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# PLANE OF HUGHES

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**Abstract.** In 1957 year Hughes constructed a significant example of finite non-Desarguesian projective plane. In this paper there is given a new representation of the Hughes plane and some its properties are considerate.

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**Key words:** projective plane, point, straight line, collineation, near-field, subplane, incidence matrix

An axiomathic definition for projective plane is given by three primary notions and three axioms. The primary notions are the following:

**E1** Set $P$ of points;

**E2** Set $S$ of straight lines;

**E3** Binomial relation of incidence betwin $P$ and $S$.

The axioms are the following:

**P1** Through two different points goes just one straight line.

**P2** Two different straight lines intersect in just one point.

**P3** There exsist at least one set of four points, no one but three of them do not lie on a straight line.

We call a projective plane Desarguesian if for it is true the Desargue's theorem for perspective triangles. We will note, that from the above three axioms does not follow the Desargue's theorem, which means that there exist non-Desarguesian planes. In every Desarguesian plane one can introduce a coordinate system. The coordinates of points are elements of an associative body (in particular field) but the equations of the straight lines define by standart way, which is well-known by the course of analytic and projective geometry.

By the same way one can introduce a coordinate method in non-Desarguesian planes, but here the coordinates of points belong to an algebraic system contains 0 and 1 in which act ternary operation. Such kind of system is not associative body and it is called ternary ring. The ternary operation satisfies requirements making sure the validity of axioms P1.P2.P3 (see [4], p. 382, Theorem 20.3.1). The construction of a ternary ring of the requirements of the cited theorem appears to be a very hard problem and such kind constructions are made only in some particular cases. These constructions are obtained by Veblen O., Wederburn J. H. M. [9], Hall M. Jr. [3], Moufang R. [6]. These descriptions are given also in the book of Marshall Hall, Jr. [4].

One close to field algebraic system by means of which one can obtain non-Desarguesians projective planes is the notion near - field. A non-empty set $K$ of elements is called near-field if in it are defined binary algebraic operations addition and multiplication, such that the following axioms hold.

**A1** With regard to addition $K$ is an abelian group.

**A2** Non-zero elements of $K$ form non-abelian group in respect to mutiplication.

**A3** The right distributive holds i.e.

$$(a+b)c = ac + bc$$

for every $a, b, c \in K$ The left distributive law in the general case does not hold and one can add a suplement axiom. It is

**A4** For $k \neq 1, k \in K, b \in K$ the equation $xk = x+b$ has an unique solution in $K$.

If near-field $K$ is finite, then A4 follows from the rest three axioms. (see [4], p.422, Corollary 20.8.1).

The big interest present the finite projective planes. A projective plane is called finite if it consist of finite numbers of points and consequently it consist of finite number of straight lines. For these planes the following properties hold:

1. The plane contains just $n^2 + n + 1$ straight lines.
2. The numbers of points is just $n^2 + n + 1$.
3. Every straight line consist of $n + 1$ points.
4. Through every point go $n + 1$ straight lines.
5. Throught two different points goes just one straight line.
6. Two different straight lines intersect in just one point, where $n \geq 2$ is a natural number. This number is called order of the plane.

In [4] there is proved that the conditions 1, 3, 5 are equivalent to duality conditions 2, 4, 6 and every one of these triple conditions determines a projective plane by order $n$ satisfying the other triple too.

An elementary example for finite projective plane is the plane which consists of 7 points and 7 straight lines. It is by order 2. If we denote the point of this plane by letters A, B, C, D, E, F, G, then the straight lines the matrix columns

| | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G |
| B | C | D | E | F | G | A |
| D | E | F | G | A | B | C . |

A point lies on a given straight line if the point lies in the column defining the straight line. By this way determine the relation of incidence in this plane. It is easy to see that the axioms of a projective plane are satisfied, the conditions 1, 3, 5 are fulfiled and the conditions 2, 4, 6 are fulfiled too, such that this is a finite projective plane. Formally this plane can be presented with fig. 1.

