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ON THE SEMIPRIMITIVITY OF CROSSED PRODUCTS OF GROUPS AND SIMPLE RINGS *

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Abstract. Let $K * G$ be a crossed product of the group G over the central simple F-algebra K of characteristic $p \geq 0$. Suppose that the kernel G_{ker} of $K * G$ has no p-elements when $p > 0$ and let P be the minimal subfield of F which contains the factor set of the natural twisted group subring $F * G_{ker}$. If either F is not an algebraic extension of P, or $|F| > |H|$ for all finitely generated subgroups H of G_{ker}, then we prove that $K * G$ is a semiprimitive ring.

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Let $K * G = K_\rho^\sigma G$ be a *crossed product* [5, 13] of the multiplicative group G over the associative ring K with respect to the *factor set*

$$\rho = \{\rho(g,h) \in K^* \mid g,h \in G\}$$

and the *mapping* $\sigma : G \to AutK$, where K^* is the *multiplicative group of* K and $AutK$ is the *automorphism group of* K. Then $K * G$ is simultaneously an associative ring and a free right K-module with a basis

$$\overline{G} = \{\bar{g} \in K * G \mid g \in G\}.$$

The elements of $\overline{G}$ satisfy the conditions

$$\bar{g}\bar{h} = \overline{gh}\rho(g,h), \qquad \alpha\bar{g} = \bar{g}\alpha^{g\sigma} \quad (g,h \in G,\ \alpha \in K),$$

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where $\alpha^{g\sigma}$ is the image of $\alpha \in K$ under the action of the automorphism $g\sigma \in Aut\, K$. Thus each element $a \in K * G$ is uniquely a finite sum

$$a = \sum_{g \in G} \bar{g}\alpha_g \qquad (\alpha_g \in K)$$

and $Supp\, a = \{g \in G \mid \alpha_g \neq 0\}$ is the *support of* a. Since $\bar{f}(\bar{g}\bar{h}) = (\bar{f}\bar{g})\bar{h}$ and $(\alpha\bar{g})\bar{h} = \alpha(\bar{g}\bar{h})$, we have

$$\begin{aligned} \rho(f, gh)\rho(g, h) &= \rho(fg, h)\rho(f, g)^{h\sigma}, \\ \alpha^{g\sigma . h\sigma} &= \rho(g, h)^{-1}\alpha^{(gh)\sigma}\rho(g, h) \end{aligned} \tag{1}$$

for all $f, g, h \in G$ and $\alpha \in K$.

Certain special cases of crossed products have their own names [13]. If $\sigma = 1$, that is $g\sigma$ is the identity automorphism of K for all $g \in G$, then $K * G = K_\rho G$ is a *twisted group ring.* If $\rho = 1$, i.e. $\rho(g, h) = 1$ for all $g, h \in G$, then $K * G = K^\sigma G$ is a *skew group ring.* Finally, if there is no action and twisting, i.e. $\sigma = 1$ and $\rho = 1$, then $K * G = KG$ is the *ordinary group ring.* The *conditions for associativity* (1) show that for every twisted group ring $K_\rho G$ the factor set ρ is *central*, i.e. $\rho(g, h)$ is a central element of K for all $g, h \in G$.

Let $Inn\, K$ be the group of the *inner automorphisms of* K. If $K * G$ is any crossed product, then the *kernel*

$$G_{ker} = \{g \in G \mid g\sigma \in Inn\, K\}$$

of $K * G$ is a normal subgroup of G [5]. If H is any subgroup of G, then it is clear that

$$K * H = \{a \in K * G \mid Supp\, a \subseteq H\}$$

is a subring of $K * G$ and $H_{ker} = H \cap G_{ker}$. By analogy, if $S \leq K$, then we put

$$S * G = \left\{ \sum_{g \in G} \bar{g}\alpha_g \mid \alpha_g \in S \right\}.$$

Unlike the group rings, the crossed products do not have natural bases. Indeed, if $\theta : G \to K^*$ is an arbitrary mapping, then $\widetilde{G} = \{\tilde{g} = \bar{g}\theta(g) \mid g \in G\}$ yields an alternate K-basis for $K * G$ which still exhibits the basic crossed product structure. The basis $\widetilde{G}$ is said to be *diagonally equivalent of* $\overline{G}$ [13]. Every crossed product $K * G$ with a basis $\overline{G}$ has a diagonally equivalent K-basis $\widetilde{G}$, such that $\tilde{1}$ $(1 \in G)$ is the identity element of $K * G$ and the subring $K * G_{ker}$ with basis $\widetilde{G}_{ker}$ is a twisted group ring [7, 9]. Then we call that the basis $\widetilde{G}$ is *normalized*. Therefore we can and we shall assume that the basis $\overline{G}$ of $K * G$ is normalized.

