1{,}96\frac{20}{\sqrt{50}}$$

parameters of a random sample: $\bar{x}$ = 23,7; n = 50

variance (known): $\sigma^2 = 20^2$

the significance level: 95% ($k_\alpha = 1{,}96 \Rightarrow \alpha = 0{,}05 \Rightarrow 1 - \alpha = 0{,}95 = 95\%$)

$$n = \left(\frac{k_\alpha \sigma}{E}\right)^2 = \left(\frac{1{,}96 \cdot 20}{20}\right)^2 = 1{,}96^2 = 3{,}84 \approx 4$$

LO: SP71, SP77, SP78

2. How is the length of confidence interval influenced by a level of test significance α?
   a) The smaller α is, the wider confidence interval is.
   a) The bigger α is, the wider confidence interval is.
   b) The bigger α is, the narrower confidence interval is.
   c) The smaller α is, the narrower confidence interval is.

   Solution: The correct answer is a).

   LO: SP71, SP74.
3. Which of the below relations defines the probability of type II error?
   a) $\alpha = P(\text{reject} H_0 | H_0 \text{is false})$
   b) $\alpha = P(\text{failtoreject} H_0 | H_0 \text{is true})$
   c) $\beta = P(\text{reject} H_0 | H_0 \text{is true})$
   d) $\beta = P(\text{failtoreject} H_0 | H_0 \text{is false})$

   Solution: The correct answer is d).

   LO: SP74, SP75.
4. What relation is between *P–value* and the level of significanceα, if hypothesis $H_0$is rejected at the level of significanceα?
   a) $P - value \geq \alpha$
   b) $P - value \geq 2\alpha$
   c) $P - value \geq \alpha/2$
   d) $P - value \leq \alpha$
   e) $P - value \leq 2\alpha$
   f) $P - value \leq \alpha/2$

   Solution: The correct answer is d).

   LO: SP73, SP74.
5. *z*–test is a test of hypothesis on
   a) mean, variance unknown
   b) variance, mean known
   c) variance, mean unknown
   d) mean, variance known

   Solution: The correct answer is d).

   LO: SP12, SP13.

6. χ^2– test is used for hypothesis tests on
   a) mean of single normal population
   b) the variance of two normal populations
   c) the variance of single normal population
   d) mean of two normal populations

   Solution: The correct answer is c).

   LO: SP7.

7. The figure presents the result of testing
   a) $H_0: \mu_1 = \mu_2 \rightarrow H_1: \mu_1 > \mu_2$
   b) $H_0: \mu_1 = \mu_2 \rightarrow H_1: \mu_1 < \mu_2$
   c) $H_0: \mu_1 < \mu_2 \rightarrow H_1: \mu_1 = \mu_2$
   d) $H_0: \sigma_1^2 = \sigma_2^2 \rightarrow H_1: \mu_1 = \mu_2$

   Solution: The correct answer is a).

   LO: SP79.

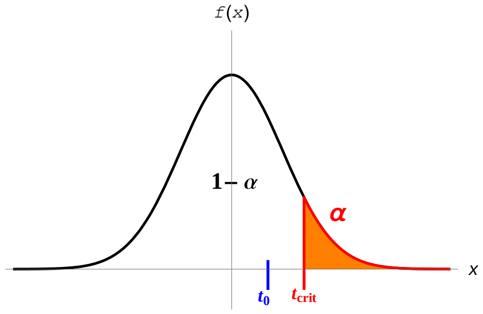

Figure 141

8. The figure presents the result
   a) $H_0: \mu_1 = \mu_2$ fail to reject
   b) $H_0: \mu_1 = \mu_2$ reject
   c) $H_0: \mu_1 < \mu_2$ reject
   d) $H_0: \sigma_1^2 = \sigma_2^2$ fail to reject

   Solution: The correct answer is a).

   LO: SP77, SP79.

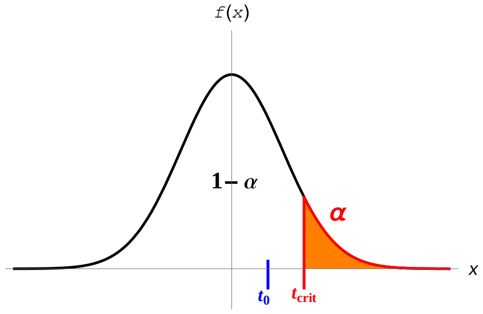

Figure 142

4.2 Questions 1

Task A

The joint seal thickness X (mm) of the gas pipe must satisfy the standard $\sigma_N^2 = 0{,}06$. The following table presents a random sample from a normal population $X \sim N(\mu, \sigma^2)$ where seven pipe connections were randomly checked.

i	1.	2.	3.	4.	5.	6.	7.
x_i	1,3	1,8	1,4	1,2	0,9	1,5	1,7

1.

a) Estimate parameters of joint seal thickness normal population

Solution:

i	x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
1.	1,3	– 0,1	0,01
2.	1,8	0,4	0,16
3.	1,4	0	0
4.	1,2	– 0,2	0,04
5.	0,9	– 0,5	0,25
6.	1,5	0,1	0,01
7.	1,7	0,3	0,09
$\sum_{i=1}^{7}$	**9,8**		**– 0,56**

$$\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{9{,}8}{7} = 1{,}4$$

$$\widehat{\sigma^2} = s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 =$$
$$= \frac{0{,}56}{6} = 0{,}09333 = 0{,}3055^2$$

LO: SP71, SP77, SP78.

a) Does the thickness in the inspected section satisfy the standard? Justify your claim via the appropriate confidence interval for α = 0,05.

Solution:

We construct the 95% two-sided confidence interval on σ^2

n = 7

$\bar{x} = 1{,}4$

s^2 = 0,09333

$1-\alpha = 0{,}95 \Rightarrow \alpha = 0{,}05$

$\nu = n-1 = 6$

$\chi^2(\nu, \alpha/2) = \chi^2(6; 0{,}025) = 14{,}4$

$\chi^2(\nu, 1-\alpha/2) = \chi^2(6; 0{,}975) = 1{,}24$

$$P\left(\frac{(n-1)s^2}{\chi^2\left(\nu, \frac{\alpha}{2}\right)} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2\left(\nu, 1\text{-}\frac{\alpha}{2}\right)}\right) = 1-\alpha$$

$$P\left(\frac{(n-1)s^2}{\chi^2\left(\nu, \frac{\alpha}{2}\right)} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2\left(\nu, 1\text{-}\frac{\alpha}{2}\right)}\right) = 0{,}95$$

$$\frac{6 \times 0{,}09333}{14{,}4} \le (\sigma^2 = \sigma_N^2 = 0{,}06) \le \frac{6 \times 0{,}09333}{1{,}24}$$

$$0{,}03888 \le (\sigma_N^2 = 0{,}06) \le 0{,}45197$$

Answer: The thickness of the joint seals in the inspected section satisfies the standard.

LO: SP71, SP77.

b) With a probability of 95%, estimate the endpoints of the interval that covers the unknown mean value of the pipe joint seal thickness.

