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SOME PROPERTIES OF THE CHARACTERISTIC EXPONENTS OF IMPULSE DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract. In the paper is introduced the notion of ω-limit operators. There are founded relations between the spectra of ω-limit operators and the general exponents of linear differential impulse equations in an arbitrary Banch space.

Key words: Linear Impulse Differential Equations, ω-limit Operator, General Exponent
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1. Introduction

In the paper we consider linear differential impulse equations in an arbitrary Banach space. There are founded relations between the spectra of ω-limit operators and the general exponents of inhomogeneous linear differential impulse equations in an arbitrary Banch space.

2. Problem statement

Let X be an arbitrary complex Banach space with identity I. We consider following impulse differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \quad \text{for } t \neq t_n \tag{1}$$

(2) $$x(t_n^+) = Q_n x(t_n) \quad \text{for } n = 1, 2, ...$$

where $x(t) \in X \ (t \geq 0),\ A(t) : X \to X \ (t \geq 0),\ Q_n : X \to X \ (n \in \mathbb{N})$.

By $L(X)$ we shall denote the space of all bounded operators operating in X.

We suppose the validity of the following condition (H) :

H1. $A(t) \in L(X) \ (t \geq 0)$, where the operator-function $A(t)$ is continuous extendable for every interval $[t_j, t_{j+1}] \ (j \in \mathbb{N})$,

H2. $Q_n \in L(X) \ (n \in \mathbb{N})$,

H3. $t_n < t_{n+1} \ (n \in \mathbb{N}), \lim t_n = \infty \ (n \to \infty)$.

By $W(t, s)$ we shall denote the Cauchy operator of (1), (2).

Definition 1. *([6]) By general exponent χ_g of the impulse differential equation (1), (2) we shall denote the greates lower bound of all numbers ρ, such, that for every solution $x(t) = W(t, t_0)x_0$ of (1), (2) holds the inequality*

$$||x(t)|| \leq N\rho e^{\rho(t-\tau)}||x(\tau)|| \quad (t \geq \tau \geq 0)$$

where the number N_ρ do not depend from the choice of the initially value of x_0 .

We shall note, that in [6] are found necessary and sufficient conditions for $\chi_g < \infty$.

Definition 2. *([6]) The operator $C : X \to X$ we shall call ω-limit for the operator-function $A(t)(t \geq 0)$, if there exists a sequence $\xi_n \to \infty(n \to \infty)$ for which $A(\xi_n) \to C \ (n \to \infty)$.*

Definition 3. *([6]) We shall say that the operator-function $A(t)(t \geq 0)$ satisfies the condition $S_{\epsilon,L}(\epsilon, L > 0)$, if there exists a number $T > 0$ such that by $s, t \geq T, |s - t| \leq L$ the inequality $||A(s) - A(t)|| < \epsilon$ holds.*

Definition 4. *([6]) We shall say that the operator-function $A(t)(t \geq 0)$ is stationary on infinity, if it satisfies the condition $S_{\epsilon,L}$ by any arbitrary small $\epsilon > 0$ and any positive $L > 0$.*

Definition 5. *([6]) We shall say that the operator-function $A(t)(t \geq 0)$ is compact, if every sequence $\{A(t_n)\}$ has a subsequence, which is convergent to some element of $L(X)$.*

3. Main results

Lemma 1. *Let $\varphi(t)(t \geq 0)$ be a nonnegative scalar piecewise continuous function with discontinuities of the first kind at the points t_n, $v(t)$ is a local intergable nonnegative scalar function, and $c \geq 0$, $d_i \geq 0$ are constants. Let the following inequality holds*

$$\varphi(t) \leq c + \sum_{t_0<t_i<t} \beta_i \varphi(t_i) + \int_{t_0}^{t} \varphi(\tau) v(\tau) \mathrm{d}\tau \quad (t_0 \geq 0)$$

Then the following estimate is valid

$$\varphi(t) \leq c \prod_{t_0<t_i<t} (1+\beta_i) e^{\int_{t_0}^{t} v(\tau)\mathrm{d}\tau}$$

Proof. We consider operator K, operating in the space $D(R,X)$ of all piecewise continuous function with discontinuities of the first kind at the points t_n, with its values belongs to X, and defined by the formula

$$(K_\varphi)(t) = \sum_{t_0<t_i<t} \beta_i \varphi(t_i) + \int_{t_0}^{t} \varphi(\tau) v(\tau) \mathrm{d}\tau$$

