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ON THE L^p-STABILITY AND THE L^p-INTEGRAL STABILITY OF NON-LINEAR DIFFERENTIAL EQUATIONS IN TERMS OF TWO MEASURES

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Abstract. New sufficient conditions of the L^p-stability and the L^p-integral stability in terms of two measures of a nonlinear system of differential equations based on the use of Lyapunov's second method and the comparison principle are established.

Key words: $L^p-(h_0,h)$-stability, $L^p-(h_0,h)$-integral stability, Lyapunov function, h-positive definite, h_0-decrescent, weakly h_0-decrescent
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1. Introduction

Many authors have discussed the integral stability [5]–[10] and the L^p-stability [11]–[14].

In paper [15] the author introduces a new stability notion called L^p-integral stability.

Lyapunov's second method is a very useful and powerful instrument in discussing the stability of the solutions of differential equations. Its power and usefulness lie in the fact that a decision is made by investigating the differential equations themselves and not by finding solutions of the differential equations. However, it is very difficult to find Lyapunov function satisfying certain conditions. Therefore, it is important to obtain a weak sufficient condition for a stability theorem.

In this paper, by using Lyapunov's second method and the comparison principle, we will state some sufficient conditions of the L^p-stability and the

L^p-integral stability of solutions of the ordinary differential equations in terms of two measures.

2. Notations and definitions

Let $R_+ = [0, \infty), R^n$ denote Euclidean n-space. For $x \in R^n$, let $\|x\|$ be any norm of x and $S_H = \{x : \|x\| < H\},\ H = \text{const} > 0$.

We shall denote by $C[R_+ \times R^n, R^n]$ the set of all continuous functions defined on $R_+ \times R^n$ valued in R^n.

We consider the system of differential equations

$$\dot{x} = f(t, x), \quad f(t, 0), \equiv 0 \tag{2.1}$$

where $f(t, x) \in C[R_+ \times R^n, R^n]$, and its perturbed system

$$\dot{x} = f(t, x) + F(t, x), \tag{2.2}$$

where $F(t, x) \in C[R_+ \times R^n, R^n]$.

Suppose that $f(t, x)$ and $F(t, x)$ are smooth enough to ensure existence, uniqueness and continuous dependence of the solutions of the initial valued problem associated with the systems (2.1), (2.2).

Furthermore, consider a scalar differential equation

$$\dot{u} = g(t, u), \quad g(t, 0) \equiv 0, \tag{2.3}$$

where $g(t, u) \in C[R_+^2, R]$, and its perturbed equation

$$\dot{u} = g(t, u) + G(t, u), \tag{2.4}$$

where $G(t, u) \in C[R_+^2, R]$.

Suppose that $g(t, u)$ and $G(t, u)$ are smooth enough to ensure existence, uniqueness and continuous dependence of the solutions of the initial value problem associated with equations (2.3), (2.4).

Throughout this paper, a solution of system (2.1) through a point $(t_0, x_0) \in R_+ \times R^n$ will be denoted by such a form as $x(t) = x(t; t_0, x_0)$, where $x(t_0) = x(t_0; t_0, x_0) = x_0$.

Let the function $V \in C[R_+ \times R^n, R_+]$ and we define the function

$$D^+ V_{(2.1)}(t, x) = \lim_{\tau \to 0^+} \sup \frac{1}{\tau} \{V(t + \tau, x + \tau f(t, x)) - V(t, x)\}$$

for each $(t, x) \in R_+ \times R^n$.

Let us list the following classes of functions and definitions for convenience.

$$\begin{aligned} K &= \{\sigma \in C[R_+, R_+] : \sigma(\nu) \text{ is strictly increasing and } \sigma(0) = 0\}, \\ CK &= \{\sigma \in C[R_+^2, R_+] : \sigma(t, \nu) \in K \text{ for each } t \in R_+\}, \\ \Gamma &= \{h \in C[R_+ \times R^n, R_+] : \inf h(t, x) = 0, (t, x) \in R_+ \times R^n\}, \\ \Gamma_0 &= \{h \in \Gamma : \inf_{x \in R^n} h(t, x) = 0 \text{ for each } t \in R_+\}. \end{aligned}$$

Definition 2.1. [1]. Let $h_0, h \in \Gamma$. Then we say that:

(i) h_0 is finer than h if there exist a $\rho > 0$ and a function $\varphi \in CK$ such that $h_0(t, x) < \rho$ implies $h(t, x) \leq \varphi(t, h_0(t, x))$;

(ii) h_0 is uniformly finer than h if in (i) φ is independent of t, that is, $\varphi \in K$.

