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POLYSTABILITY CRITERIA FOR DIFFERENTIAL EQUATIONS IN TERMS OF TWO MEASURES

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Abstract. In this work, we will introduce the notion of polystability of systems of differential equations in terms of two measures sufficient conditions for this kind of stability are obtained by the method of vector Lyapunov functions.

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***Key words*:** polystability in terms of two measures, Lyapunov function, h-positive definite, h_0-decrescent, weakly h_0-decrescent.

1. Introduction

In [1] the authors present the definition for polystability of movement on part of the variables and apply the Direct Lyapunov Method for analysis of the polystability. Later Russinov I. K. in [7-9] analyzes the polystability of a flag of integral manifolds using scalar, vector and matrix Lyapunov function. Developing this idea in [6] the exponential polystability of a flag of invariant sets is analyzed.

On the other hand a notion of new type of stability emerges, called stability in terms of two measures, which is considered in [2,3,5,10].

The present paper presents the notion of polystability of dynamical systems in terms of two measures, analyzing this polystability using scalar Lyapunov functions.

2. Preliminary notes

We shall consider the system of differential equations

$$\dot{x} = X(t,x), \;\; X(t,0) \equiv 0, \tag{1}$$

where $x = (x^1, \dots, x^n)^T \in R^n$ and $t \in R^+$, under the assumption $X \in C[R^+ \times R^n, R^n]$.

Let us list definitions and classes of functions for convenience:

$K = \{\sigma \in C[R^+, R^+] : \sigma(u)$ is strictly increasing in x and $\sigma(0) = 0\}$,

$CK = \{\sigma \in C[R^+ \times R^+, R^+] : \sigma(t, \bullet) \in K$ for each $t \in R^+\}$,

$\Gamma = \{h \in C[R^+ \times R^n, R^+] : \inf\limits_{x \in R^n} h(t,x) = 0$ for each $t \in R^+\}$.

Definition 1 [2]. Let $h_0^i,\ h^i \in \Gamma$ for $i = 1,2$. Then we say that h_0^i is uniformly finer than h^i if there exists a $\rho^i > 0$ and a function $\Phi^i \in K$ such that $h_0^i(t,x) < \rho^i$ implies $h^i(t,x) \le \Phi^i(h_0^i(t,x))$ for $i = 1,2$.

Definition 2. Let $h_0^i,\ h^i \in \Gamma$. Then system (1) is said to be polystable in terms of two measures, if:

1) it is (h_0^1, h^1)-uniformly stable, i.e. for each $\varepsilon > 0$ and $t_0 \in R^+$ there exists a function $\delta = \delta(t_0, \varepsilon) > 0$, which is continuous in t_0 for each ε such that $h_0^1(t_0, x_0) < \delta$ implies $h^1(t, x(t)) < \varepsilon, t \ge t_0$, for any solution $x(t) = x(t; t_0, x_0)$ of the system (1);

2) it is (h_0^2, h^2)-uniformly asymptotically stable, i.e. it is (h_0^2, h^2)-uniformly stable and (h_0^2, h^2)-uniformly attractive, i.e. for each $t_0^* \in R^+$ there exists a $\Delta > 0$ such that for each solution $x(t) = x(t; t_0^*, x_0^*)$ with initial conditions $x_0^*(t) = x(t_0^*; t_0^*, x_0^*)$ definite for each $t \ge t_0^*$ and for some sequence $t_i \to \infty$, depending on t_0^* and x_0^*, $h_0^2(t_0^*, x_0^*) < \Delta$ implies $\lim\limits_{i \to \infty} h^2(t_i; t_0^*, x_0^*) = 0$.

We use a scalar function $V(t,x)$, where $V \in C[R^+ \times R^n, R^+]$ and $V(t,0) \equiv 0$.

Definition 3. Let $V \in C[R^+ \times R^n, R^+]$. Then for $(t,x) \in R^+ \times R^n$, the upper right Dini derivative of $V(t,x)$ with respect to the system (1) is defined as

$$D^+V(t,x) = \lim_{h \to 0^+} \sup[V(t+h, x + hX(t,x)) - V(t,x)]/h \tag{2}$$

Definition 4 [2]. Let $V^i \in C[R^+ \times R^n, R^+]$ and $h_0^i, h^i \in \Gamma$ for $i = 1, 2$.

1) Then $V^i(t,x)$ is said to be h^i-positive definite if there exists a $\rho^i > 0$ and a function $b^i \in K$ such that $h^i(t,x) < \rho^i$ implies $b^i(h^i(t,x)) \leq V^i(t,x)$ for $i = 1, 2$;

2) Then $V^i(t,x)$ is said to be h_0^i-decrescent if there exists a $\rho_0^i > 0$ and a function $a^i \in K$ such that $h_0^i(t,x) < \rho_0^i$ implies $V^i(t,x) \leq a^i(h_0^i(t,x))$ for $i = 1, 2$;

3) Then $V^i(t,x)$ is said to be weakly h_0^i-decrescent if there exists a $\rho_0^i > 0$ and a function $a^i \in CK$ such that $h_0^i(t,x) < \rho_0^i$ implies $V^i(t,x) \leq a^i(t, h_0^i(t,x))$ for $i = 1, 2$.