It holds the inequality $\varphi(t) \leq \psi(t)$, where $\psi(t)$ is a solution of the equation

$$\psi(t) = c + \sum_{t_0<t_i<t} \beta_i \psi(t_i) + \int_{t_0}^{t} \psi(\tau) v(\tau) \mathrm{d}\tau$$

It may be directly shown, that

$$\psi(t) = c \prod_{t_0<t_i<t} (1+\beta_i) e^{\int_{t_0}^{t} v(\tau)\mathrm{d}\tau}$$

Lemma 1 is proved. □

We consider following nonhomogenous impulse differential equation :

(3) $$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t) \quad \text{for } t \neq t_n$$

(4) $$x(t_n^+) = Q_n x(t_n) + h_n \quad \text{for } n = 1, 2, ...$$

where $f(t) \in X, h_n \in X$ (n= 1, 2, ...).

Lemma 2. *Let the operators ${Q_n}^{-1} (n \in \mathbb{N})$ exist and ${Q_n}^{-1} \in L(X)$. Then the solution of the equation (3), (4) is determined by the formula*

$$x(t) = \begin{cases} W(t, {t_0}^+)x_0 + \int\limits_{t_0}^{t} W(t,\tau)f(\tau)\mathrm{d}\tau + \sum\limits_{t_0<t_j<t} W(t, {t_j}^+)h_j & , t \geq t_0 \\ W(t, {t_0}^+)x_0 + \int\limits_{t_0}^{t} W(t,\tau)f(\tau)\mathrm{d}\tau - \sum\limits_{t<t_j<t_0} W(t, {t_j}^+)h_j & , t < t_0 \end{cases}$$

The proof of Lemma 2 is obtained by directly calculation.

Let $W_k(t, s) (k = 1, 2) (a \leq s, t \leq b)$ are the evolution operators of the impulse equations respectively

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A_1(t)x \quad \text{for } t \neq t_n$$

$$x(t_n^+) = Q_n x(t_n) \quad \text{for } n = 1, 2, ...$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A_2(t)x \quad \text{for } t \neq t_n$$

$$x(t_n^+) = \tilde{Q}_n x(t_n) \quad \text{for } n = 1, 2, ...$$

Lemma 3. *Let there exist constants $N > 0$ and $\nu_1 \in \mathbb{R}$ such, that*

$$||W_1(t, s)|| \leq N e^{-\nu_1(t-s)} \quad (a \leq s, t \leq b)$$

Then the follow estimate holds

$$||W_1(t, s)|| \leq N e^{-\nu_1(t-s)} e^{N\int_s^t ||A_1(\tau)-A_2(\tau)||\mathrm{d}\tau} (1 + \prod_{s<t_j<t} ||Q_j - \tilde{Q}_j||)$$

Proof. It is sufficient to proof the lemma only for the case $a = s = 0$. The operator $V_2(t) = W_2(t, 0)$ is solution of the initialy impulse equation

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} - A_1 W_2 = (A_2 - A_1)W_2 \quad \text{for } t \neq t_n$$

$$V_2(t_n^+) = Q_n V_2(t_n) + (\tilde{Q}_n - Q_n)V_2(t_n) \quad \text{for } n = 1, 2, ...$$

$$V_2(0) = I$$

Let consider the equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} - A_1 X = F(t) \quad \text{for } t \neq t_n$$

$$X(t_n^+) = \tilde{Q}_n X(t_n) + (\tilde{Q}_n - Q_n)V_2(t_n) \quad \text{for } n = 1, 2, ...$$

$$X(0) = I,$$

where $F(t) = (A_2(t) - A_1(t))V_2(t)$. Following Lemma 1 for $X(t)$ there holds the presentation

$$X(t) = W_1(t,0) + \int\limits_0^t W_1(t,\tau)(A_2(\tau) - A_1(\tau))V_2(\tau)\mathrm{d}\tau + \\ + \sum_{0<t_j<t} W_1(t, t_j{}^+)(\tilde{Q}_j - Q_j)V_2(t_j)$$

i.e.