Definition 2.2. [2]. Let $V \in C[R_+ \times R^n, R_+]$ and $h_0, h \in \Gamma$. Then V is said to be:

(i) h-positive definite if there exist a $\rho > 0$ and a function $b \in K$ such that $h(t, x) < \rho$ implies $b(h(t, x)) \leq V(t, x)$;

(ii) h_0-decrescent if there exist a $\rho_0 > 0$ and a function $a \in K$ such that $h_0(t, x) < \rho_0$ implies $V(t, x) \leq a(h_0(t, x))$;

(iii) weakly h_0-decrescent if there exist a $\rho_0 > 0$ and a function $a \in CK$ such that $h_0(t, x) < \rho_0$ implies $V(t, x) \leq a(t, h_0(t, x))$.

Definition 2.3. [3]. System (2.1) is said to be:

(i) (h_0, h)-equistable, if for each $\varepsilon > 0$ and $t_0 \in R_+$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$, that is continuous in t_0 for each ε such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$ for all $t \geq t_0$;

(ii) uniformly (h_0, h)-stable, if (i) holds with δ being independent of t_0.

Definition 2.4. System (2.1) is said to be:

(i) $L^p-(h_0, h)$-equistable, where p is a positive integer, if it is (h_0, h)-equistable and there exists a $\delta_0 = \delta_0(t_0) > 0$ such that $h_0(t_0, x_0) \leq \delta_0$ implies

$$\int_{t_0}^{\infty} h^p(t, x(t))dt < \infty; \tag{2.5}$$

(ii) uniformly $L^p-(h_0, h)$-stable, where p is a positive integer, if it is uniformly (h_0, h)-stable, the δ_0 in (i) is independent of t_0 and the integral (2.5) converges uniformly in t_0;

(iii) $L^p-(h_0, h)$-integrally stable, where p is a positive integer, if it is (h_0, h)-equistable and for each $\varepsilon > 0$ and $t_0 \in R_+$ there exist a $\delta_1 = \delta_1(t_0, \varepsilon) > 0$

and a $\delta_2 = \delta_2(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta_1$ and $\int\limits_{t_0}^{\infty} \sup\limits_{h(t,x)\le\varepsilon} \|F(t,x)\| dt < \delta_2$ implies $\int\limits_{t_0}^{\infty} h^p(t, x^*(t))dt < \infty$, where $x^*(t) = x^*(t; t_0, x_0)$ denotes a solution of the perturbed system (2.2) satisfying an initial condition $x^*(t_0; t_0, x_0) = x_0$.

Definition 2.5. The trivial solution $u = 0$ of equation (2.3) is said to be:

(i) L^1-stable, if it is stable and there exists a $\delta_0^* = \delta_0^*(t_0) > 0$ such that $u_0 \le \delta_0^*$ implies

$$\int\limits_{t_0}^{\infty} u(t; t_0, u_0)dt < \infty, \tag{2.6}$$

where $u(t) = u(t; t_0, u_0)$ is a solution of equation (2.3), satisfying an initial condition $u(t_0; t_0, u_0) = u_0$;

(ii) uniformly L^1-stable, if it is uniformly stable, the δ_0^* in (i) is independent of t_0 and the integral (2.6) converges uniformly in t_0;

(iii) L^1-integrally stable, if it is stable and if for each $\varepsilon > 0$ and $t_0 \in R_+$ there exist a $\eta_1 = \eta_1(t_0, \varepsilon) > 0$ and a $\eta_2 = \eta_2(t_0, \varepsilon) > 0$ such that $u_0 < \eta_1$ and $\int\limits_{t_0}^{\infty} \sup\limits_{u\le\varepsilon} |G(t, u)| dt < \eta_2$ implies $\int\limits_{t_0}^{\infty} u^*(t; t_0, u_0)dt < \infty$, where $u^*(t) = u^*(t; t_0, u_0)$ is a solution of the perturbed equation (2.4) satisfying an initial condition $u^*(t_0; t_0, u_0) = u_0$.