Solution:

n = 7 $\quad 1-\alpha = 0{,}95 \Rightarrow \alpha = 0{,}05$

$\bar{x} = 1{,}4 \quad \nu = n-1$

$s = 0{,}3055 \quad t(\nu,\alpha) = \chi^2(6;\ 0{,}05) = 2{,}447$

$$P\left(\bar{X} - t(\nu,\alpha)\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t(\nu,\alpha)\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(\bar{X} - t(\nu,\alpha)\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t(\nu,\alpha)\frac{S}{\sqrt{n}}\right) = 0{,}95$$

$$1{,}4 - 2{,}447\frac{0{,}3055}{\sqrt{7}} \le \mu \le 1{,}4 + 2{,}447\frac{0{,}3055}{\sqrt{7}}$$

$$1{,}11744 \le \mu \le 1{,}68256$$

LO: SP71, SP77, SP78.

2. Suppose the seal thickness selection is a random sample from a normal distribution $X \sim N(\mu; 0{,}09)$

a) Estimate the upper bound of an interval which covers the unknown mean value of the pipe joint seal thickness with a probability of 99 %.

Solution:

We construct the 95% two-sided confidence interval on σ^2

n = 7 $\quad 1-\alpha = 0{,}99 \Rightarrow \alpha = 0{,}01$

$\bar{x} = 1{,}4 \quad k_{2\alpha} = k_{0{,}02} = 2{,}316$

$\sigma^2 = 0{,}3^2$

$$P\left(\mu \le \bar{X} + k_{2\alpha}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(\mu \le \bar{X} + k_{2\alpha}\frac{\sigma}{\sqrt{n}}\right) = 0{,}99$$

$$\mu \le 1{,}4 + 2{,}316\frac{03}{\sqrt{7}}$$

$$\mu \le 1{,}66374 \Rightarrow \mu \in (-\infty; 1{,}66374]$$

LO: SP71, SP77, SP78.

b) How many randomly selected seals must be checked to avoid the error greater than 0,02 mm when calculating the 95% two sided confidence interval for an unknown mean of the seal thickness?

Solution:

We construct the 95% two-sided confidence interval on σ^2

n = ? $1-\alpha = 0{,}95 \Rightarrow \alpha = 0{,}05$

$\bar{x} = 1{,}4$ $k_\alpha = k_{0{,}05} = 1{,}96$

$\sigma^2 = 0{,}3^2$ E = 0,02

$$P\left(\bar{X} - k_\alpha \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + k_\alpha \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$P\left(\bar{X} - k_\alpha \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + k_\alpha \frac{\sigma}{\sqrt{n}}\right) = 0{,}95$$

$$n = \left(\frac{k_\alpha \sigma}{E}\right)^2 = \left(\frac{1{,}96 \times 0{,}3}{0{,}02}\right)^2 = 864{,}36 \doteq 865$$

Answer: Calculating the 95% two sided confidence interval for an unknown mean of the seal thickness, 865 seals have to be checked to avoid the error greater than 0,02 mm.

LO: SP71, SP77, SP78.

c) Estimate the value of seal thickness for next observation with probability 97 %.

Solution:

We construct a two-sided prediction interval for the next observation of the seal thickness X_8, σ^2 known:

n = 7 $1-\alpha = 0{,}97 \Rightarrow \alpha = 0{,}03$

$\bar{x} = 1{,}4$ $k_\alpha = k_{0{,}03} = 2{,}17$

$\sigma^2 = 0{,}3^2$

$$P\left(\bar{X} - k_\alpha \sigma \sqrt{1+\frac{1}{n}} \le X_{n+1} \le \bar{X} + k_\alpha \sigma \sqrt{1+\frac{1}{n}}\right) = 1-\alpha$$

$$P\left(\bar{X} - k_\alpha \sigma \sqrt{1+\frac{1}{n}} \le X_8 \le \bar{X} + k_\alpha \sigma \sqrt{1+\frac{1}{n}}\right) = 0{,}97$$

$$1{,}4 - 2{,}17 \times 0{,}3 \sqrt{1+\frac{1}{7}} \le X_8 \le 1{,}4 + 2{,}17 \times 0{,}3 \sqrt{1+\frac{1}{7}}$$

$$0{,}704052 \le X_8 \le 2{,}09595$$

LO: SP71, SP77, SP78.

d) Calculate two sided interval that covers at least 95% seal thickness values from the population with probability 90 %.

Solution:

We construct a two-sided prediction interval for the next observation of the seal thickness X_8, σ^2 known:

n = 7 $p = 95\% = 0{,}95$

$\bar{x} = 1{,}4$ tolerance factor:

$\sigma^2 = 0,3^2$ $k(n, m, p, 1-\alpha) = k(7; 1; 0{,}95; 0{,}90) = 3{,}45$

$1 - \alpha = 0{,}90 \Rightarrow \alpha = 0{,}01$

$$P(P(\bar{x} - k\sigma < X < \bar{x} + k\sigma) \geq p) = 1 - \alpha$$

$$P(P(\bar{x} - k\sigma < X < \bar{x} + k\sigma) \geq 0{,}95) = 0{,}90$$

$$1{,}4 - 3{,}45 \times 0{,}3 < X < 1{,}4 - 3{,}45 \times 0{,}3$$

$$0{,}365 < X < 2{,}435$$

LO: SP71, SP77, SP78.

e) Product inspection of joint seal thicknesses found out that 24 from 300 randomly inspected pieces were defective. Find the 95% confidence interval with the lower limit for the proportion of defective pipe joint seal thicknesses.

Solution:

We construct a one–sided 95% confidence interval with the lower limit for the proportion of defective pipe joint seal thicknesses p.

n = 300 $1 - \alpha = 0{,}95 \Rightarrow \alpha = 0{,}05$

$x = 24$ $k_{2\alpha} = k_{0,1} = 1{,}645$

estimator $\hat{P}$ of the parameter p: $\hat{P} = \frac{x}{n} = \frac{24}{300} = 0{,}08$

Confirm: $n \times \hat{p} \times (1 - \hat{p}) > 9 \Rightarrow (300 \times 0{,}08 \times 0{,}92 = 22{,}08) > 9$..true

$$P\left(\hat{P} - k_{2\alpha}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} \leq p\right) = 1 - \alpha$$

$$P\left(\hat{P} - k_{2\alpha}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} \leq p\right) = 0.95$$

$$0{,}08 - 1{,}645\sqrt{\frac{0{,}08(1-0{,}08)}{300}} \leq p$$

$$0{,}05425 \leq p$$

$$p \geq 5{,}425\% \Rightarrow p \in [0{,}05425; \infty)$$

LO: SP71, SP77, SP78.

f) How many randomly selected pipe joints must be checked to ensure that an error greater than 0,02 mm does not occur when calculating the 95% confidence interval for the proportion of defective pipe joint seal thicknesses?

i) Solve the task for an unknown proportion of defective pipe joint seal thicknesses.

Solution:

$$1-\alpha = 0{,}95 \Rightarrow \alpha = 0{,}05 \qquad E = 0{,}02$$

$$k_\alpha = k_{0{,}05} = 1{,}96$$

$$n = 0{,}25\left(\frac{k_\alpha}{E}\right)^2 = 0{,}25\left(\frac{1{,}96}{0{,}02}\right)^2 = 2401$$

Answer: It is necessary to check 2401 pipe joints to satisfy the given task.

ii) Solve the task for given proportion of defective pipe joint seal thicknesses $p = 5\%$.

Solution:

$$1-\alpha = 0{,}95 \Rightarrow \alpha = 0{,}05 \qquad E = 0{,}02$$

$$k_\alpha = k_{0{,}05} = 1{,}96$$

$$n = 0{,}25\left(\frac{k_\alpha}{E}\right)^2 = 0{,}25\left(\frac{1{,}96}{0{,}02}\right)^2 = 2401$$

$$n = \left(\frac{k_\alpha}{E}\right)^2 p(1-p) = \left(\frac{1{,}96}{0{,}02}\right)^2 0{,}05(1-0{,}05) = 456{,}19 \doteq 457$$

Answer: It is necessary to check 457 pipe joints to satisfy the given task.