$$V_2(t) = W_1(t,0) + \int\limits_0^t W_1(t,\tau)(A_2(\tau) - A_1(\tau))V_2(\tau)\mathrm{d}\tau + \\ + \sum_{0<t_j<t} W_1(t, t_j{}^+)(\tilde{Q}_j - Q_j)V_2(t_j)$$

Let $\varphi(t) = ||V_2(t)||$, $p(t) = ||A_2(t) - A_1(t)||$, $q_j = ||\tilde{Q}_j - Q_j||$. Then for $\varphi(t)$ we get the estimate

$$\varphi(t) \leq Ne^{-\nu_1 t} + N\int\limits_0^t e^{-\nu_1(t-\tau)}p(\tau)\varphi(\tau)\mathrm{d}\tau + \sum_{t_0<t_j<t} e^{-\nu_1(t-t_j)}q_j\varphi(t_j) \tag{5}$$

The proof of Lemma 3 follows from (5) and Lemma 2. □

Theorem 1. *Let the conditions hold:*

1. The impulse equation (1), (2) has negative general exponent, i.e. there exist positive constants N, ν *for which the estimate holds*

$$||W(t,\tau)|| \leq Ne^{-\nu(t-\tau)} \quad (0 \leq \tau \leq t < \infty) \tag{6}$$

2. The operator-function $A(t)(t \geq 0)$ satisfied the condition $S_{\epsilon,L}$ by sufficient small ϵ and large L.

3. The operators $Q_n(n \in \mathbb{N})$ are surections and have bounded reverse ${Q_n}^{-1}$, i.e. there exist numbers $q_n > 0$ for which are fulfilled the estimates

$$||Q_n z|| \geq q_n \quad (n \in \mathbb{N}, ||z|| = 1)$$

4. $(q_l q_{l+1} ... q_{n-1} q_n)^{-1} \leq K e^{\chi\omega}$, where the points $q_l, q_{l+1}, ..., q_{n-1}, q_n$ lie in the interval $[a, b]$ with lenght $= \omega$, χ is a constant and $\nu > \chi$.

5. $A(t)Q_n = Q_n A(t) \ \ (n \in \mathbb{N}, t \geq 0)$.

Then the spectra of all ω-limit operators of the operator $A(t)$ are in the halfplane $Re\lambda \leq -\nu_0 \ \ (\nu_0 > 0)$.

Proof. Let $C \in L(X)$ is an arbitrary ω-limit operator of the operator $A(t)$, i.e. there exist a sequence $\{\tau_k\}$ for which $A(\tau_k) \to C \ (k \to \infty)$ in the space $L(X)$, i.e. by sufficient small $\delta > o$ and large k the estimate holds

$$||A(\tau_k) - C|| < \delta \tag{7}$$

But from other side by sufficient large k it holds the estimate

$$||A(\tau_k) - A(t)|| \leq \epsilon \quad (\tau_k \leq t \leq \tau_k + L) \tag{8}$$

Hence by sufficient large k from (7) and (8) it follows the inequality

$$||A(t) - C|| < \epsilon + \delta \quad (\tau_k \leq t \leq \tau_k + L)$$

Applying Lemma 3 for the equation (1), (2) and for the impulse equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Cx \quad \text{ for } t \neq t_n \tag{9}$$

$$x(t_n^+) = Q_n x(t_n) \quad \text{ for } n = 1, 2, ... \tag{10}$$

with the corresponding evolution operator $W_1(t, \tau)(s = \tau_k, t = \tau_k + L)$ we obtain the estimate

$$||W_1(t, s)|| \leq N e^{-\nu L} e^{NL(\epsilon+\delta)} \tag{11}$$

i.e.

$$||e^{C(t-t_n)} Q_n e^{C(t_n - t_{n-1})} Q_{n-1} ... Q_l e^{C(t_l - s)}|| \leq N e^{-\nu L} e^{NL(\epsilon+\delta)} \tag{12}$$

where $t_{l-1} < s \leq t_l < t_n < t \leq t_{n+1}$.