3. Preliminary results

Theorem 3.1. *Let $V \in C[R_+ \times R^n, R_+]$ and $V(t,x)$ be locally Lipschitzian in x. Assume that function $D^+V(t,u)$ satisfies $D^+V_{(2.1)}(t,x) \le g(t, V(t,x))$, $(t,x) \in R_+ \times R^n$, where $g \in C[R_+^2, R]$. Let $r(t) = r(t; t_0, u_0)$ be the maximal solution of the equation (2.3).*

Then $V(t_0, x_0) \le u_0$ implies $V(t, x(t)) \le r(t)$ for each $t \ge t_0$, where $x(t) = x(t; t_0, x_0)$ is any solution of system (2.1).

For the proof, see [3].

Theorem 3.2. *Assume that:*

(i) $h_0, h \in \Gamma$ and h_0 is finer than h;

(ii) $V \in C[R_+ \times R^n, R_+]$, $V(t,x)$ is locally Lipschitzian in x, V is h-positive definite and weakly h_0-decrescent;

(iii) $g \in C[R_+^2, R]$ *and* $g(t,0) \equiv 0$*;*

(iv) $D^+V_{(2.1)}(t,x) \leq g(t, V(t,x))$, $(t,x) \in S(h,\rho)$ *for some* $\rho = \mathrm{const} > 0$, *where* $S(h,\rho) = \{(t,x) \in R_+ \times R^n : h(t,x) < \rho\}$.

Then, the stability of the trivial solution $u = 0$ *of equation (2.3) implies the* (h_0, h)*-equistability of system (2.1).*

For the proof, see [3].

Theorem 3.3. *Assume that:*

(i) $h_0, h \in \Gamma$ *and* h_0 *is uniformly finer than* h*;*

(ii) $V \in C[R_+ \times R^n, R_+]$, $V(t,x)$ *is locally Lipschitzian in* x, V *is* h*-positive definite and* h_0*-decrescent;*

(iii) $g \in C[R_+^2, R]$ *and* $g(t,0) \equiv 0$*;*

(iv) $D^+V_{(2.1)}(t,x) \leq g(t, V(t,x))$ *for* $(t,x) \in S(h,\rho)$.

Then, the uniform stability of the trivial solution $u = 0$ *of equation (2.3) implies the uniform* (h_0, h)*-stability of system (2.1).*

For the proof, see [3].

Theorem 3.4. *Assume that:*

(i) $V \in C[R_+ \times R^n, R_+]$, $h \in \Gamma$, $V(t,x)$ *is locally Lipschitzian in* x *and* h*-positive definite;*

(ii) $D^+V_{(2.1)}(t,x) \leq 0$ *for* $(t,x) \in S(h,\rho)$.

Then

(A) if, in addition, $h_0 \in \Gamma$, h_0 *is finer than* h *and* $V(t,x)$ *is weakly* h_0*-decrescent, then system (2.1) is* (h_0, h)*-equistable;*

(B) if, in addition, $h_0 \in \Gamma, h_0$ *is uniformly finer than* h *and* $V(t,x)$ *is* h_0*-decrescent, then system (2.1) is uniformly* (h_0, h)*-stable.*

For the proof, see [4].

4. Main results

Theorem 4.1. *Assume that:*

(i) $h_0, h \in \Gamma$ *and* h_0 *is finer than* h*;*

(ii) $g \in C[R_+^2, R]$ *and* $g(t,0) \equiv 0$*;*

(iii) $V \in C[R_+ \times R^n, R_+]$, $V(t,0) = 0$, $V(t,x)$ *is locally Lipschitzian in* x *and weakly* h_0*-decrescent and* $Ah^p(t,x) \leq V(t,x)$, $A = \mathrm{const} > 0$, *for* $(t,x) \in S(h,\rho)$*;*

(iv) $D^+V_{(2.1)}(t,x) \leq g(t, V(t,x))$ *for* $(t,x) \in S(h,\rho)$.

