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PROVING THAT CERTAIN POINTS LIE ON A CIRCLE

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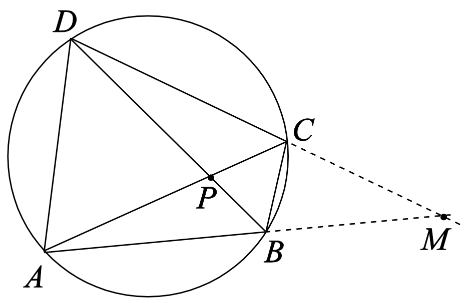
Abstract. The necessary and sufficient conditions that we know from the different Mathematics classes for four points to lie on a circle are reminded via a theorem. Their application is shown using appropriate problems. In the first problems the proof is derived from using one of these conditions. For the last three problems more conditions are needed to obtain the needed proof.

Key words: secants and chords from a point to a circle; perpendicular bisectors of segments; center and diameter of circle.

Theorem. A circle can be circumscribed around quadrangle *ABCD* exactly when one of the following conditions is true:



A) The perpendicular bisectors of its sides cross in one point.

B) $\sphericalangle DAB + \sphericalangle BCD = 180^\circ$ $(\sphericalangle ABC + \sphericalangle CDA = 180^\circ)$.

C) $MA.MB = MD.MC$.

D) $AP.PC = BP.PD$.

E) One side of the quadrangle is seen from the other two vertices at the same angle.

With the problems described in the paper we aim to improve the students' knowledge by conducting logical rationalization. It helps students acquire mathematical thinking by displaying certain aspects of the mind: flexibility, range, depth, criticality, activity, thought transference et al.

The problems are chosen in such a way that the students have to form hypotheses. Each of them has to rationalize which of the five conditions of the given theorem these problems come down to. The chosen problems are not stereotypical. They allow different conditions for points belonging to a circle to be used which makes it possible to assess solution rationalization.

To achieve the goals that are set, we divided the problems in two groups.

GROUP I: Problems that are solved using one condition from the theorem.

Problem 1. In $\triangle ABC$ ($AB > BC$) point *P* is chosen on the side *AB* so that $BP = BC$. Angle bisector BM ($M \in AC$) for $\sphericalangle ABC$ crosses the circumcircle around $\triangle ABC$ in point *N*. Prove that points *A*, *P*, *M* and *N* lie on the same circle. (The problem is from an Olympiad in Moscow, 9th grade, 1997)

Note: It is given that $\triangle PBM \cong \triangle CBM$ (figure 1). The equal corresponding angles of the congruent triangles and the equal inscribed angles for the circumcircle around $\triangle ABC$ led the students towards using condition B) from the theorem, namely $\sphericalangle BPM = \sphericalangle BCM = \sphericalangle MNA$ and $\sphericalangle MNA + \sphericalangle APM = 180°$.

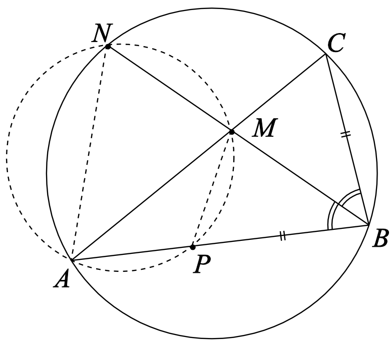


Figure 1.

Problem 2. In acute $\triangle ABC$ the altitudes BB_1 and CC_1 ($B_1 \in AC$, $C_1 \in AB$) are built. Let *M* and *N* are the midpoints of BB_1 and CC_1, $P = AM \cap CC_1$ and $Q = AN \cap BB_1$. Prove that points *M, N, P, Q* lie on the same circle. (The problem is from the Winter Mathematics Competition, Bulgaria, 10th grade, 2007)

Note: It is easy to find out that $\triangle ACC_1 \sim \triangle ABB_1$ (figure 2) and *AN* and *AM* are the medians in these triangles. Then $\sphericalangle A > 90^\circ$ and $\sphericalangle QNP + \sphericalangle QMP = 180^\circ$.

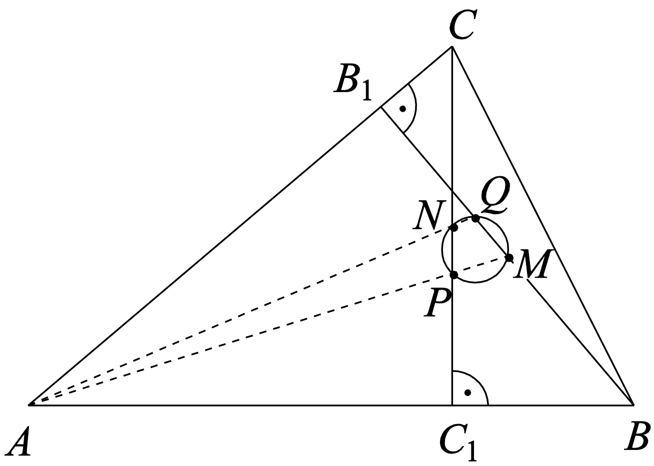


Figure 2.