LO: SP71, SP77, SP78.

Task B

Instructions

Express hypotheses in symbolic and also in verbal notation.

Find necessary sample data via a statistical software.

Test hypotheses using

a) a statistic

b) a confidence interval

c) a *P–value*

d) the graph of a probability density function (mark the critical value, α, and *P–value*).

A company tested the life length of Infinity Light Bulbs (*X*; unit: hour). The following table presents a random sample of 30 bulb life lengths x_i.

i	x_i	***i***	x_i	***i***	x_i
1	727	11	831	21	725
2	755	12	742	22	735

3	714	13	784	23	770
4	840	14	807	24	792
5	772	15	820	25	765
6	750	16	812	26	749
7	814	17	804	27	829
8	820	18	754	28	821
9	753	19	715	29	816
10	796	20	845	30	743

Using statistical software we receive following data

- sample size : $n = 30$
- sample mean : $\bar{x} = 780$
- sample variance : $s^2 = 40{,}0164^2$

Problem 1A

Suppose that the life length of an Infinity light bulb (X; unit: hour) follows the normal distribution with X- $N(\mu, \sigma^2)$. The value of variance of an Infinity light bulb specified by the norm is 40^2 hours.

Verify at the level of significance α = 0,05, if the standard deviation of the life length satisfies the norm.

Solution:

Hypothesis test on σ^2; two–sided test at the level of significance $\alpha = 0{,}05$

Null hypothesis		Alternative hypothesis
$H_0: \sigma^2 = 40^2$ $(H_0: \sigma^2 = \sigma_0^2)$ *The standard deviation is equal to the norm.*	→	$H_1: \sigma^2 \neq 40^2)$ $(H_1: \sigma^2 \neq \sigma_0^2)$ *The standard deviation differs from the norm.*

a) Testing hypothesis using the statistic

The test statistic: $\chi_0^2 = \frac{(n-1)\cdot s^2}{\sigma_0^2} = \frac{(30-1)\cdot 40{,}0164^2}{40^2} = 29{,}0238$

The critical values:

$$\chi^2(n-1; \alpha/2) = \chi^2(29; 0{,}025) = 45{,}7$$
$$\chi^2(n-1; 1-\alpha/2) = \chi^2(29; 0{,}975) = 16{,}0$$

Reject H_0 at $\alpha = 0{,}05$, when

$$\chi_0^2 > \chi^2(n-1;\alpha/2) \vee \chi_0^2 < \chi^2(n-1;1-\alpha/2)$$

29,0238 > 45,7 or 29,0238 < 16,0 is false $\Rightarrow$ fail to reject H_0

<u>Conclusion:</u> **Fail to reject** $H_0: \sigma^2 = 40^2$ at the level of significance α = 0,05:

We conclude that at the level of significance α = 0,05 (95% confidence) the standard deviation of the life length of an Infinity Light Bulb satisfies the norm. The test did not provide the sufficient evidence that the standard deviation differs from the norm.

b) Testing hypothesis using the confidence interval

Reject H_0 at α = 0,05, when $(\sigma_0^2 = 40^2) \notin \left[\frac{(n-1)\cdot s^2}{\chi^2(n-1;\alpha/2)}; \frac{(n-1)\cdot s^2}{\chi^2(n-1;1-\alpha/2)}\right]$

$(\sigma_0^2 = 40^2) \notin [31{,}8693^2; 53{,}7946^2]$ is false $\Rightarrow$ ***fail to reject*** H_0

c) Testing hypothesis using P-value

Using statistical software we recieve *P-value* = 0,0927656

Reject H_0 at α = 0,05, when ***P*-value** $\leq \alpha$

$0{,}927656 \leq 0{,}05$ is false $\Rightarrow$ ***fail to reject*** H_0

d) Testing hypothesis using graph

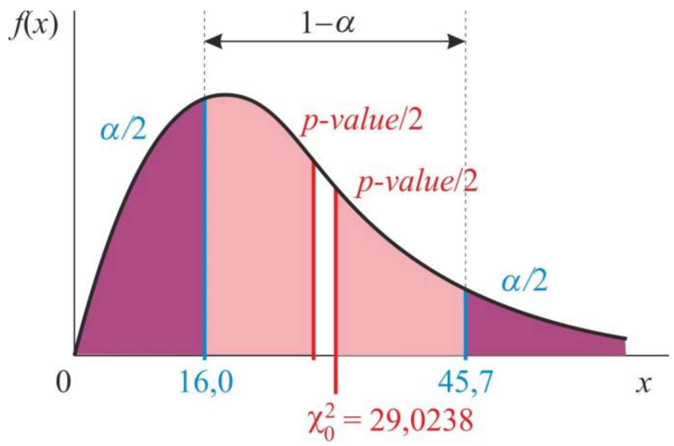

Figure 143

The statistic and *P-value* area are part of the acceptance region $\Rightarrow$ fail to reject H_0

LO: SP7.

Problem 1B

Determine the random sample size *n* for the hypothesis test in the problem 1A, which is necessary to detect whether the true value of the standard deviation of the life length is 50 hours with a test strength of 0.8.

Solution:

The two-sided χ^2–test at the level of significance $\alpha = 0{,}05$ was used in the problem 1C, we will use OC–*i* curve.

$$\sigma_0 = 40$$
$$\sigma = 50$$

parameter: $\lambda = \frac{\sigma}{\sigma_0} = \frac{50}{40} = 1{,}25$

Power of test: $1 - \beta = 0{,}8 \Rightarrow \beta = 0{,}2$

$$\left.\begin{array}{l} \lambda = \frac{\sigma}{\sigma_0} = \frac{50}{40} = 1{,}25 \\ 1 - \beta = 0{,}8 \Rightarrow \beta = 0{,}2 \end{array}\right\} \Rightarrow OC - i: n \approx 75(est.)$$

<u>Conclusion:</u> *To detect whether the true value of the standard deviation of the life length is 50 hours with the test strength 0.8 (e.i. there is 80% probability to detect), it is necessary to test a random sample with approximately 75 pieces of light bulbs.*