Together with the parent system (1) we also consider the equations

$$\dot{u} = f(t,u),\ f(t,0) \equiv 0,\ f \in C[R^+ \times R^+, R], \tag{3}$$

$$\dot{v} = g(t,v),\ g(t,0) \equiv 0,\ g \in C[R^+ \times R^+, R], \tag{4}$$

and f, g are nondecreasing.

Let $u(t; t_0, u_0)$ and $v(t; t_0^*, v_0^*)$ be the solutions of the equations (3) and (4) respectively, as $u_0 = u(t_0; t_0, u_0) \in R^+$ and $v_0^* = v(t_0^*; t_0^*, v_0^*) \in R^+$. We denote by $\bar{u}(t_0; t_0, u_0)$ and $\bar{v}(t_0^*; t_0^*, v_0^*)$ the maximal solutions of the scalar equations (3) and (4) respectively.

We put

$S(h^i, \rho^i) = [(t,x) \in R^+ \times R^n : h^i(t,x) < \rho^i]$ for $i = 1, 2$.

3. Main result

Theorem. *Assume that*

1) $h_0^i, h^i \in \Gamma$ and h_0^i is uniformly finer than h^i for $i = 1, 2$;

2) $V^i \in C[S(h^i, \rho^i), R^+]$, $V^i(t,x)$ is locally Lipschitzian in x, h^i-positive definite, h_0^i-decrescent for $i = 1, 2$;

3) $D^+V^1(t,x) \leq f(t, V^1(t,x))$ for $(t,x) \in S(h^1, \rho^1)$;

4) $D^+V^2(t,x) \leq g(t, V^2(t,x))$ for $(t,x) \in S(h^2, \rho^2)$;

5) the trivial solutions of (3) and (4) are uniformly stable and uniformly asymptotically stable respectively.

Then, the differential system (1) is polystable in terms of two measures in sense of definition 2.

Proof. First we shall prove that the system (1) is (h_0^1, h^1)-uniform stability.

Since $V^1(t,x)$ is h^1-positive definite, there exist $\lambda \in (0, \rho^1]$ and $b^1 \in K$ such that

$$b^1(h^1(t,x)) \le V^1(t,x), \text{ whenever } h^1(t,x) < \lambda. \tag{5}$$

Let $0 < \varepsilon < \lambda$ and $t_0 \in R^+$ be given and suppose that the trivial solution of (3) is uniformly stable. Then for the given $b^1(\varepsilon) > 0$ and $t_0 \in R^+$ there exists a function $\delta^1 = \delta^1(\varepsilon) > 0$ such that

$$u_0 < \delta^1 \text{ implies } u(t; t_0, u_0) < b^1(\varepsilon), \ t \ge t_0. \tag{6}$$

We choose $u_0 = V^1(t_0, x_0)$. Since $V^1(t,x)$ is h_0^1-decrescent there exist $\rho_0^1 (0 < \rho_0^1 \le \rho^1)$ and a function $a^1 \in K$ such that

$$h_0^1(t,x) < \rho_0^1 \text{ implies } V^1(t,x) \le a^1(h_o^1(t,x)). \tag{7}$$

Since h_0^1 is uniformly finer than h^1, there exist $\rho_1^1 (0 < \rho_1^1 \le \rho_0^1)$ and $\Phi^1 \in K$ such that

$$h_0(t,x) < \rho_1^1 \text{ implies } h^1(t,x) \le \Phi^1(h_o^1(t,x)), \tag{8}$$

where ρ_1^1 is such that $\Phi^1(\rho_1^1) < \rho_0^1$.

Since $a^1, \ \Phi^1 \in K$ there exists a function $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$a^1(\delta_1) > \delta^1 \text{ and } \Phi^1(\delta^1) < \varepsilon. \tag{9}$$

Since $a^1 \in K$ and $V(t,0) \equiv 0$, by (7) it follows, that there exists a function $\delta_2 = \delta_2(\varepsilon) > 0$ such that $\delta_2 = \min(\delta_1, \rho_0^1)$. Then

$$h_0^1(t_0, x_0) < \delta_2 \text{ implies } V^1(t_0, x_0) \le a^1(h_0^1(t_0, x_0)) < \delta^1. \tag{10}$$

Choose $\delta = \min(\delta_1, \delta_2)$, $a^1(\delta) < \delta^1$ and assume that $h_0^1(t_0, x_0) < \delta$. In consequence of (8) and (9) we have

$$h^1(t_0, x_0) \le \Phi^1(h_0^1(t_0, x_0)) \le \Phi^1(\delta) \le \Phi^1(\delta_1) < \varepsilon. \tag{11}$$