Then the L^1*-stability of the trivial solution* $u = 0$ *of equation (2.3) implies the* $L^p-(h_0, h)$*-equistability of system (2.1).*

Proof. Since the trivial solution $u = 0$ of equation (2.3) is L^1-stable, it is stable and there exists a $\delta_0^* = \delta_0^*(t_0) > 0$ such that $u_0^* \le \delta_0^*$ implies (2.6). By Theorem 3.2, the system (2.1) is (h_0, h)-equistable. Thus, for each $\varepsilon > 0$ and $t_0 \in R_+$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$ for all $t \ge t_0$.

We shall show that there exists a $\delta_0 = \delta_0(t_0) > 0$ such that $h_0(t_0, x_0) \le \delta_0$ implies (2.5). Since $V(t, x)$ is weakly h_0-decrescent, for a given δ_0^* and any $t_0 \in R_+$, there exist a $\delta_0 = \delta_0(t_0) > 0$ and a function $a \in CK$ such that for $(t_0, x_0) \in S(h_0, \delta_0)$, $V(t_0, x_0) \le a(t_0, h_0(t_0, x_0))$.

Choose $\delta = \delta(t_0, \varepsilon)$ such that $\delta \in (0, \delta_0]$, $a(t_0, \delta) < \delta_0^*$ and let $h_0(t_0, x_0) < \delta_0$. We set $u_0 = V(t_0, x_0)$. By using condition (iii) of the theorem and Theorem 3.1, we have

$$V(t, x(t; t_0, x_0)) \le r(t; t_0, u_0), \quad t \ge t_0,$$

where $x(t) = x(t; t_0, x_0)$ is any solution of system (2.1) such that $h_0(t_0, x_0) \le \delta_0$ and $r(t) = r(t; t_0, u_0)$ is the maximal solution of the equation (2.3). From this, it follows that

$$Ah^p(t, x(t)) \le V(t, x(t)) \le r(t; t_0, u_0)$$

and hence

$$\int_{t_0}^{\infty} h^p(t, x(t))dt \le 1/A \int_{t_0}^{\infty} r(t, t_0, u_0)dt < \infty.$$

Thus, the proof is completed. □

Theorem 4.2. *Assume that:*

(i) $h_0, h \in \Gamma$ and h_0 is uniformly finer than h;

(ii) $g \in C[R_+^2, R]$ and $g(t, 0) \equiv 0$;

(iii) $V \in C[R_+ \times R^n, R_+]$, $V(t, 0) = 0$, $V(t, x)$ is locally Lipschitzian in x and h_0-decrescent and

$$Ah^p(t, x(t)) \le V(t, x), \; A = \text{const} > 0 \;\; \textit{for} \;\; (t, x) \in S(h, \rho);$$

(iv) $D^+V_{(2.1)}(t, x) \le g(t, V(t, x))$ for $(t, x) \in S(h, \rho)$.

Then the uniform L^1-stability of the trivial solution u=0 of equation (2.3) implies uniform $L^p-(h_0, h)$-stability of system (2.1).

Proof. From Theorem 3.3, the system (2.1) is uniformly (h_0, h)-stable. Since the trivial solution $u = 0$ of equation (2.3) is uniformly L^1-stable, there exists a $\delta_0^* > 0$ such that $u_0 \le \delta_0^*$ implies (2.6) and the integral (2.6) converges uniformly in t_0. To prove that the integral (2.5) converges uniformly in t_0, we follow the proof of Theorem 4.1 and choose $u_0 = a(h_0(t_0, x_0))$, where $a \in K$, thereby deducing $\delta = a^{-1}(\delta_0^*)$, where $\delta \in (0, \delta_0]$. Let $h_0(t_0, x_0) < \delta_0$. It is evident that δ_0 is independent of t_0 and integral (2.5) converges uniformly in t_0. □