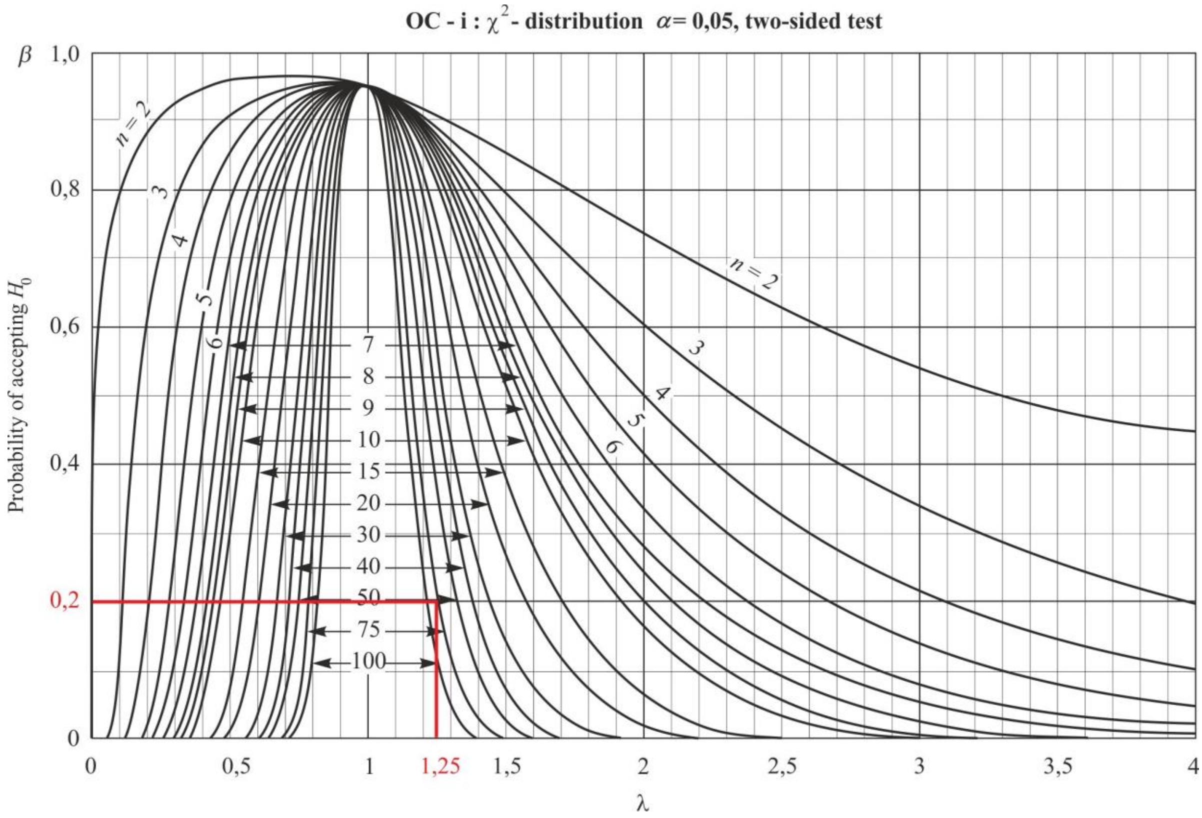

Figure 144. Graph of the operating characteristics OC-i curves

LO: SP7.

Problem 2A

Suppose that the life length of Infinity Light Bulbs is a random variable (X; unit: hour) which follows the normal distribution X~ $N(\mu, 1600)$. The production guarantees 765 hours to be the mean of the life length of Infinity Light Bulbs.

Examine at the level of significance α = 0,01 if it is possible to claim with respect to results of hypothesis test that the mean of the life length exceeds the value garanteed by the production.

Solution:

Variance of the normal population is known: $\sigma^2 = 40^2$

Hypothesis test on μ, σ^2 known; upper–sided test at the level of significance α = 0,01

Null hypothesis		Alternative hypothesis
$H_0: \mu = 765$ $(H_0: \mu = \mu_0)$ *The mean of the life length is equal to the mean garanteed by the production.*	→	$H_1: \mu > 765$ $(H_1: \mu > \mu_0)$ *The mean of the life length exceeds the mean garanteed by the production.*

a) Testing hypothesis using the statistic

The test statistic: $z_0 = \frac{\bar{x}-\mu_0}{\sigma}\sqrt{n} = \frac{780-765}{40}\sqrt{30} = 2{,}05396$

The critical value: $k_{2\alpha} = k_{2\cdot 0{,}01} = k_{0{,}02} = 2{,}326$

Reject H_0 at $\alpha = 0{,}01$, when

$$z_0 > k_{2\alpha}$$

$2{,}05396 > 2{,}326$ is false $\Rightarrow$ fail to rejectH_0

Conclusion: **Fail to reject** $H_0: \mu = 765$ at the level of significance α = 0,01:

We can claim that the mean of the life length of an Infinity light bulb exceeds the value garanteed by the production 765 hours with the probability of 99%.

b) Testing hypothesis using the confidence interval

Reject H_0 at α = 0,01, when$\bar{x} - k_{2\alpha} \cdot \frac{\sigma}{\sqrt{n}} > \mu_0$

$$780 - 16{,}9893 > \mu_0$$

$763{,}011 > (765 = \mu_0)$is false, ***fail to reject*** H_0

$$(765 = \mu_0) \in [763{,}011; \infty)$$

c) Testing hypothesis using P-value

Using statistical software we receive *P*-value = 0,01999

Reject H_0 at α = 0,01, when ***P*-value** $\leq \alpha$

$0{,}01999 \leq 0{,}01$ **–** is false $\Rightarrow$ ***fail to reject***H_0

d) Testing hypothesis using graph

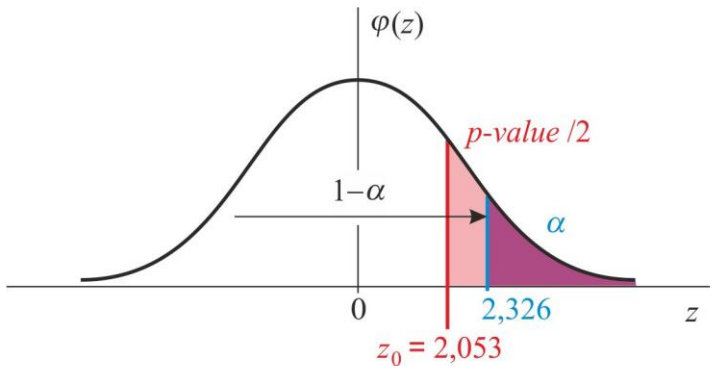

Figure 145

The statistic and *P-value* area are part of the acceptance region $\Rightarrow$ fail to reject H_0

LO: SP7.

Problem 2B

Determine the random sample size *n* required for test hypothesis in the problem 2A to detect the difference between the sample mean 785 hours and the hypothetical life length mean 765 hours with power of the test 0,9 (Use the formula or the OC–curve.)

Solution:

1. OC– curve

The upper–sided *z* –test at the level of significance α = 0,01 was used in the problem1G, so we will use OC–*d* curve.