We claim that

$$h^1(t,x(t)) < \varepsilon,\ t \geq t_0,\ \text{whenever } h_0^1(t_0,x_0) < \delta, \tag{12}$$

where $x(t) = x(t;t_0,x_0)$. Suppose that this is not true. Then there exists a solution $x(t) = x(t;t_0,x_0)$ of the system (1) with $h_0^1(t_0,x_0) < \delta$ and a $t^* > t_0$ such that

$$h^1(t^*,x(t^*)) = \varepsilon < \rho^1 \text{ and } h^1(t,x(t)) < \varepsilon \text{ for } t \geq t_0. \tag{13}$$

Setting $m^1(t) = V^1(t,x(t))$ for $t \in [t_0,t^*]$ and using the conditions 2) and 3) of the theorem we obtain

$$D^+m^1(t) \leq f(t,m^1(t)),\ t \in [t_0,t^*]\,.$$

Thus we get by the comparison theorem [4], the estimate

$$m^1(t) \leq \bar{u}(t;t_0,m^1(t_0)),\ t \in [t_0,t^*], \tag{14}$$

where $\bar{u}(t;t_0,u_0)$ is the maximal solution of (3). Then, using (5), (6) and the choice of δ, we have

$$b^1(\varepsilon) \leq b^1(h^1(t^*,x(t^*))) \leq V^1(t^*,x(t^*)) \leq \bar{u}(t^*,t_0,u_0) < b^1(\varepsilon),$$

which is a contradiction. Thus (12) is true, proving the (h_0^1,h^1)-uniform stability of the system (1).

Let the trivial solution of (4) be uniformly asymptotically stable, which implies that the system (1) is (h_0^2,h^2)-uniformly stable. To prove the (h_0^2,h^2)-uniform attractivity, let $0 < \varepsilon \leq \lambda^*$ and $t_0^* \in R^+$ be given and designate $\tilde{\delta}_0 = \tilde{\delta}_0(\lambda^*)$. Since the trivial solution of (4) is attractive, given $b^2(\varepsilon^1) > 0$ and $t_0^* \in R^+$, there exist $\delta_{10}^* = \delta_{10}^* > 0$ and $T = T(\varepsilon^1) > 0$ such that

$$v_0^* < \delta_{10}^* \text{ implies } v(t;t_0^*,v_0^*) < b^2(\varepsilon^1),\ t \geq t_0^* + T. \tag{15}$$

Choosing $v_0^* = V^2(t_0^*,x_0^*)$ as before, where $x_0^* = x(t_0^*;t_0^*,x_0^*)$ we find $\delta_0^* > 0$ such that $a^2(\delta_0^*) < \delta_{10}^*$. Let $\delta_0 = \min(\delta_0^*,\tilde{\delta}_0)$ and $h_0^2(t_0^*,x_0^*) < \delta_0$. This implies that $h^2(t,x(t)) \leq \Phi^2(h_0^2(t,x)) < \rho^1,\ t \geq t_0^*$ for all solutions $x(t) = x(t;t_0^*,x_0^*)$ of the system (1). Hence, setting $m^2(t) = V^2(t,x(t))$ we obtain

$$D^+m^2(t) \leq g(t,m^2(t)),\ t \geq t_0^*\,.$$

Thus we get, by the comparison theorem [4], the inequality

$$m^2(t) \le \bar v(t; t_0^*, m^2(t_0^*),\ t \ge t_0^*. \tag{16}$$

Suppose now that there exists a sequence $\{t^{(n)}\},\ t^{(n)} \ge t_0^* + T,\ t^n \to \infty$, as $n \to \infty$ such that the estimate

$$\varepsilon^1 \le h^2(t^{(n)}, x(t^{(n)})) < \rho^1 \tag{17}$$

hold, where $x(t) = x(t; t_0^*, x_0^*)$ is a solution of the system (1) with $h_0(t_0^*, x_0^*) < \delta_0$. Thus we receive

$$b^2(\varepsilon^1) \le b^2(h^2(t^{(n)}, x(t^{(n)}))) \le V^2(t^{(n)}, x(t^{(n)})) \le \bar v(t^{(n)}, t_0^*, v_0^*) < b^2(\varepsilon^1),$$

which contradicts with (15)–(17). Hence it follows that the system (1) is (h_0^2, h^2)-uniformly asymptotically stable and the proof is complete. □

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КРИТЕРИЙ ЗА ПОЛИУСТОЙЧИВОСТ НА ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ПО ОТНОШЕНИЕ НА ДВЕ МЕРКИ

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Резюме. В тази работа ще представим идеята за полиустойчивост на системи от диференциални уравнения по отношение на две мерки. Получени са достатъчни условия за този вид устойчивост с помощта на метода на векторните функции на Ляпунов.

Erratum

The words "integral manifold" sould be regarded as "invariant set" in the following papers of I. K. Russinov, published as follows:

1. Comptes Rendus de l'Academie Bulgare des Sciences, 46(2), 1993, 21–24;
2. Mathematica Balkanica, 7(3–4), 1993, 323–331;
3. Scient Works at Plovdiv University – Math.: 26(3), 1988, 75–86; 27(3), 1989, 159–171 and 173–188; 28(3), 1990, 123–134 and 135–145; 31(3), 1994, 89–95; 32(3), 1995, 9–16 and 17–21.

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