From condition 5 of Theorem 1 follows the equality $CQ_n = Q_nC$, then we obtain

$$||Q_nQ_{n-1}...Q_{l+1}Q_le^{CL}|| \leq Ne^{-\nu L}e^{NL(\epsilon+\delta)} \tag{13}$$

From conditions 3 and 4 of Theorem 1 follows the inequality

$$q_nq_{n-1}...q_{l+1}q_l||e^{CL}|| \leq Ne^{-\nu' L} \tag{14}$$

i.e.

$$||e^{CL}|| \leq Ne^{-\nu' L}(q_nq_{n-1}...q_{l+1}q_l)^{-1},$$

where $\nu' = \nu - N(\epsilon+\delta)$. Following Lemma 2.2 [8] the spectre $Sp(CL)$ is lieing in the halfplane

$Re\lambda \leq \frac{\ln KN}{L} - (\nu' - \chi) = -(-\frac{\ln KN}{L} + \nu' - \chi) = -(-\frac{\ln KN}{L} + \nu - N(\epsilon+\delta) - \chi)$,

i.e.

$Re\lambda \leq -(-\frac{\ln KN}{L} + \nu - N\epsilon - N\delta - \chi)$

By sufficient small ϵ, δ and sufficient large L there holds the inequality

$$\nu_0 = -\frac{\ln KN}{L} + \nu - N\epsilon - N\delta - \chi > 0$$

Theorem 1 is proved. □

Remark 1. *Theorem 1 still holds, if there exist numbers $n' \in \mathbb{N}, t, a > 0$, such that the conditions 3 and 5 of Theorem 1 are fulfilled only by $n \geq n', t \geq t'$, and condition 4 by $a \geq a'$.*

Corollary 1. *Theorem 1 still holds, if the conditions 1, 2, 5 of the Theorem 1 are fulfilled and the condition*

$$1 + \prod_{t_n \in [a,b]} ||Q_n - I|| \leq Ke^{\chi\omega} \tag{15}$$

holds for every interval $[a, b]$ with lenght $\leq \omega$, (a is large enough), where K and χ are constants and $\nu > \chi$.

Proof. Analogously to the proof of the Theorem 1 by sufficient large k we obtain the inequality

$$||A(t) - C|| < \epsilon + \delta \quad (\tau_k \leq t \leq \tau_k + L)$$

We apply Lemma 3 for the impulse equation (1), (2) and the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Cx \tag{16}$$

$(s = \tau_k, t = \tau_k + L)$ and obtain the estimate

$$||e^{CL}|| \leq Ne^{-\nu L(\epsilon+\delta)}(1 + \prod_{\tau_k < t_j < \tau_{k+L}} ||Q_j - I||) \tag{17}$$

From the estimates (15), (17) follows the estimate

$$||e^{CL}|| \leq NKe^{-\nu' L}e^{\chi L} = NKe^{-L(\nu'-\chi)},$$

where $\nu' = \nu - N(\epsilon + \delta)$.

Corollary 1 is proved. □

By $C(A(t))$ we shall note the set of all ω-limit operator of the operator-functions $A(t)$, and by $B(A(t))$ - the set of the general exponents of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Cx$$

$$x(t_n^+) = Q_n x(t_n) \quad (n \in \mathbb{N}),$$

where $C \in C(A(t))$.

Theorem 2. *Let the operator-function $A(t)(t \geq 0)$ is compact and stationary on infinity.*

Then $\chi_g = \chi_{A(t)}$,

where χ_g is general exponent of (1), (2) and $\chi_{A(t)} = \sup\{k : k \in B(A(t))\}$.

Proof. Let consider the impulse equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (A(t)x + \lambda I)x \quad \text{for } t \neq t_n \tag{18}$$

$$x(t_n^+) = Q_n x(t_n) \quad (n \in \mathbb{N}) \tag{19}$$

It is not difficult to check, that the impulse equation (18), (19) has a general exponent $\chi_g + \lambda$ and $\chi_{A(t)+\lambda I} = \chi_{A(t)} + \lambda$.

From Theorem 8 [6] and Theorem 1 follows, that the numbers $\chi_g + \lambda$ and $\chi_{A(t)} + \lambda$ can be negative only together, i.e. $\chi_g = \chi_{A(t)}$.

Theorem 2 is poved. □

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НЯКОИ СВОЙСТВА НА ХАРАКТЕРИСТИЧНИТЕ ПОКАЗАТЕЛИ НА ИМПУЛСНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ В БАНАХОВО ПРОСТРАНСТВО

А. Георгиева, Х. Кискинов

Резюме. В работата е въведено понятието ω-гранични оператори. Намерени са връзки между спектрите на ω-граничните оператори и генералните показатели на линейни импулсни диференциални уравнения в произволно Банахово пространство.