Theorem 4.3. *Assume that:*
(i) $h_0, h \in \Gamma$ *and* h_0 *is uniformly finer than* h*;*
(ii) $g \in C[R_+^2, R]$ *and* $g(t, 0) \equiv 0$*;*
(iii) $V \in C[R_+ \times R^n, R_+]$, $V(t, 0) = 0$, $V(t, x)$ *is locally Lipschitzian in* x*,* V *is* h*-positive definite and weakly* h_0*-decrescent;*
(iv) $D_+V_{(2.1)}(t, x) \le -Ch^p(t, x)$, $C = \text{const} > 0$, *for* $(t, x) \in S(h, \rho)$.
Then system (2.1) is $L^p-(h_0, h)$*-equistable.*

Proof. By Theorem 3.4, it follows that the system (2.1) is (h_0, h)-equistable. Thus, for each $\varepsilon > 0$ and $t_0 \in R_+$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$ for all $t \ge t_0$.

We shall show that there exists a $\delta_0 = \delta_0(t_0) > 0$ such that $h_0(t_0, x_0) < \delta_0$ implies (2.5). We define

$$m(t) = V(t, x(t)) + C \int_{t_0}^{t} h^p(t, x(t))dt.$$

By condition (iv) of the theorem, we have $D^+m(t) = D^+V_{(2.1)}(t, x(t)) + Ch^p(t, x(t)) \le 0$.

This implies that $m(t)$ is nonincreasing, therefore $m(t) \le m(t_0)$ and

$$\int_{t_0}^{\infty} h^p(t, x(t))dt \le m(t_0)/C = V(t_0, x_0)/C \quad \text{for} \quad t \ge t_0.$$

Thus, the proof is completed. □

Theorem 4.4. *Assume that:*
(i) $h_0, h \in \Gamma$ *and* h_0 *is uniformly finer than* h*;*
(ii) $g \in C[R_+^2, R]$ *and* $g(t, 0) \equiv 0$*;*
(iii) $V \in C[R_+ \times R^n, R_+]$, $V(t, 0) = 0$, $V(t, x)$ *is locally Lipschitzian in* x*,* h*-positive definite and* h_0*-decrescent;*
(iv) $D^+V_{(2.1)}(t, x) \le -Ch^p(t, x)$, $C = \text{const} > 0$, *for* $(t, x) \in S(h, \rho)$.
Then system (2.1) is uniformly $L^p-(h_0, h)$*-stable.*

Proof. Condition (iv) of the theorem, in virtue of condition (iii), reduces to $D^+V_{(2.1)}(t,x) \le g(t,V(t,x))$, where $g(t,u) = -(C/B)u$, and hence it is easy to check that the trivial solution $u = 0$ of equation (2.3) is uniformly L^1-stable. Therefore, from Theorem 4.2, the system (2.1) is uniformly $L^p-(h_0,h)$-stable. □

Theorem 4.5. *Let $h_0, h \in \Gamma$. System (2.1) is $L^p-(h_0,h)$-integrally stable if and only if for each $\varepsilon > 0$ and $t_0 \in R_+$ there exist a $\delta_1 = \delta_1(t_0,\varepsilon) > 0$ and a $\delta_2 = \delta_2(t_0,\varepsilon) > 0$ such that if $\Phi(t)$ is any continuous function defined on $[t_0,\infty]$ and satisfies $\int\limits_{t_0}^{\infty}\|\Phi(t)\|dt < \delta_2$ then for any solution $y(t) = y(t,t_0,y_0)$ of the system*

$$\dot{y} = f(t,y) + \Phi(t) \tag{4.1}$$

for which $h_0(t_0,y_0) < \delta_1$ the inequality $\int\limits_{t_0}^{\infty} h^p(t,y(t))dt < \infty$ is verified.

Proof. The necessity of the condition is clear. Therefore it suffices to prove that if the property from the statement occurs, the system (2.1) is $L^p-(h_0,h)$-integrally stable.

Let $F(t,x)$ be such that $\int\limits_{t_0}^{\infty} \sup\limits_{h(t,x)\le\varepsilon} \|F(t,x)\|dt < \delta_2$, where $\delta_2 = \delta_2(t_0,\varepsilon) > 0$ is one given the condition. Consider (t_0,x_0) with $h_0(t_0,x_0) < \delta_1$ and the solution $x^*(t) = x^*(t;t_0,x_0)$ of the perturbed system (2.2).