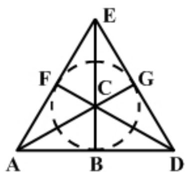


Figure 1

The straight line BFG is imagined by a circle and the other straight lines are given through straight lines. The above figure is called configuration of Fanao. Plane in which the diagonal points of every full quadrangle lie on a straight line is called Fanao's plane (see [2]).

A projective plane can be determined by means of incidence matrix. This is a quadratic matrix by order $n^2 + n + 1$ containing of zeros and unities. The matrix rows imitate the points of the plane and the columns - the straight lines. A point and straight line are incidence if the corresponding row and column intersect in 1 and non-incidence if they intersect in 0. The incidence matrix of the plane by order 2 is the following:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| B | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| C | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| D | 1 | 0 | 1 | 1 | 0 | 0 | 0 |
| E | 0 | 1 | 0 | 1 | 1 | 0 | 0 |
| F | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| G | 0 | 0 | 0 | 1 | 0 | 1 | 1 |

In every row and every column there are 3 unites and 4 zeros.

It is established that from every from the following orders 2, 3, 4, 5, 7, 8 there exisits one projective plane and it is Desarguesian. In 1900 year Tarry G. [8] proved that the plane by order 6 does not exist. From the order 9 there is one Desarguesian plane and 3 non-Desarguesian planes. The big problem appears to be the existence of a plane by order 10, known as the problem for the ten. This problem is very hard and as it is known of the author it is stile unsolved. A significant result in the theory of finite projective planes is the theorem of Bruck R. H., Ruser H. J. [1], in which is argued that if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ and $n \neq a^2 + b^2$, $a, b \in N$, then projective plane by order $n$ does not exist. From it follows that planes by order 6, 14, 21, 22, 30, 33, ... do not exist, but it cannont solve the problem for the ten, since $10 = 3^2 + 1^2$.

A significant example of non-Desarguesian planes had been constructed by Hughes [5]. Let $q = p^r$ be degree of odd prime number. Then there exsist near - field $K$ by order $q^2$, the center $Z$ of which is Galoi's field $GF(q)$. The Hughes planes have order $q^2$. Let fix an collineation $\alpha$ of the Desarguesian plane by order $q$ which coordinates from field $Z = GF(q)$, such that the order of $\alpha$ to be $q^2 + q + 1$. Such kind of collineation exists by a theorem of Singer J. [7]. A Hughes plane obtain as the collineation $\alpha$ spreds on the points with coordinates from near-field $K$. Let consider the straight lines with equations

$$x + ty + z = 0 \tag{1}$$

in homogeneous coordinates, where $t = 1$ or $t \notin Z$. These straight lines are called basic straight lines. We apply the degrees of collineation $\alpha$ over the basic straight lines. By this way we obtain straight lines with the following equations

$$a_1x + b_1y + c_1z = t(a_2x + b_2y + c_2z), \tag{2}$$

where $a_1, b_1, c_1, a_2, b_2, c_2 \in Z$ and $t = 1$ or $t \notin Z$. This set of straight lines contains the basic straight lines, since the identity is a degree of $\alpha$. The number of straight lines from kind (2) is $(q^2+q+1)(q^2-q+1) = q^4+q^2+1 = n^2+n+1$, where $n = q^2$. The relation of incidence determines with the condition the coordinates of the point satisfy the equality the straight line. The point have homogeneous coordinates within non-zero right factor of K. At this chosen sets of point and straight lines and the relation of incidence it is proved in [4] that it is a projective plane $\pi$ which is called Hughes plane. The points with coordinates from $Z$ and straight lines (2) for $t = 1$ make up Desarguesian subplane $\pi_1$ of the plane $\pi$. We shall explain that the coordinates of the points from $\pi$ and the equations of the straight lines refer to coordinate system with a coordinate quadrangle XYOI lieing in subplane $\pi_1$. The vertices of this quadrangle have homogenuous coordinates $X(1,0,0), Y(0,1,0), O(0,0,1), I(1,1,1)$. For the points outside from the straight line $XY$ we introduce non-homogenuous coordinates, but for the straight lines different from $XY$-non-homogenuous equalities. This can be made when we set in (2) $z = 1$.

