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# BOUNDARY CROSSING PROBABILITIES FOR A BROWNIAN MOTION AND PARALLEL EXPONENTIAL BOUNDARIES

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**Abstract.** We consider the probability that a Brownian motion hits a two-sided exponential boundary by a certain moment. We find formulae for the crossing probabilities when the upper and lower parts of the boundary are parallel curves.

**Mathematics Subject Classification 2000:** Primary 60J65, Secondary 60G40
**Key words:** Brownian motion, hitting time, Laplace transformation

## 1. Introduction

Let $(B_s)_{s \geq t}$ be a Brownian motion with unit volatility and no drift, $B_t = x$. Let $T$ be an arbitrary fixed time-horizon, $T \geq t$, and $g(s) < f(s)$ be two smooth real functions, defined at least for $s \in [t;T]$, such that $g(t) \leq x \leq f(t)$. Consider the hitting time $\tau = \inf\{s \in [t;T] \mid B_s = f(s) \text{ or } B_s = g(s)\}$, where $\inf \varnothing = T$, and the random events $\mathcal{F} = \{B_\tau = f(\tau)\}$, $\mathcal{G} = \{B_\tau = g(\tau)\}$, $\mathcal{H} = \{g(s) < B_s < f(s) \text{ for all } s \in [t;T]\}$. We are interested in the probabilities $P_{t,x}(\mathcal{F})$, $P_{t,x}(\mathcal{G})$, and $P_{t,x}(\mathcal{H})$.

In 1960 T. W. Anderson [1] discovered the crossing probabilities for rectilinear boundaries with no horizon — two straight lines that are parallel or cross on the left of the starting point. In 1964 A. V. Skorokhod [2] found the probability of going out of the domain through a little "door" at the horizon; his formula holds for rectilinear boundaries. In 1967 L. A. Shepp [3] found a formula for the expectation of the first hitting time for a two-sided symmetric square-root boundary with no horizon. In 1971 A. A. Novikov [4] solved the same problem for a one-sided square-root boundary. In 1981 he published a formula [5] for the probability of going out of the domain through the horizon; it holds for curvilinear boundaries that are close to each other. A little later in the same year A. V. Mel'nikov and D. I. Hadžiev [6] published a solution to a similar problem for Gaussian martingales. In 1999 A. Novikov, V. Frishling and N. Kordzakhia [7] found approximate formulae for the crossing probabilities for both a one-sided and a two-sided boundary with a horizon; they were able to derive exact formulae for a one-sided and a two-sided symmetric square-root boundary.

In this paper we consider parallel exponential boundaries $f(s) = be^{as} + c_2$, $g(s) = be^{as} + c_1$, $c = c_2 - c_1 > 0$, and find formulae for $P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{F})$, $P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{G})$, and $P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{H})$.

## 2. Analysis of the problem

Obviously, the non-crossing probability can be calculated via the formula

$$P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{H}) = 1 - P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{F}) - P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{G})$$

and for reasons of symmetry

$$P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{G}) = P_{t,\,-x|a,\,-b,\,-c_2,\,-c_1,\,T}(\mathcal{F}).$$

Therefore, it is enough to calculate $P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{F})$. (By translation along the $\overrightarrow{Ox}$ axis one may also prove the equality

$$P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{F}) = P_{t,\,x+c_0|a,\,b,\,c_1+c_0,\,c_2+c_0,\,T}(\mathcal{F}) \text{ for all } c_0\,,$$

but we shall not make use of this fact.)

Let $v(t,x) = P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{F})$, that is consider $t$ and $x$ as arguments and $a$, $b$, $c_1$, $c_2$, $T$ as parameters. According to [8], the function $v(t,x)$ is a solution to the problem

$$\left|\begin{array}{ll} \dfrac{\partial v}{\partial t} + \dfrac{1}{2}\cdot\dfrac{\partial^2 v}{\partial x^2} = 0, & t < T,\ x \in (g(t); f(t)) \\ v(T,x) = 0, & x \in (g(T); f(T)) \\ v(t, g(t)) = 0, & t \le T \\ v(t, f(t)) = 1, & t \le T. \end{array}\right.$$

The equation is simple enough, but the boundary is complicated. To get a rectangular boundary and an initial condition instead of the final one, set

$$v(t,x) = u\left(T - t\,,\ \frac{x - c_1 - be^{at}}{c}\right).$$

Then the function $u(t,x)$ is a solution to the problem

$$\left|\begin{array}{l} -\dfrac{\partial u}{\partial t} - \dfrac{abe^{aT}e^{-at}}{c}\cdot\dfrac{\partial u}{\partial x} + \dfrac{1}{2c^2}\cdot\dfrac{\partial^2 u}{\partial x^2} = 0, \quad t>0, \quad 0<x<1 \\ u(0,x)=0, \quad 0<x<1 \\ u(t,0)=0, \quad t\geq 0 \\ u(t,1)=1, \quad t\geq 0. \end{array}\right.$$