Let K be a central simple F-algebra, that is K is a simple ring with center F. Some results of [8, 9] assert that if G_{ker} has no p-elements when $char\, F = p > 0$ and G_{ker} has a finite subnormal series, such that all factors of this series are either locally finite, or locally solvable, then $K * G$ is a semiprimitive ring, that is $\mathcal{J}(K * G) = O$, where $\mathcal{J}(K * G)$ if the Jacobson's radical of the ring $K * G$. Here we show that for some fields F the second condition for G_{ker} is not necessary.

Namely, if $F(G_{ker})$ is the minimal subfield of F which contains the factor set of the normalized twisted group ring $K * G_{ker}$, then we have the following

Theorem 1. *Let $K * G$ be any crossed product of a multiplicative group G over a central simple F-algebra K of characteristic $p \geq 0$. If G_{ker} has no p-elements when $p > 0$ and either F is not an algebraic extension of $F(G_{ker})$, or $|F| > |H|$ for all finitely generated subgroups H of G_{ker}, then $\mathcal{J}(K*G) = O$.*

The formulated theorem is a crossed product analog of well known results of Amitsur [2] and Passman [11] for semiprimitive group ring KG, where K is a field. In effect, if $K * G = KG$ is a group ring, then $G_{ker} = G$, $\rho = 1$ and $F(G_{ker})$ is the prime subfield of K. Thus the results of Amitsur and Passman follow from Theorem 1.

The proof of Theorem 1 uses the methods of Amitsur and Passman and a recent result of Dimitrova [6, 7]. So we commence with following

Lemma 2. *Let $K * G$ be a crossed product of a group G over a central simple F-algebra K of characteristic $p \geq 0$.*

(i). If $\mathcal{J}(K * G) \neq O$, then $\mathcal{J}(F * G_{ker}) \neq O$;

(ii). If G_{ker} has no p-elements when $p > 0$, then $K * G$ has no nil ideals.

Proof. Since K is a prime ring without nil ideals and $G_{ker} = G_{inn}$ [5, 13], in view of [7, Lemma 2.2 (ii)] we obtain that

$$I = \mathcal{J}(K * G) \cap F * G_{ker}$$

is a nonzero ideal of $F * G_{ker}$. Thus it suffices to show that I is a quasiregular ideal of $F * G_{ker}$.

Indeed, by [12, Lemma 7.1.5] we have

$$I = \mathcal{J}(K * G) \cap F * G_{ker} \subseteq \mathcal{J}(K * G) \cap K * G_{ker} \subseteq \mathcal{J}(K * G_{ker}).$$

Therefore every nonzero element $a \in I$ has a quasi-inverse element $b \in K * G_{ker}$, such that $a + b + ab = 0$.

Since K is a linear space over F, we write $K = F \oplus V$ as direct sum of F-modules, where V is a complementary F-subspace of K. Now write $b = b_0 + b_1$ with $b_0 \in F * G_{ker}$ and $b_1 \in V * G_{ker}$. Then

$$0 = a + b + ab = (a + b_0 + ab_0) + (b_1 + ab_1),$$

where $a+b_0+ab_0 \in F*G_{ker}$ and $b_1+ab_1 \in V*G_{ker}$. Thus we conclude that both of these summands must be zero. In particular, the element $b_0 \in F * G_{ker}$ is also quasi-inverse for $a \in I$. Hence I is a nonzero quasiregular ideal of $F*G_{ker}$ and $\mathcal{J}(F * G_{ker}) \neq O$. Since the part (ii) follows from [6, Theorem A], the lemma is proved. □