We shall show that the equations of the straight lines from the Hughes plane can be presented in another more couviniance form. For this purpose we shall use non-homogenuous coordinates. We shall prove the following main result.

**Theorem 1.** *Let $XYOI$ be a coordinate quadrangle in the Hughes plane $\pi$, lieing in the subplane $\pi_1$ and $l$ be an arbitrary straight line from $\pi$, different from $XY$. Then for non-homogenuous equation of $l$ just one of the cases holds.*

*a) if $l$ intersects $XY$ in a point not belonging to $\pi_1$, then $l$ has non-homogenuous equation by the kind*

$$y - y_0 = k(x - x_0), \tag{3}$$

*where $k \in K \backslash Z, x_0, y_0 \in Z$.*

*b) if $l$ intersects $XY$ in a point belonging to $\pi_1$ and different from $Y$, then $l$ has non-homogenuous equation by the kind*

$$y = kx + b, \tag{4}$$

*where $k \in Z, b \in K$.*

*c) if $l$ goes through $Y$, then its non-homogenuous equation is*

$$x = x_1, x_1 \in K. \tag{5}$$

**Proof.** The straight line $l$ will have non-homogenuous equation by the kind (2) as we put $z = 1$. If $t = 1$, then $l \in \pi_1$ and since $\pi_1$ is a Desarguesian, then the equation of $l$ turus to (4) with $b \in Z$ or turus to (5) with $x_1 \in Z$. Let now $t \neq 1$. Then $t \in K \backslash Z$. The straight lines with the equations

$$l_1 : a_1 x + b_1 y + c_1 = 0$$

$$l_2 : a_2 x + b_2 y + c_2 = 0$$

lie in $\pi_1$ and imagines of the different straight lines $x + z = 0$ and $y = 0$ through collineation of $\pi_1$. Consequently $l_1 \neq l_2$. We shall consider the following cases:

$a'$) Let $l_1$ and $l_2$ intersect in a point not lieing on $XY$. Then $a_1b_2-a_2b_1 \neq 0$ and the system (6) will have an unique solution $(x_0, y_0); x_0, y_0 \in Z$. This solution gives non-homogenuous variant of (2) can be presented as

$$a_1(x - x_0) + b_1(y - y_0) = t(a_2(x - x_0) + b_2(y - y_0)), \tag{7}$$

which is the equation of the straight line $l$. Since $a_1b_2 - a_2b_1 \neq 0$ then at least $b_1$ or $b_2$ is different from zero. Without limitation of generating we can accept $b_2 \neq 0$. We shall note in details that at the reveal of clamps with factor from $Z$ one can use both distributive laws. Then (7) can be reprezented in the kind

$$\delta(x - x_0) = (b_2t - b_1)(a_2(x - x_0) + b_2(y - y_0)), \tag{8}$$

where $\delta = a_1b_2 - a_2b_1 \neq 0$. Since $b_1, b_2 \in Z, b_2 \neq 0, t \notin Z$ then it follows $b_2t - b_1 \notin Z$ and it seems that $b_2t - b_1 \neq 0$. Putting $\lambda = (b_2t - b_1)^{-1}$ we reduce (8) to the type

$$y - y_0 = \frac{\lambda\delta - a_2}{b_2}(x - x_0). \tag{9}$$

As we set $k = \dfrac{\lambda\delta - a_2}{b_2}$ the equation (9) takes the kind (3). Consequently the straight line $l$ has the equation from the type (3) with $k \in K\backslash Z$ and $x_0, y_0 \in Z$

$b'$) Let $l_1$ and $l_2$ intersect on $XY$ in point different from $Y$ with direction $k$. Then we will have $a_1 + b_1k = 0, a_2 + b_2k = 0$, at that $b_1 \neq 0$ or $b_2 \neq 0$. Without limitattion of generating we can accept $b_2 \neq 0$. Then (2) will take the type

$$(b_2t - b_1)(b_2(-kx + y) + c_2) = b_2c_1 - b_1c_2. \tag{10}$$

We put now $\lambda = (b_2t - b_1)^{-1}$ as in the case $a'$) and then (10) will take the type

$$y = kx + b, \tag{11}$$

where $b = \dfrac{\lambda\delta_1 - c_2}{b_2}$ and $\delta_1 = b_2c_1 - b_1c_2$. By the way we obtain (4).