Let $\kappa = \dfrac{1}{2c^2} > 0$, $\lambda = \dfrac{abe^{aT}}{c}$, $U(p,x) = L[u(t,x)]$, where $L$ is the Laplace transformation. Then we get the problem

$$\left|\begin{array}{ll} \kappa\cdot\dfrac{\partial^2 U}{\partial x^2}(p,x) - \lambda\,.\dfrac{\partial U}{\partial x}(p+a,x) - p\,.\,U(p,x) = 0, & \mathrm{Re}(p)>0, \quad 0<x<1 \\ U(p,0)=0\,, & \mathrm{Re}(p)>0 \\ U(p,1)=\dfrac{1}{p}\,, & \mathrm{Re}(p)>0. \end{array}\right.$$

Since $U(p,x)$ is defined for $\mathrm{Re}(p)>0$, we have to impose on $a$ the constraint $a\geq 0$. If $a=0$ or $b=0$, then the boundaries become horizontal lines. This case is well-studied, that is why we assume that $a>0$ and $b\neq 0$ (so $\lambda\neq 0$).

This is a functional differential equation. It is functional in $p$ and differential in $x$. Such equations are hard to solve. Their solutions can hardly ever be written in a closed form. On the other hand, the solution $U(p,x)$ is analytical in $x$, and its power series converge for all $x\in\mathbb{R}$. Therefore we shall find the coefficients of the power series representation $U(p,x) = \sum_{n=0}^{\infty} Q_n(p)\,x^n$.

The differential equation turns into the recurrent equation

$$Q_{n+2}(p) = \frac{\lambda}{\kappa(n+2)}\,Q_{n+1}(p+a) + \frac{p}{\kappa(n+1)(n+2)}\,Q_n(p), \quad \mathrm{Re}(p)>0, \quad n\in\mathbb{N}_0.$$

To find the sequence $(Q_n(p))_{n=0}^{\infty}$, we need to know its first two terms.

From $U(p,0)=0$ it follows that $Q_0(p)=0$ for all $p$ with $\mathrm{Re}(p)>0$. Therefore, $U(p,x) = \sum_{n=1}^{\infty} Q_n(p)\,x^n$.

Unfortunately, we do not know $Q_1(p)$. To find it, we must use the equation $U(p,1)=\dfrac{1}{p}$, which is equivalent to $\dfrac{1}{p} = \sum_{n=1}^{\infty} Q_n(p)$.

Let $Q(p) = Q_1(p)$. Using the recurrent equation, we find

$$Q_2(p) = \frac{\lambda}{2\kappa}\, Q(p+a), \qquad Q_3(p) = \frac{\lambda^2}{6\kappa^2}\, Q(p+2a) + \frac{p}{6\kappa}\, Q(p),$$

$$Q_4(p) = \frac{\lambda^3}{24\kappa^3}\, Q(p+3a) + \frac{\lambda p}{12\kappa^2}\, Q(p+a), \qquad \text{etc.}$$

By induction it follows that $Q_n(p) = \sum\limits_{m=0}^{n-1} \gamma_{n,m}(p)\, Q(p+ma)$. Substituting this into the recurrent equation above, we get another recurrent equation:

$$\gamma_{n+2,m}(p) = \begin{cases} \frac{\lambda}{\kappa(n+2)}\, \gamma_{n+1,m-1}(p+a) + \frac{p}{\kappa(n+1)(n+2)}\, \gamma_{n,m}(p), & m = \overline{1,\ n-1} \\ \frac{\lambda}{\kappa(n+2)}\, \gamma_{n+1,m-1}(p+a), & m = n,\ n+1 \\ \frac{p}{\kappa(n+1)(n+2)}\, \gamma_{n,m}(p), & m = 0. \end{cases}$$

It holds for $n \in \mathbb{N}$. The important difference is that we know $\gamma_{n,m}(p)$ for both $n=1$ and $n=2$: from $Q_1(p) = Q(p)$ and $Q_2(p) = \frac{\lambda}{2\kappa}\, Q(p+a)$ it follows that

$$\gamma_{1;0}(p) = 1, \quad \gamma_{2;0}(p) = 0, \quad \gamma_{2;1}(p) = \frac{\lambda}{2\kappa}\,.$$