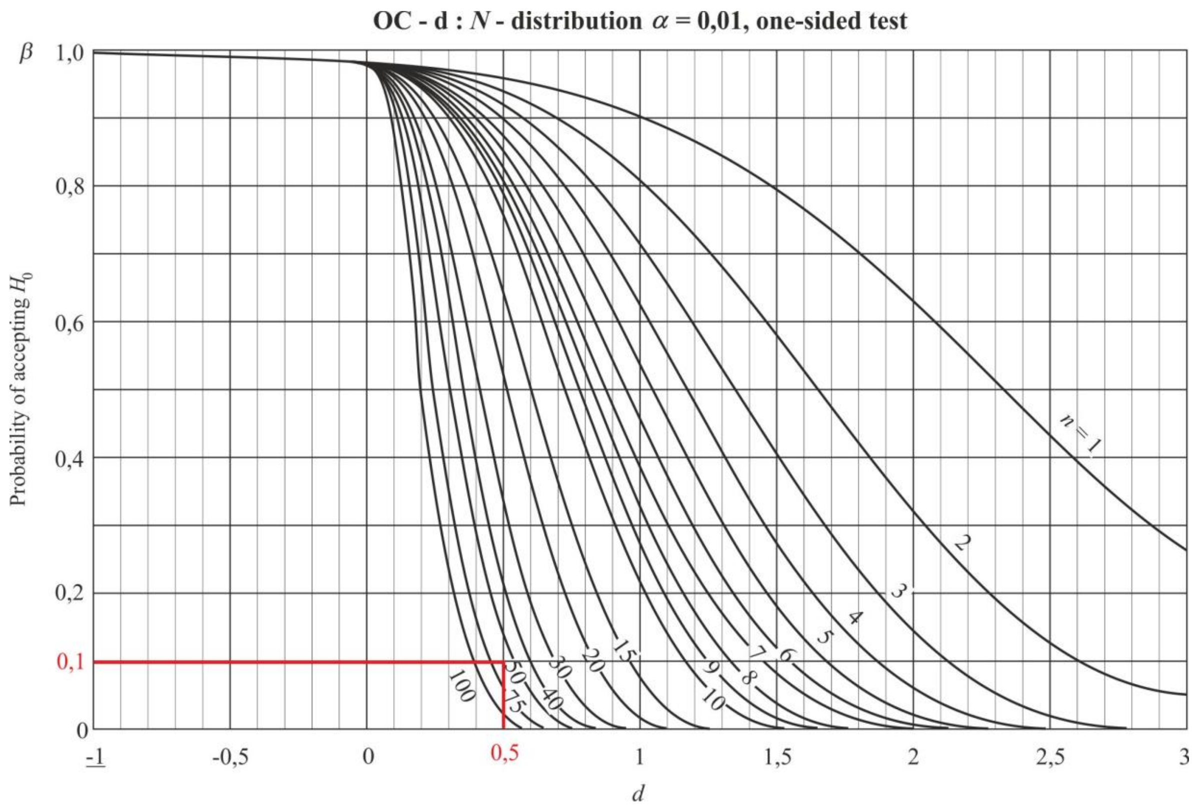

Figure 146. Graph of the operating characteristics OC–d curve

$$\left.\begin{aligned}\mu &= 785\\ \mu_0 &= 765\end{aligned}\right\} \Rightarrow \delta = \mu - \mu_0 = 785 - 765 = 20$$

Power of test:

2. Sample size formula

$$n = \frac{(k_{2\alpha} + k_{2\beta})^2 \cdot \sigma^2}{\delta^2} = \frac{(2{,}326 + 1{,}282)^2 \cdot 40^2}{20^2} = 52{,}0707 \approx 53$$

According to the OC–*d* curve we determine required sample size to be approximately *n* = 55 which is close to the value *n* = 53 calculated using the sample size formula.

Conclusion: *To be able to detect the difference between the sample mean 785 hours and the hypothetical life length mean 765 hours with 90% probability, it is necessary to test the sample with the size n = 53 bulbs.*

LO: SP7.

Task C

Analyse the results obtained from a statistical software:

Hypothesis Tests

Sample mean = 775,0
Sample standard deviation = 39,855
Sample size = 25

95,0% confidence interval for mean: 775,0 +/- 16,4514 [758,549;791,451]

Null Hypothesis: mean = 758,0
Alternative: not equal
Computed t statistic = 2,13273
P-Value = 0,043377

Solution:

Given one random sample: sample size : $n = 25$

sample mean : $\bar{x} = 775$

sample variance : $s^2 = 39{,}855^2$

- Hypothesis test on μ, σ^2 unknown; two–sided test at the level of significance $\alpha = 0{,}05$,

$$H_0: \mu = 758 \rightarrow H_1: \mu \neq 758$$

- The test statistic: $t_0 = 2{,}13273$
- *P – value* = 0,043377
- **Reject** H_0 at $\alpha = 0{,}05$ because: $P - value \leq \alpha$ and $(\mu_0 = 758) \notin [758{,}549; 791{,}451]$ (confidence interval)

LO: SP7.

4.3 QUESTIONS 2

A company tested the life length of an Infinity Light Bulb (X_1; unit: hour) and the life length of a Forever Light Bulb (X_2; unit: hour). The life lengths were tested on two random samples: The sample of randomly selected n_1 = 30 pieces of Infinity Light Bulbs and n_2 = 25 pieces of Forever bulbs. The life lengths of both samples are seen in the following table.

Table 5: Two samples of Life length: life length of an Infinity Light Bulb (X_1; unit: hour) and the life length of a Forever Light Bulb (X_2; unit: hour)

i	**Life lengths**		***i***	**Life lengths**	
	X_1	X_2		X_1	X_2
1	727	789	16	812	756
2	755	835	17	804	787
3	714	765	18	754	788
4	840	796	19	715	794
5	772	797	20	845	822
6	750	776	21	725	837
7	814	769	22	735	798
8	820	836	23	770	837
9	753	847	24	792	841
10	796	769	25	765	766
11	831	755	26	749	
12	742	813	27	829	
13	784	828	28	821	
14	807	771	29	816	
15	820	829	30	743	

Suppose that X_1 and X_2 are normally distributed. Test at the level of significance $\alpha = 0{,}05$ in terms of precision and accuracy. Show the results using all indicators (statistic, a confidence interval, a *P–value,* the graph of a probability density function (mark the critical value, α, and *P–value*).) We would like to know

a) whether Infinity and Forever light bulbs differ in their life lengths,

b) the least necessary sample sizes to detect the 20 hours between life length means (to have 80% chance), and

c) the least necessary sample sizes to detect the true ratio of brand life lengths variances $1,5^2$ hours.

d) In case the Infinity variance is 30^2, Forever variance is 40^2, and the level of significance $\alpha = 0,01$ is the mean of Infinity bulbs lifetime less than the mean of Forever bulbs? How many bulbs do we need to detect the true difference 30 hours between means of bulb life lengths with power of test 0,9?

Using statistical software we recieve following data:

light brand life time	**sample size**	**sample mean**	**sample variance**
Infinity (X_1)	$n_1 = 30$	$\bar{x}_1 = 780,00$	$s_1^2 = 40,0164^2$
Forever (X_2)	$n_2 = 25$	$\bar{x}_2 = 800,04$	$s_2^2 = 30,0048^2$

a) whether Infinity and Forever light bulbs differ in their life lengths

Solution:

$X_1 \sim N(\mu_1; \sigma_1^2); X_2 \sim N(\mu_2; \sigma_2^2)$, the level of significance $\alpha = 0,05$.

At first we will find out whether the unknown variations differ or not: $\sigma_1^2 = \sigma_2^2$ or $\sigma_1^2 \neq \sigma_2^2$.

1st STEP: We set and examine hypothesis test on σ_1^2/σ_2^2; two-sided test at the level of significance $\alpha = 0,05$

Null hypothesis		Alternative hypothesis
$H_0: \sigma_1^2 = \sigma_2^2$ $(H_0: \sigma_1^2/\sigma_2^2 = 1)$ *There is not a statistically significant difference between unknown variances of Infinity bulb life length and Forever bulb life length.*	→	$H_1: \sigma_1^2 \neq \sigma_2^2$ $(H_1: \sigma_1^2/\sigma_2^2 \neq 1)$ *There is a statistically significant difference between unknown variances of Infinity bulb life length and Forever bulb life length.*

a) Testing hypothesis using the statistic

The test statistic:

$$f_0 = \frac{s_1^2}{s_2^2} = \frac{40,0164^2}{30,0048^2} = 1,77867$$

Critical values:

$$f(n_1 - 1; n_2 - 1; \alpha/2) = f(29; 24; 0,025) = 2,21744$$
$$f(n_2 - 1; n_1 - 1; \alpha/2) = f(24; 29; 0,025) = 2,15401$$
$$f(n_1 - 1; n_2 - 1; 1 - \alpha/2) = f(29; 24; 0,975) = \frac{1}{f(24; 29; 0,025)} = \frac{1}{2,15401} = 0,464251$$