If we would not have $\int\limits_{t_0}^{\infty} h^p(t,x^*(t))dt < \infty$, there exists the first point $t_1 > t_0$, $\int\limits_{t_0}^{t_1} h^p(t,x^*(t))dt = \infty$ and $\int\limits_{t_0}^{t} h^p(t,x^*(t))dt < \infty$ for all $t \in [t_0,t_1)$. For any $t \in [t_0,t_1)$ take $\Phi(t) = F(t,x^*(t))$, we have

$$\int\limits_{t_0}^{t_1} \|\Phi(t)\|dt \le \int\limits_{t_0}^{t_1} \sup_{h(t,x^*(t))\le\varepsilon} \|F(t,x^*(t))\|dt < \delta_2.$$

We extend $\Phi(t)$ continuously on the whole semiaxis $t \ge t_0$ such that $\int\limits_{t_0}^{t_1}\|\Phi^*(t)\|dt \le \delta_2$, where $\Phi^*(t)$ is a extended function, for this it is sufficient to take $t_2 \ge t_1$ such that $t_2 - t_1 < \dfrac{2[\delta_2(t_0,\varepsilon) - \int\limits_{t_0}^{t_1}\|\Phi(t)\|dt]}{1 + \|\Phi(t_1)\|}$, $\Phi^*(t) = \Phi(t)$ for all

$t \in [t_0, t_1)$ linear on $[t_1, t_2)$, where we put $\Phi^*(t_2) = 0$, and zero for all $t \geq t_1$.

We consider the following system

$$\dot{y} = f(t, y) + \Phi^*(t) \tag{4.2}$$

Now let $y^*(t) = y^*(t; t_0, x_0)$ be a solution of system (4.2).

From $h_0(t_0, x_0) < \delta_1$ and $\int_{t_0}^{\infty} \Phi^*(t)dt < \delta_2$, we have $\int_{t_0}^{\infty} h^p(t, y^*(t))dt < \infty$. Hence $\int_{t_0}^{t_1} h^p(t, y^*(t))dt < \infty$. But we have $y^*(t; t_0, x_0) \equiv x^*(t; t_0, x_0)$ on (t_0, t_1) hence $\int_{t_0}^{t_1} h^p(t, x^*(t))dt < \infty$ which is contradictory. Therefore, the system (2.1) is $L^p-(h_0, h)$-integrally stable. □

Theorem 4.6. *Assume that:*

(i) $h_0, h \in \Gamma$ and h_0 is finer than h;

(ii) $g \in C[R_+^2, R]$ and $g(t, 0) \equiv 0$;

(iii) $V \in C[R_+ \times R^n, R_+]$, $V(t, 0) = 0$, V is weakly h_0-decrscent and $Ah^p(t, x) \leq V(t, x)$, $A = \text{const} > 0$, for $(t, x) \in S(h, \rho)$;

(iv) $|V(t, x) - V(t, x')| \leq M\|x - x'\|$ for any (t, x), $(t, x') \in S(h, H)$ $(H = \text{const} > 0)$, where $M = \text{const} > 0$;

(v) $D^+V_{(2.1)}(t, x) \leq g(t, V(t, x))$ for $(t, x) \in S(h, \rho)$.

If the trivial solution $u = 0$ of equation (2.3) is L^1-integrally stable, then system (2.1) is $L^p-(h_0, h)$-integrally stable.

Proof. By Theorem 3.2 system (2.1) is (h_0, h)-equistable. Thus, for each $\varepsilon > 0$ and $t_0 \in R_+$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t)) < \varepsilon$ for all $t \geq t_0$.

Since the trivial solution $u = 0$ of equation (2.3) is L^1-integrally stable, it is stable and for each $\varepsilon > 0$ and $t_0 \in R_+$ there exist a $\eta_1 = \eta_1(t_0, \varepsilon) > 0$ and a $\eta_2 = \eta_2(t_0, \varepsilon) > 0$ such that $u_0 < \eta_1$ and $\int_{t_0}^{\infty} \sup_{u \leq \varepsilon} |G(t, u)| dt < \eta_2$ implies $\int_{t_0}^{\infty} u^*(t, t_0, u_0)dt < \infty$, where $u^*(t) = u^*(t; t_0, u_0)$ is the solution of the perturbed equation (2.4).