Incrementing the index $n$ step by step, we can calculate as many functions of the family $\{\gamma_{n,m}(p)\}_{n=1}^{\infty}{}_{m=0}^{n-1}$ as we need. Then

$$U(p\,,x) = \sum_{n=1}^{\infty} Q_n(p)\, x^n = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \gamma_{n,m}(p)\, x^n\, Q(p+ma) =$$

$$\sum_{m=0}^{\infty} \left[ Q(p+ma) \sum_{n=m+1}^{\infty} \gamma_{n,m}(p)\, x^n \right]$$

if we are allowed to change the order of summation. However, since this operation is problematic, we shall prove a similar equality through a devious path. By definition,

$$U(p\,,x) = \sum_{n=1}^{\infty} Q_n(p)\, x^n = \lim_{N\to\infty} \sum_{n=1}^{N} Q_n(p)\, x^n = \tag{1}$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \sum_{m=0}^{n-1} \gamma_{n,m}(p)\, x^n\, Q(p+ma) = \lim_{N\to\infty} \sum_{m=0}^{N-1} \left[ Q(p+ma) \sum_{n=m+1}^{N} \gamma_{n,m}(p)\, x^n \right]$$

(now we can change the order of summation, because the sums are finite). Substituting $x = 1$ and making use of the equation $U(p\,,1) = \dfrac{1}{p}$ , we get the following functional equation:

$$\frac{1}{p} = \lim_{N\to\infty} \sum_{m=0}^{N-1} \left[ Q(p+ma) \sum_{n=m+1}^{N} \gamma_{n,\,m}(p) \right], \quad \mathrm{Re}(p) > 0. \tag{2}$$

There is a natural procedure to approximately solve this equation for $Q$. This procedure will be described immediately after Theorem 1.

## 3. Main result

**Theorem 1.** *If* $g(t) = be^{\,at} + c_1$ , $f(t) = be^{\,at} + c_2$ , $\forall t \le T \in \mathbb{R}$, $c = c_2 - c_1 > 0$, $a > 0$, $b \neq 0$, $\kappa = \dfrac{1}{2c^2} > 0$ , $\lambda = \dfrac{abe^{\,aT}}{c}$ , *then for all admissible* $t$ *and* $x$*:*

*a)* $P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{H}) = 1 - P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{F}) - P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{G})$;

*b)* $P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{G}) = P_{t,\,-x|a,\,-b,\,-c_2,\,-c_1,\,T}(\mathcal{F})$;

*c)* $v(t,x) := P_{t,\,x|a,\,b,\,c_1,\,c_2,\,T}(\mathcal{F}) = u\left(T - t\,,\ \dfrac{x - c_1 - be^{\,at}}{c}\right)$, *where*
$u(t,x) = L^{-1}\left[U(p\,,x)\right]$ *is the inverse Laplace transformation of*
$U(p\,,x) = \displaystyle\sum_{n=1}^{\infty} Q_n(p)\, x^n$*; here* $Q_n(p) = \displaystyle\sum_{m=0}^{n-1} \gamma_{n,\,m}(p)\, Q(p+ma)$ $\forall n \in \mathbb{N}$,
*the family* $\{\gamma_{n,\,m}(p)\}_{\,n=1}^{\,\infty}\,{}_{m=0}^{\,n-1}$ *is defined through the initial conditions* $\gamma_{1;\,0}(p) = 1$, $\gamma_{2;\,0}(p) = 0$, $\gamma_{2;1}(p) = \dfrac{\lambda}{2\kappa}$ *and the recurrent equation* ($n \in \mathbb{N}$)

$$\gamma_{n+2,\,m}(p) = \begin{cases} \frac{\lambda}{\kappa(n+2)}\,\gamma_{n+1,\,m-1}(p+a) + \frac{p}{\kappa(n+1)(n+2)}\,\gamma_{n,\,m}(p), & m = \overline{1,\,n-1} \\ \frac{\lambda}{\kappa(n+2)}\,\gamma_{n+1,\,m-1}(p+a), & m = n,\ n+1 \\ \frac{p}{\kappa(n+1)(n+2)}\,\gamma_{n,\,m}(p), & m = 0; \end{cases}$$

*and the function* $Q(p)$ *is implicitly defined as a solution to the equation*

$$\frac{1}{p} = \lim_{N\to\infty} \sum_{m=0}^{N-1} \left[ Q(p+ma) \sum_{n=m+1}^{N} \gamma_{n,\,m}(p) \right], \quad \mathrm{Re}(p) > 0.$$

## 4. Numerical calculations

Theorem 1 offers a convenient way for calculating the crossing and non-crossing probabilities for any set of input data. Most of its prescriptions can be directly implemented. A few moments need explaining.

The inverse Laplace transformation $w(t) = L^{-1}[W(p)]$ of a function $W(p)$ can be calculated by means of the well-known formula

$$w(t) = \frac{1}{2\pi i} \ \text{P. V.} \int_{r - i\infty}^{r + i\infty} e^{p\,t}\, W(p) \ \mathrm{dp}, \quad r > 0.$$

Tabulating the $W$ function and calculating the integral is no problem.

