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SOME PRESENTATIONS OF GENERALIZED POLYNOMIALS BASED ON THE THEORY OF INTERPOLATION

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Abstract. In this short communication some presentations of polynomials based on algebraic, trigonometric, exponential and generalized fundamental polynomials using in the interpolation are given.

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1. Interpolation with algebraic polynomials

Let knots $x_0,\ x_1,\ \ldots,\ x_n \in [a;b]$, $x_i \neq x_j, i \neq j$ and values of a function $f(x)$ at these knots be given. It is well known fact that the problem (Lagrange interpolation formula) for finding an algebraic polynomial $A_n(x)$ (A-polynomial) on the basic functions $\{x^k\}_{k=0}^n$ for which the equalities

$$A_n(x_i) = f(x_i), \quad i = \overline{0,n} \tag{1}$$

are fulfilled has [1,2] an unique solution

$$A_n(x) = \sum_{k=0}^{n} f(x_k) l_k(x)\,, \tag{2}$$

where

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)} \tag{3}$$

are so called fundamental algebraic polynomials possessing the properties

$$l_k(x_i) = \delta_{ki} = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases} \tag{4}$$

Using the fundamental theorem of algebra it is easy to show that for every A-polynomial of degree n, the identity

$$A(x) \equiv \sum_{k=0}^{n} A(x_k) l_k(x) \tag{5}$$

holds true.

2. Interpolation with trigonometric polynomials

Let interpolating knots $x_0 < x_1 < \ldots < x_{2n} < x_0 + 2\pi$ and the values $f(x_i)$, $i = \overline{0, 2n}$ of a periodic function (with period 2π) be given. Correspondingly to (1) exists [1, 2] an unique trigonometric polynomial $T_n(x)$ (T-polynomial) on the basic functions $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx$ which satisfies the following conditions

$$T_n(x_i) = f(x_i) \quad i = \overline{0, 2n}\,. \tag{6}$$

This interpolating polynomial $T_n(x)$ can be presented in the form

$$T_n(x) = \sum_{k=0}^{2n} f(x_k) \lambda_k(x)\,, \tag{7}$$

where $\lambda_k(x)$ are fundamental basic T-polynomials having the properties (4)

One of the possible way for presentation of basic T-polynomials $\lambda_k(x)$, $k = \overline{0, 2n}$ is [1,2]

$$\lambda_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{2n} \frac{\sin \frac{x - x_i}{2}}{\sin \frac{x_k - x_i}{2}}\,. \tag{8}$$

Similarly to (5), we have a presentation

$$T(x) \equiv \sum_{k=0}^{2n} T(x_k)\lambda_k(x) \tag{9}$$

for every T-polynomials of order n.

In the particular case when the knots $x_k = 2k\pi/(2n+1)$, $k = \overline{0,2n}$ are equidistant in the interval $[0; 2\pi]$ the fundamental trigonometric polynomials $\lambda_k(x)$ can be obtained using the core of Dirichlet

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin \frac{2n+1}{2} x}{2 \sin \frac{x}{2}}$$

in the form

$$\lambda_k(x) = \frac{\sin \frac{2n+1}{2}(x - x_k)}{(2n+1) \sin \frac{x - x_k}{2}} \, .$$

Then, for every T-polynomial of order n the identity

$$T(x) \equiv \frac{1}{2n+1} \sum_{k=0}^{2n} T(x_k) \frac{\sin \frac{2n+1}{2}(x - x_k)}{\sin \frac{x - x_k}{2}} \tag{10}$$

holds true.

3. Interpolation with exponential polynomials

Correspondingly to 2. we could make a presentation for every exponential polynomial (E-polynomial) $E_n(x)$ on the basis $(1, \operatorname{sh} x, \operatorname{ch} x, \operatorname{sh} 2x, \operatorname{ch} 2x, \ldots, \operatorname{sh} nx, \operatorname{ch} nx)$ or $(1, e^x, e^{-x}, e^{2x}, e^{-2x}, \ldots, e^{nx}, e^{-nx})$ in the form

$$E(x) \equiv \sum_{k=0}^{2n} E(x_k) h_k(x) \, , \tag{11}$$

where the fundamental E-polynomials $h_k(x)$ can be written as follows

$$h_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{2n} \frac{\operatorname{sh} \frac{x - x_i}{2}}{\operatorname{sh} \frac{x_k - x_i}{2}} \, , \quad h_k(x_i) = \delta_{ki} \, .$$