Since $V(t, x)$ is weakly h_0-decrescent, for given $\eta_1 > 0$ and $t_0 \in R_+$, there exist a $\delta_1 = (\varepsilon, \eta_1, t_0) = \delta_1(\varepsilon, t_0) > 0$ and a function $a \in CK$ such that for $(t_0, x_0) \in S(h_0, \delta_1), V(t_0, x_0) \leq a(t_0, h_0(t_0, x_0))$.

Choose $\delta = \delta(t_0, \varepsilon)$ such that $\delta = (0, \delta_1]$, $a(t_0, \delta) < \eta_1$ and let $h_0(t_0, x_0) < \delta_1$. We set $u_0 = V(t_0, x_0)$. Hence $0 < u_0 < \eta_1$.

Given $\eta_2 < 0$, there exists a $\delta_2 = \delta_2(t_0, \varepsilon) > 0$ such that $\delta_2 < \min(\eta_2/M, \delta)$. By using condition (v) of the theorem

$$D^+V_{(2.2)}(t,x) \le D^+V_{(2.1)}(t,x) + M\|F(t,x)\| \le g(t, V(t,x)) + M\|F(t,x)\|.$$

We put $\lambda(t) = M\|F(t, x^*(t; t_0, x_0))\|$, where $x^*(t) = x^*(t; t_0, x_0)$ is the solution of the perturbed system (2.2) such that $h_0(t_0, x_0) < \delta_1$.

For this function $\lambda(t)$, we consider the following scalar equation

$$\dot{u} = g(t,u) + \lambda(t). \tag{4.3}$$

Because $h(t, x(t)) < \varepsilon$ for all $t \ge t_0$ satisfying the condition $h_0(t_0, x_0) < \delta_1$, we get

$$\int_{t_0}^{\infty} \lambda(t)dt = M \int_{t_0}^{\infty} \|F(t, x^*(t; t_0, x_0))\|dt \le$$

$$\le M \int_{t_0}^{\infty} \sup_{h(t,x^*(t)) \le \varepsilon} \|F(t, x^*(t))\|dt \le M\delta_2 < \eta_2.$$

By Theorem 4.5, we have

$$\int_{t_0}^{\infty} r^*(t; t_0, u_0)dt < \infty, \tag{4.4}$$

where $r^*(t) = r^*(t; t_0, u_0)$ is the maximal solution of the equation (4.3) such that $u_0 < \eta_1$.

Moreover, from Theorem 3.1 we obtain that $V(t_0, x_0) \le u_0$ implies

$$V(t, x^*(t)) \le r^*(t; t_0, u_0) \quad \text{for all} \quad t \ge t_0. \tag{4.5}$$

For each $\varepsilon > 0$ and $t_0 \in R_+$, using δ_1 and δ_2 defined above, we claim that system (2.1) is $L^p-(h_0, h)$-integrally stable, whenever $h_0(t_0, x_0) < \delta_1$ and $\int_{t_0}^{\infty} \sup_{h(t,x) \le \varepsilon} \|F(t,x)\|dt < \infty$.

By using condition (iii) of the theorem and relations (4.4) and (4.5), it follows that $Ah^p(t, x^*(t)) \le V(t, x^*(t; t_0, x_0)) \le r^*(t; t_0, u_0)$, and hence

$$\int_{t_0}^{\infty} h^p(t, x^*(t))dt \le 1/A \int_{t_0}^{\infty} r^*(t; t_0, u_0)dt < \infty.$$

This proves that system (2.1) is $L^p-(h_0, h)$-integrally stable. □

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ВЪРХУ L^p-УСТОЙЧИВОСТТА И L^p-ИНТЕГРАЛНАТА УСТОЙЧИВОСТ НА НЕЛИНЕЙНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ПО ОТНОШЕНИЕ НА ДВЕ МЕРКИ

Иван К. Русинов

Резюме. Намерени са нови достатъчни условия за L^p-устойчивост и L^p-интегрална устойчивост на нелинейни системи от диференциални уравнения по отношение на две мерки, като се използват втория метод на Ляпунов и принципа на сравнението.