One also has to replace the infinite sum in the expression

$$U(p\,, x) = \sum_{n=1}^{\infty} Q_n(p)\, x^n$$

with a finite one:

$$U(p\,, x) \approx \sum_{n=1}^{N} Q_n(p)\, x^n$$

taking some great integer $N$.

The definition of the $Q(p)$ function is the only implicit definition. The equation (2) can be solved for $Q$ replacing $p$ with $p + ka$:

$$\frac{1}{p + ka} = \lim_{N \to \infty} \sum_{m=0}^{N-1} \left[ Q\left(p + (k+m)a\right) \sum_{n=m+1}^{N} \gamma_{n,\,m}(p + ka) \right], \qquad \text{i.e.}$$

$$\frac{1}{p + ka} = \lim_{N \to \infty} \sum_{m=k}^{N+k-1} \left[ Q\left(p + ma\right) \sum_{n=m-k+1}^{N} \gamma_{n,\,m-k}(p + ka) \right], \quad \text{Re}(p) > 0.$$

Actually, this is a system with infinitely many equations ($k \in \mathbb{N}_0$) and infinitely many unknowns $Q(p + ma)$, $m \in \mathbb{N}_0$. It is natural to search for its solution replacing the limit with the $N$-th term of the sequence and taking finitely many equations, i.e. $k = 0, 1, 2, \ldots, K$:

$$\frac{1}{p + ka} \approx \sum_{m=k}^{N+k-1} \left[ Q\left(p + ma\right) \sum_{n=m-k+1}^{N} \gamma_{n,\,m-k}(p + ka) \right], \quad \text{Re}(p) > 0.$$

For any fixed $p$, $\mathrm{Re}(p)>0$, this is a (finite) linear system with $K+1$ equations and $N+K$ unknowns $Q(p+ma)$, $m=0,1,2,\ldots,N+K-1$. To solve it, we need $N-1$ additional equations. These could be

$$Q(p+ma)=0, \qquad m=K+1,\ K+2,\ \ldots,\ N+K-1,$$

if $K$ is a great integer, because $\lim\limits_{m\to\infty} Q(p+ma)=0$. It follows from the equality $\lim\limits_{p\to\infty} Q(p)=0$, which holds, because $Q$ is an image of a function:

$$Q(p)=Q_1(p)=\frac{\partial}{\partial x}U(p,x)\Big|_{x=0}=\frac{\partial}{\partial x}L\,[u(t,x)]\Big|_{x=0}=L\left[\frac{\partial}{\partial x}u(t,x)\Big|_{x=0}\right].$$

Now we have a linear system with $N+K$ equations and $N+K$ unknowns. Solving it is straightforward. Both $N$ and $K$ must be great enough so that the solution to this system could be closer to the actual value of $Q(p)$.

The careful reader may have noticed that we can avoid calculating $Q_n(p)$ if we use (1), i.e.

$$U(p,x)=\lim_{N\to\infty}\sum_{m=0}^{N-1}\left[Q(p+ma)\sum_{n=m+1}^{N}\gamma_{n,m}(p)\,x^n\right] \tag{3}$$

instead of

$$U(p,x)=\sum_{n=1}^{\infty}Q_n(p)\,x^n=\lim_{N\to\infty}\sum_{n=1}^{N}Q_n(p)\,x^n. \tag{4}$$

However, if you are tabulating $U(p,x)$, you can group the calculations by $p$: while the value of $p$ is fixed, $x$ may take on different values.

If you choose to use (3), you may cache the values of $\{Q(p+ma)\}_{m=0}^{N-1}$ and $\{\gamma_{n,m}(p)\}_{m=0}^{N-1}\,{}_{n=m+1}^{N}$, which will require $O(N^2)$ amount of memory. Then each value of $x$ will require $O(N^2)$ steps to calculate $U(p,x)$.

If you use (4), you may cache the values of $\{Q_n(p)\}_{n=1}^{N}$, which will require $O(N)$ amount of memory and $O(N)$ steps for each value of $x$.

Obviously, the second implementation scheme is more effective.

The formulae in this paper were programmed and tabulated. The results were compared with the values of the crossing probabilities calculated by means of the Monte Carlo method and dynamical programming. The idea of the last method is to calculate the crossing probabilities, beginning from the horizon and moving to the starting moment step by step.

The three results concur, which gives a numerical support to the formulae.

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**ВЕРОЯТНОСТ ЗА ИЗХОД НА БРАУНОВО ДВИЖЕНИЕ ПРЕЗ УСПОРЕДНИ ЕКСПОНЕНЦИАЛНИ ГРАНИЦИ**

Добромир Кралчев

**Резюме**. Разглеждаме вероятността брауново движение да достигне двустранна експоненциална граница преди определен момент. Извеждаме формули за вероятността за изход на процеса през горната, респективно долната граница, в случай че те са успоредни.