Reject H_0 at $\alpha = 0,05$, when

$$f_0 > f(n_1 - 1; n_2 - 1; \alpha/2) \vee f_0 < f(n_1 - 1; n_2 - 1; 1 - \alpha/2)$$

$1{,}77867 > 2{,}21744$ or $1{,}77867 < 0{,}464251$ both expressions are false $\Rightarrow$ fail to reject H_0

<u>Conclusion:</u> **Fail to reject** $H_0: \sigma_1^2 = \sigma_2^2$ at the level of significance $\alpha = 0{,}05$:

At the level of significance $\alpha = 0{,}05$ (95% confidence), we are not able to conclude that the Infinity light bulbs and Forever light bulbs differ in their variances of the life length, i. e. $\sigma_1^2 = \sigma_2^2$. The test executed on given samples does not provide sufficient statistical evidence that the brands of light bulbs differ in their variances.

b) Testing hypothesis using the confidence interval

Reject H_0 at $\alpha = 0{,}05$, when

$$\left(\frac{\sigma_1^2}{\sigma_2^2} = 1\right) \notin \left[\frac{s_1^2}{s_2^2} \cdot \frac{1}{f\left(n_1 - 1; n_2 - 1; \frac{\alpha}{2}\right)}; \frac{s_1^2}{s_2^2} \cdot \frac{1}{f\left(n_1 - 1; n_2 - 1; 1 - \frac{\alpha}{2}\right)}\right]$$

$\left(\frac{\sigma_1^2}{\sigma_2^2} = 1\right) \notin [0{,}802125; 3{,}83126]$ is false $\Rightarrow$ fail to reject H_0

c) Testing hypothesis using P-value

Using statistical software (or calculation formula) we receive *P-value* = 0,15361

Reject H_0 at $\alpha = 0{,}05$, when *P*-value $\leq \alpha$

$0{,}15361 \leq 0{,}05$ is false $\Rightarrow$ fail to reject H_0

d) Testing hypothesis using the graph

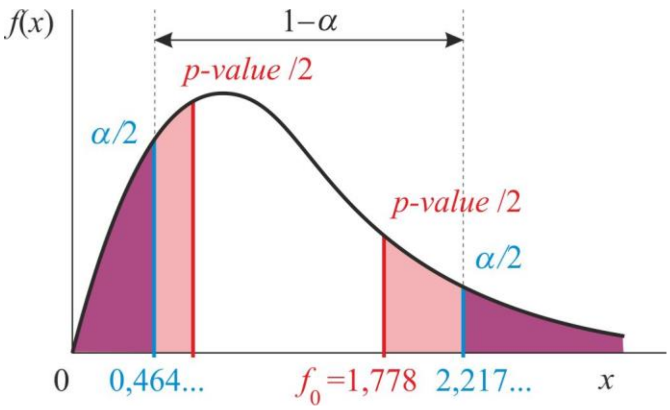

Figure 147

The statistic and *P-value* area are part of the acceptance region $\Rightarrow$ fail to reject H_0.

2nd STEP: Set and examine hypothesis test on $\mu_1 - \mu_2$, σ_1^2, σ_2^2 are unknown, $\sigma_1^2 = \sigma_2^2$; two-sided test at the level of significance $\alpha = 0{,}05$

Null hypothesis		Alternative hypothesis
$H_0: \mu_1 = \mu_2$ $(H_0: \mu_1 - \mu_0 = 0)$ *There is not a statistically significant difference between means of Infinity bulb life length and Forever bulb life length; e.i.* The types of light bulbs do not differ in their life length means.	→	$H_1: \mu_1 \neq \mu_2$ $(H_1: \mu_1 - \mu_0 \neq 0)$ *There is a statistically significant difference between means of Infinity bulb life length and Forever bulb life length; e.i.* The types of light bulbs do differ in their life length means.

a) Testing hypothesis using the statistic

$$s_p^2 = \frac{(n_1-1)\cdot s_1^2 + (n_2-1)\cdot s_2^2}{n_1+n_2-2} = \frac{(30-1)\cdot 40{,}0164^2 + (25-1)\cdot 30{,}0048^2}{30+25-2} = 1283{,}87$$
$$= 35{,}8311^2$$

The test statistic: $t_0 = \frac{(\bar{x}_1-\bar{x}_2)-\delta_0}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{(780-800{,}04)-0}{35{,}8311\sqrt{\frac{1}{30}+\frac{1}{25}}} = -2{,}06532.$

The critical value: $t(\nu;\alpha) = t(30+25-2;0{,}05) = t(53;0{,}05) = 2{,}00575.$

Reject H_0 at $\alpha = 0{,}05$, when

$$|t_0| > t(\nu;\alpha)$$
$$|-2{,}06532| > 2{,}00575$$

$2{,}06532 > 2{,}00575$ is true $\Rightarrow$ reject $H_0: \mu_1 = \mu_2$

Conclusion: **Reject** $H_0: \mu_1 = \mu_2$ at the level of significance $\alpha = 0{,}05$:

At the level of significance $\alpha = 0{,}05$ *(confidence 95 %) it can be said that Infinity light bulbs and Forever light bulbs differ in their life length means. The kind of light bulb has an effect on the life length. (The test executed on given samples provides the strong evidence)*

b) Testing hypothesis using the confidence interval

Reject H_0 at $\alpha = 0{,}05$, when

$$(\mu_1-\mu_2=0) \notin \left[(\bar{x}_1-\bar{x}_2) - t(\nu,\alpha)s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}};(\bar{x}_1-\bar{x}_2) + t(\nu,\alpha)s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right]$$

$$(\mu_1-\mu_2=0) \notin [20{,}04-19{,}462; 20{,}04+19{,}462]$$

$(\mu_1-\mu_2=0) \notin [-39{,}502; -0{,}577995]$ is true $\Rightarrow$ reject H_0

c) Testing hypothesis using the P-value

Using statistical software we recieve *P-value* = 0,0438

Reject H_0 at $\alpha = 0{,}05$, when *P*-value $\leq \alpha$

$0{,}0438 \leq 0{,}05$ is true $\Rightarrow$ reject H_0

d) Testing hypothesis using the graph

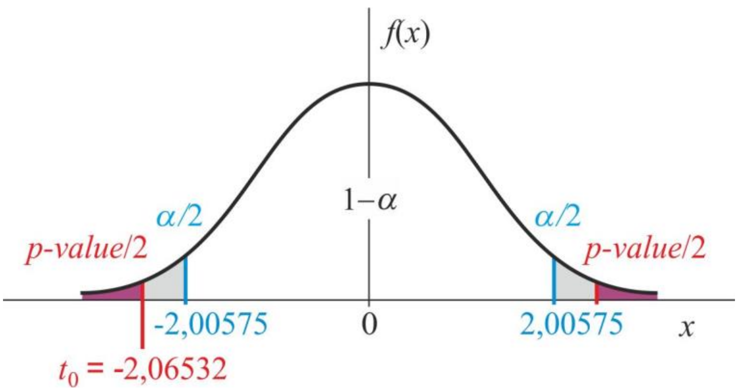

Figure 148

The statistic and *P-value* area lie in the rejection region $\Rightarrow$ reject H_0.

b) the least necessary sample sizes to detect the 20 hours between life length means

Solution:

The power of test 0,8 is considered to determine the least sizes of samples

true difference : $\delta = \mu_1 - \mu_2 = 20$

hypothetical difference : $\delta_0 = \mu_1 - \mu_2 = 0$

parameter: $$d = \frac{|\delta - \delta_0|}{2\hat{\sigma}} = \frac{|\delta - \delta_0|}{2s_P} = \frac{20}{2 \times 35{,}8311} = 0{,}279 \doteq 0{,}28$$

Power of test: $$1 - \beta = 0{,}8 \Rightarrow \beta = 0{,}2$$

$$\Rightarrow OC - e: (n = n_1 = n_2) \approx 100$$

The value of the sample size is estimated from OC – *e* curve to be $n \approx 100$

Conclusion: *To detect the statistically significant difference between life length means of value 20 hours with 0,8 power of test, it is necessary to examine approximately the random samples of 100 pieces of Infinity bulbs and 100 pieces of Forever bulbs.* These sample sizes give 80% chance of detection the difference by the test, and they can be considered to be the smallest sizes to show the difference.