4. The most general interpolation problem

Let X be a linear space of functions and $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x) \in X$. Let also $L_0, L_1, \ldots, L_n$ be linear functionals defined in X. It is well known [1,2,3] that the necessary and sufficient condition for the general interpolation problem

$$L_k(G_n) = L_k(f), \quad k = \overline{0, n}\,, \tag{12}$$

it is have an unique solution as a generalized polynomial (G-polynomial) $G_n(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \ldots + a_n\varphi_n(x)$ on the basis $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$ for every $f(x) \in X$ is

$$\Delta = det[L_k(\varphi_i)] \neq 0\,. \tag{13}$$

If we chose the functionals L_k as $L_k(g) = g(x_k)$, $k = \overline{0, n}$ when $x_0, x_1, \ldots, x_n$ are different points in interval $[a, b]$ then the condition (13) shows that the basic functions $\{\varphi_k(x)\}_{k=0}^{n}$ form a Chebishev system for $[a, b]$. Many special interesting cases of a choice of $L_0, L_1, \ldots, L_n$ are considered in [3].

Lemma. *If $\bar{G}(x)$ and $\bar{\bar{G}}(x)$ are G-polynomials on the basis $\varphi_0(x), \varphi_1(x), \ldots,$ $\varphi_n(x)$ and $L_k(\bar{G}) = L_k(\bar{\bar{G}})$, $k = \overline{0, n}$ then $\bar{G}(x) \equiv \bar{\bar{G}}(x)$.*

Proof. Let $\bar{G}(x)$ and $\bar{\bar{G}}(x)$ have the presentations $\bar{G}(x) = \alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \ldots + \alpha_n\varphi_n(x)$ and $\bar{\bar{G}}(x) = \beta_0\varphi_0(x) + \beta_1\varphi_1(x) + \ldots + \beta_n\varphi_n(x)$. From the conditions $L_k(\bar{G}) = L_k(\bar{\bar{G}})$, $k = \overline{0, n}$ it follows that

$$\sum_{i=0}^{n} \alpha_i L_k(\varphi_i) = \sum_{i=0}^{n} \beta_i L_k(\varphi_i)$$

which is

$$\sum_{i=0}^{n} (\alpha_i - \beta_i) L_k(\varphi_i) = 0\,, k = \overline{0, n}\,. \tag{14}$$

The linear system (14) with respect to the unknowns $\alpha_i - \beta_i$, $i = \overline{0, n}$ is homogeneous. Because of the condition (13), this system has only a trivial solution $\alpha_i - \beta_i = 0$. Consequently $\bar{G}(x) \equiv \bar{\bar{G}}(x)$.

The solution of the most general interpolation problem (12) can be written [1] in the form

$$G_n(x) = \sum_{k=0}^{n} L_k(f)\Phi_k(x)\,, \tag{15}$$

where $\Phi_k(x)$, $k=\overline{0,n}$, are fundamental generalized G-polynomials on the basis $\varphi_0(x),\varphi_1(x),\ldots,\varphi_n(x)$, for which the equalities

$$L_i(\Phi_k)=\delta_{ik}$$

hold true. This fact follows from the presentation (15) of $G_n(x)$ in the determinant form.

$$\Phi_k(x)=\frac{1}{det[L_k(\varphi_i)]}\begin{vmatrix} L_0(\varphi_0) & \ldots & L_0(\varphi_n) \\ \ldots & \ldots & \ldots \\ L_{k-1}(\varphi_0) & \ldots & L_{k-1}(\varphi_n) \\ \varphi_0(x) & \ldots & \varphi_n(x) \\ L_{k+1}(\varphi_0) & \ldots & L_{k+1}(\varphi_n) \\ \ldots & \ldots & \ldots \\ L_n(\varphi_0) & \ldots & L_n(\varphi_n) \end{vmatrix}$$

The main result is:

Theorem. *For every G-polynomial the identity*

$$G(x)\equiv\sum_{k=0}^{n}L_k(G)\Phi_k(x) \tag{16}$$

holds true.

The conclusion of the theorem follows from the above lemma and (15).

This result (16) generalizes the formulas (5), (9), (10), (11) for algebraic, trigonometric and exponential polynomials.

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НЯКОИ ПРЕДСТАВЯНИЯ НА ОБОБЩЕНИ ПОЛИНОМИ БАЗИРАЩИ СЕ ВЪРХУ ТЕОРИЯТА НА ИНТЕРПОЛИРАНЕТО

П. Хр. Атанасова

Резюме. В съобщението са дадени някои представяния на полиноми с използване на фундаментални алгебрични, тригонометрични, експоненциални и обобщени полиноми от теорията на интерполирането.