Let A be an F-algebra over the field F. Then the algebra A is said to be *separable* [12, p.284] if for all fields $L \geq F$ the algebra $A^L = L \otimes_F A$ is semiprimitive, i.e. $\mathcal{J}(A^L) = O$. Recall that A is a *nilpotent free* F-algebra if for all fields $L \geq F$ the algebra A^L has no nilpotent ideals [12, p.285]. It is known [12, Theorem 7.3.6] that if A is a nilpotent free F-algebra and $\mathcal{J}(A) = O$, then A is separable. Thus by preceding lemma we obtain the following

Corollary 3. *Let $F_\rho G$ be a twisted group ring of a group G over a field F of characteristic $p \geq 0$. If G has no p-elements when $p > 0$ and $\mathcal{J}(F_\rho G) = O$, then $F_\rho G$ is a separable F-algebra.*

Proof. Let L be any field extension of F. Then

$$(F_\rho G)^L = L \otimes_F (F_\rho G) = L_\rho G$$

is a twisted group ring of G over L with factor set ρ. Since $char\, L = char\, F$, by Lemma 2 (ii) we conclude that $L_\rho G$ has no nilpotent ideals. Therefore $F_\rho G$ is a nilpotent free F-algebra and the statement follows from [12, Theorem 7.3.6]. □

Proof of Theorem 1. Assume by way of contradiction that $\mathcal{J}(K * G) \neq O$. Then by Lemma 2 (i) we see that $\mathcal{J}(F * G_{ker}) \neq O$. Since $F * G_{ker} = F_\rho G_{ker}$ is a twisted group ring [7, Lemma 2.1 (ii)] and the factor set ρ is central, it follows that $F * G_{ker}$ is an F-algebra.

First, assume that $|F| > |H|$ for all finitely generated subgroups H of G_{ker}. If $0 \neq a \in \mathcal{J}(F * G_{ker})$, then the supporting subgroup $H = \langle Supp\, a \rangle$ is finitely generated and therefore $|F| > |H|$. Since $|H| = |\overline{H}|$ we conclude that the K-basis of $F * H$ satisfies the condition $|F| > |\overline{H}|$. Then applying a well known theorem of Amitsur [1] or [12, Lemma 7.1.2], we receive that $\mathcal{J}(F * H)$ is a nil ideal. But this contradicts Lemma 2 (ii), since by [9, Lemma 2.2 (ii)] we have

$$a \in \mathcal{J}(F * G_{ker}) \cap F * H \subseteq \mathcal{J}(F * H).$$

In the second place, if F is not an algebraic extension of the subfield $P = F(G_{ker})$, then there exists an element $\alpha \in F$ such that $L = P(\alpha)$ is a purely transcendental extension of the field P. Now we put

$$L_\rho G_{ker} = L \otimes_P (P_\rho G_{ker})$$

and by a theorem of Amitsur [1], or [12, Theorem 7.3.4] we have

$$\mathcal{J}(L_\rho G_{ker}) = L \otimes_P I,$$

where $I = \mathcal{J}(L_\rho G_{ker}) \cap (P_\rho G_{ker})$ is a nil ideal of $P_\rho G_{ker}$. Hence by Lemma 2 (ii) we obtain that $I = O$ and therefore $\mathcal{J}(L_\rho G_{ker}) = O$. Thus by Corollary 3 we see that $L_\rho G_{ker}$ is a separable L-algebra. Finally, since $F_\rho G_{ker} = F \otimes_L (L_\rho G_{ker})$, where F is semiprimitive and $L_\rho G_{ker}$ is separable, by a theorem of Bourbaki [4, p.223] or [12, Theorem 7.3.9] we obtain that $\mathcal{J}(F_\rho G_{ker}) = O$, which again is a contradiction. Therefore $\mathcal{J}(K * G) = O$ and the theorem is proved. □

As an immediate consequence of Theorem 1 we obtain

Corollary 4. *Let $K * G$ be a crossed product over a central simple F-algebra K of characteristic $p \geq 0$. If G_{ker} has no p-elements when $p > 0$ and either F is nondenumerable, or F is infinite and G_{ker} is locally finite, then $\mathcal{J}(K * G) = O$.*

Indeed, let H be a finitely generated subgroup of G_{ker}. If G_{ker} is a locally finite group, then $|H| < \infty$ and $|F| > |H|$. If G_{ker} is not locally finite and H is infinite, then H is a denumerable group and $|F| > |H|$, because F is a nondenumerable field. Thus the statement follows from Theorem 1.