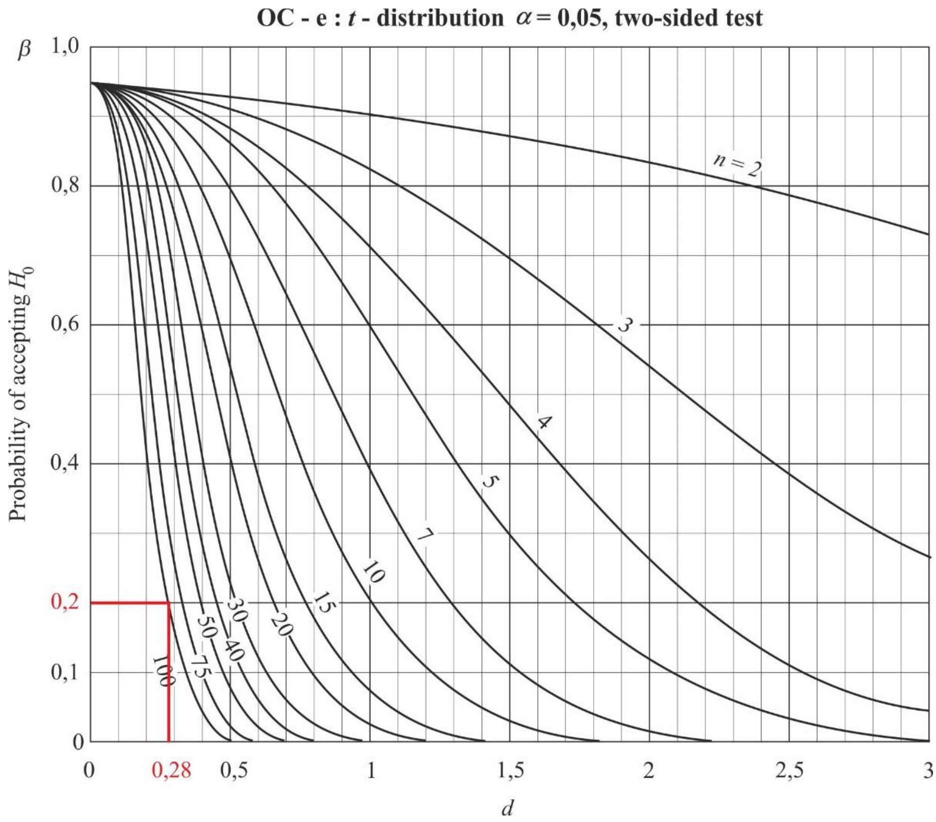

Figure 149

LO: SP7.

c) the least necessary sample sizes to detect the true ratio of brand life lengths variances $1,5^2$ hours.

Solution:

The two-sided *F* – test at the level of significance $\alpha = 0{,}05$ implies to use the OC – *o* curve.

$$\frac{\sigma_1^2}{\sigma_2^2} = 1,5^2$$

parameter: $\lambda = \frac{\sigma_1}{\sigma_2} = 1{,}5$

Power of test $1 - \beta = 0{,}8 \Rightarrow \beta = 0{,}2$

$$\Rightarrow OC - o: (n = n_1 = n_2) \approx 50(est.)$$

The value of the sample size is estimated from OC – o curve to be $n \approx 50$

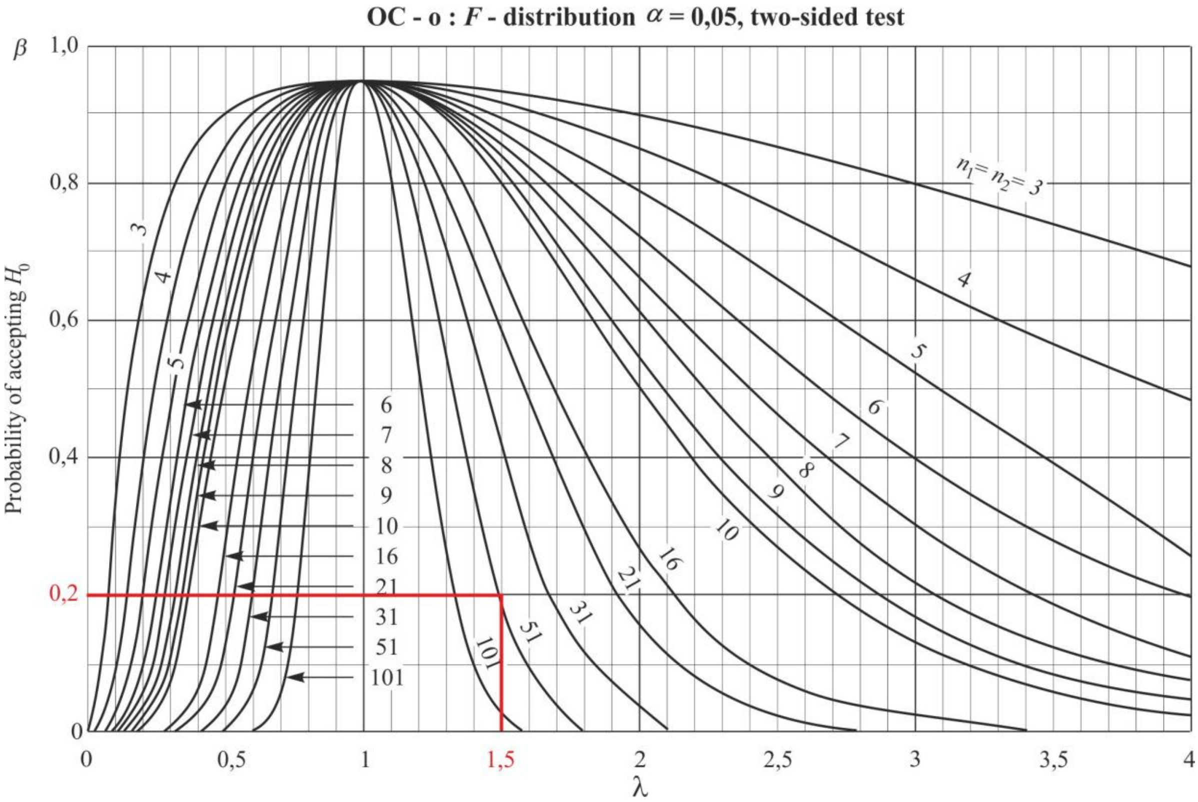

Figure 150

Conclusion: *To detect the true ratio between life length variances of Infinity and Forever light bulbs to be $1,5^2$ hours with 80% probability, it is necessary to test random samples of $n_1 = n_2 \approx 50$ bulbs. These sample sizes will be sufficient to show that the Forever bulbs are more stable and so more reliable in their lifetime than the Infinity bulbs.*

d) In case the Infinity variance is 30^2, Forever variance is 40^2, and the level of significance $\alpha = 0{,}01$ is the mean of Infinity bulbs lifetime less than the mean of Forever bulbs?