$c'$) Let $l_1$ and $l_2$ intersect in point $Y$. Then it follows $b_1 = b_2 = 0$. The equation (2) gets $a_1x + c_1 = t(a_2x + c_2)$. Moreover we can accept $a_2 \neq 0$. The last equation turus to the kind

$$x = x_1 \tag{12}$$

where $x_1 = \dfrac{\mu\delta_2 - c_2}{a_2}$, $\mu = (a_2t - a_1)^{-1}$, $\delta_2 = a_2c_1 - a_1c_2$ as we obtain (5).

From this theorem follows that the Hughes plane does not depend on the chosen collineation $\alpha$. One can choose any group of collineations of $\pi_1$ having order $q^2+q+1$ and acting transitive on $\pi_1$.So one can obtain the same Hughes plane. From the proof of this theorem one can see that the straight lines from the Hughes plane can be distributed in three types which we shall call respectively type a), type b) and type c), where their equations give with form (3), (4) and (5), respectively. Now we shall give a criterion for coincidence of two straight lines from the plane $\pi$.

**Theorem 2.** *Two straight lines coincide if and if when their equations coincide i.e. every straight line has a unique equation from any type.*

**Proof.** From the cases considered in the proof of Theorem 1one can see that two straight lines from different types cannot coincide. If $l$ is a straight line from type a) with equation (3) it intersects $XY$ in a point with coordinate $k \notin Z$. But since the crossing point of two straight lines is unique then it follows that $k$ is uniquely defined from $l$. From the fact $k \notin Z$ follows that $l \notin \pi_1$. Then $l$ cannot walk throught different points belonging to $\pi_1$ from where it follows that $x_0$ and $y_0$ in (3) are uniquely defined too. Let now the straight line $l$ is from the type b) with equation (4). Then it is different from the straight line $OY$ since $l$ does not walk through the point $Y$. But the equation of $OY$ is $x = 0$ (from type $c$) with $x_1 = 0$) and consequently $l$ intersects $OY$ in the point $(0, b)$. From this follows the uniquely of $b$ and the uniquely of $k$ follows as it proved above. At the end if $l$ is from type c) with equation (5) then every point from $l$ has abscissa $x_1$. This uniquely determines $x_1$ from the straight line $l$.

**Theorem 3.** *Every straight line from the Hughes plane not belonging to the subplane $\pi_1$ passes through just one point from $\pi_1$. Through every point not belonging to $\pi_1$ walks only one straight line from $\pi_1$.*

**Proof.** Every straight line not belonging to $\pi_1$ can pass at most through one point from $\pi_1$. If the straight line is from type a) with equation (3) then it walks through the point $(x_0, y_0)$ from $\pi_1$. If the straight line is from type b) then it passes through the point (k) which is from $\pi_1$ because $k \in Z$. If straight line is from type c), then it walks through the point $Y \in \pi_1$. The second assertion follows from the duality principle.

We shall note that this theorem is true for every finite projective plane by order $n^2$ having subplane by order $n$.

As we have already seen every finite near-field by order $q^2$ with center the Galoi's field $GF(q)$ for odd $q$ defines Hughes plane and vice versa. Because of this it is necessary to make a classification of all such near-fields. Such kind of classification is made by Zassenhaus $H$. [9] without limitation of their order.

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### РАВНИНА НА ХЮГЕС

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**Резюме**. В 1957 г. Хюгес построява един забележителен пример на крайна недезаргова проективна равнина. В тази статия се дава ново представяне на равнината на Хюгес и се разглеждат някои нейни свойства.