For skew group rings we have the following

Corollary 5. *Let $F^{\sigma}G$ be a skew group ring of the group G over the field F of characteristic $p \geq 0$. If G_{ker} has no p-elements when $p > 0$ and either F is nondenumerable, or F is not algebraic over the prime subfield of F, then $F^{\sigma}G$ is a semiprimitive ring.*

Really, $F^{\sigma}G_{ker}$ is a group ring and $F(G_{ker}) = F$. Hence, as above the assertion follows again from Theorem 1.

It is clear that if $\sigma = 1$, then Corollary 5 gives well known results of [2], [3] and [12] (see also [13, Lemma 7.1.6, Theorem 7.3.13 and Theorem 7.3.14]).

We close this paper with an application of [9, Theorem 3.6].

Let

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_{\alpha} \supseteq \cdots$$

be the commutator series of the group G, that is $G_1 = G' = [G, G]$ is the commutator subgroup of G. $G_{\alpha+1} = [G_{\alpha}, G_{\alpha}]$ for every ordinal number α and $G_{\alpha} = \bigcap_{\beta<\alpha} G_{\beta}$ for limite ordinal numbers α. Then there exists an ordinal number τ such that $G_{\tau} = G_{\alpha}$ for all $\alpha \geq \tau$ [10, p.89]. It is clear that $H = G_{\tau}$ is a normal subgroup of G and $H = H' = [H, H]$. This normal subgroup $H = H(G)$ we shall call *hypercommutant of* G. Thus we have the following

Theorem 6. *Let $K * G$ be a crossed product over a central simple F-algebra K with $\operatorname{char} F = 0$. If $H = H(G_{ker})$ is the hypercommutant of G_{ker} and F is not an algebraic extension of $F(H)$, then $\mathcal{J}(K * G) = O$.*

Proof. Recall that $F(H)$ is the minimal subfield of F which contains the factor set of the normalized twisted group ring $F * H \subseteq K * G$.

Suppose that $\mathcal{J}(K*G) \neq O$. Then Lemma 2 (i) yields $\mathcal{J}(F*G_{ker}) \neq O$. By [9, Theorem 3.6] we receive $\mathcal{J}(F * G_{ker}) \subseteq \mathcal{J}(F * H)F * G_{ker}$. But Theorem 1 shows that $\mathcal{J}(F * H) = O$. Hence $\mathcal{J}(F * G_{ker}) = O$ and we obtain a contradiction. The theorem is proved. □

It is well known that if A is a finitely generated commutative algebra over a field F, then $\mathcal{J}(A)$ is a nil ideal. The question here is whether the commutativity condition can be eliminated [12, p.291]. An affirmative answer proves that then $\mathcal{J}(F_\rho G)$ is a nil ideal in general. Thus if G has no p-elements when $char\, F = p > 0$, it would prove that always $\mathcal{J}(F_\rho G) = O$. But Theorem 1 shows that in this case $F_\rho G$ is always a subring of a semiprimitive twisted group ring. Indeed, if L is a purely transcendental or nondenumerable extension of F, then $F_\rho G$ is a subring of the semiprimitive twisted group ring $L_\rho G = L \otimes_F (F_\rho G)$.

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ВЪРХУ ПОЛУПРИМИТИВНОСТТА НА КРЪСТОСАНИ ПРОИЗВЕДЕНИЯ НА ГРУПИ И ПРОСТИ ПРЪСТЕНИ

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Резюме. Нека $K * G$ е кръстосано произведение на групата G над централно простата F-алгебра K с характеристика p. Да предположим, че ядрото G_{ker} на $K * G$ не съдържа p-елементи, когато $p > 0$ и нека P е минималното подполе на F, което съдържа системата от фактори на кръстосания групов пръстен $F * G_{ker} \subseteq K * G$. Ако F не е алгебрично разширение на P или $|F| > |H|$ за всяка крайно породена подгрупа H на групата G_{ker}, тогава доказваме, че $K * G$ е полупримитивен пръстен, т.е. неговият радикал на Джекъбсън е нулев.