Solution:

X_1 and X_2 are normally distributed:$X_1 \sim N(\mu_1, 40^2), X_2 \sim N(\mu_2, 30^2)$.

We test the hypothesis at the level of significance $\alpha = 0{,}01$ that the mean of the life lengths of an Infinity bulb is less than the mean of the life lengths of an Forever bulb.

The hypothesis lower sided test will be used for μ_1, μ_2; variances σ_1^2 and σ_2^2 are known.

Null hypothesis		Alternative hypothesis
$H_0: \mu_1 = \mu_2$ $(H_0: \mu_1 - \mu_0 = 0)$ *There is not a statistically significant difference between the means of Infinity and Forever bulbs.*	→	$H_1: \mu_1 < \mu_2$ $(H_1: \mu_1 - \mu_0 < 0)$ *The mean of an Infinity bulb life length is less than the mean of a Forever bulb life length.*

a) Testing hypothesis using the statistic

$$\delta_0 = \mu_1 - \mu_2 = 0$$

The test statistic: $z_0 = \frac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{(780 - 800{,}04) - 0}{\sqrt{\frac{40^2}{30} + \frac{30^2}{25}}} = -2{,}12027$

The critical value: $k_{2\alpha} = k_{0{,}02} = 2{,}32635$

Reject H_0 at $\alpha = 0{,}01$, when $z_0 < -k_{2\alpha}$

$-2{,}12027 < -2{,}32635$ is false $\Rightarrow$ fail to reject

Conclusion: **Fail to reject** $H_0: \mu_1 = \mu_2$ at the level of significance$\alpha = 0{,}01$:

At the level of significance$\alpha = 0{,}01$ (confidence 99%) and with given sample sizes, we are not able to claim that the mean of Infinity bulb life length is less than the mean of Forever bulb life length and so there is not statistically signifficant difference.

b) Testing hypothesis using the confidence interval

Reject H_0 at $\alpha = 0{,}01$, when

$$(\bar{x}_1 - \bar{x}_2) + k_{2\alpha}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < (\mu_1 - \mu_2 = 0)$$

$$-20{,}04 + 22{,}6796 < (\mu_1 - \mu_2 = 0)$$

$2{,}6396 < (\mu_1 - \mu_2 = 0)$ is false $\Rightarrow$ fail to reject H_0

$(\mu_1 - \mu_0 = 0) \in (-\infty; 2{,}6396]$

c) Testing hypothesis using P-value

Reject H_0 at $\alpha = 0{,}01$, when *P*-value $\leq \alpha$

$0{,}01936 \leq 0{,}01$ is false, fail to reject H_0

d) Testing hypothesis using the graph

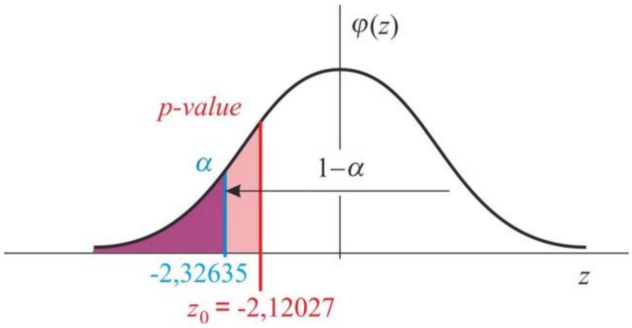

Figure 151

The statistic and *P-value* area are part of the acceptance area $\Rightarrow$ fail to reject H_0.

e) How many bulbs do we need to detect the true difference 30 hours between means of bulb life lengths with power of test 0,9?

Solution:

i) OC– curve

The lower–sided *z* –test at the level of significance $\alpha = 0{,}01$implies the OC–*d* curve.

true difference : $\delta = \mu_1 - \mu_2 = 30$

hypothetical difference: $\delta_0 = \mu_1 - \mu_2 = 0$

parameter: $d = \dfrac{|\delta - \delta_0|}{\sqrt{\sigma_1^2 + \sigma_2^2}} = \dfrac{30}{\sqrt{40^2 + 30^2}} = 0{,}6$

Power of test: $1 - \beta = 0{,}9 \Rightarrow \beta = 0{,}1$

$$\Rightarrow OC - d \Rightarrow (n = n_1 = n_2) \approx 40(est.)$$

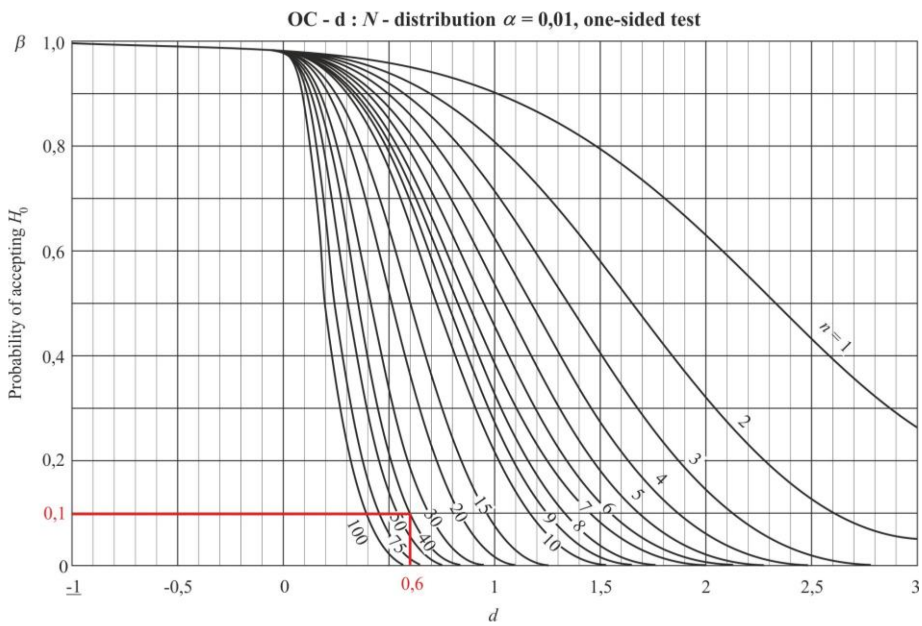

Figure 152

ii) *Sample* size formula

$$(n = n_1 = n_2) = \frac{(k_{2\alpha} + k_{2\beta})^2(\sigma_1^2 + \sigma_2^2)}{{\Delta_0}^2} = \frac{(k_{0,02} + k_{0,2})^2(40^2 + 30^2)}{30^2} =$$

$$= \frac{(2{,}326 + 1{,}282)^2 \times (40^2 + 30^2)}{30^2} = 36{,}1602 \approx 37$$

The value of the sample size can be estimated from OC – *d* curve $n \approx 40$, which coincides with $n = 37$ calculated from the sample size formula.

Conclusion: *We conclude that the sample sizes n = n_1= n_2 = 37 are adequate to detect the true difference 30 hours between means of the bulb life lengths with power of the test 0,9 (the chance is 90%).*

LO: SP7.

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