Problem 3. For $\triangle ABC$ point *O* is such a point inside the triangle that $\sphericalangle AOB = \sphericalangle AOC = 120^\circ$. Points *D* and *E* are the midpoints of *AB* and *AC* and $\sphericalangle BAC = 60^\circ$. Prove that points *A, E, O* and *D* are on the same circle. (The problem is from an Olympiad in St. Petersburg, 9th grade, 1996)

Note: In this problem there is also a lot of information about the angles of the shape in question. The students again looked for application of condition B) of the theorem (figure 3). If $\sphericalangle CAO = x$, then $\sphericalangle OBA = x$ and $\triangle OBA \sim \triangle OAC$, $\triangle OBD \sim \triangle OAE$, from where $\sphericalangle OEA + \sphericalangle ODA = 180^\circ$, which is the necessary proof.

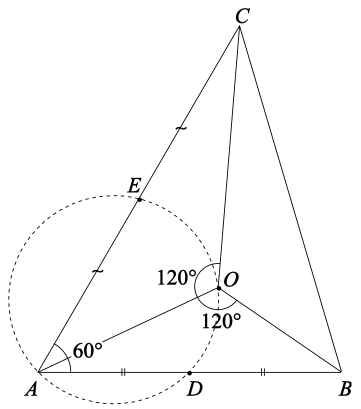


Figure 3.

Problem 4. A random point *E* was chosen from the arc AB of the circumcircle around the isosceles trapezoid *ABCD* $(AB \| CD,\ AB > CD)$. Perpendiculars *AX*, *AY*, *BZ* и *BT* are built from points *A* and *B* to the lines *DE* and *CE* (see figure 4). Prove that points *X*, *Z*, *T* and *Y* are on the same circle. (The problem is from an Olympiad in St. Petersburg, $8^{th} – 9^{th}$ grade, 1999)

Note: From the figure we can clearly see that *EX* and *EY* intersect the circle. This leads the students towards using condition C) of the theorem.

$AD = BC$ infers $AD = BC$ and $\sphericalangle AED = \sphericalangle BEC = \alpha$. From right triangles *AXE, BZE, AYE* and *BTE* we have $AX = AE.\cos\alpha$, $EZ = BE.\cos\alpha$, $AY = AE.\cos(\alpha + \sphericalangle DEC)$ and $ET = BE.\cos(\alpha + \sphericalangle DEC)$. The only thing left to do is apply condition C).

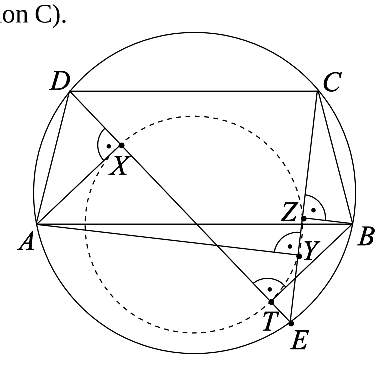


Figure 4.

Problem 5. Around acute $\triangle ABC$ with altitude CH ($H \in AB$) was circumscribed a circle with diameter *CD*. On rays $\overrightarrow{AC}$ and $\overrightarrow{HC}$ are built segments $CA_1 = CB$ and $CH_1 = CD$. Prove that points A_1, H_1, A and H lie on the same circle. (The problem is from an admission examination for Bulgarian universities, 1972)

Note: A careful read of the problem shows that AA_1 and HH_1 are two chords passing through point *C* for the circle in question (figure 5). This reminds the students to use condition D) of the theorem.

They now only have to prove that $\triangle ACH \sim \triangle DCB$ and use the equalities from the problem clause.

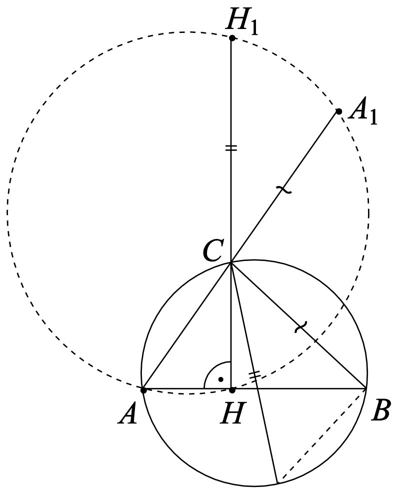


Figure 5.

Problem 6. The diagonal *AC* of trapezoid *ABCD* ($AB \parallel CD$, $AB > CD$) divides it into two similar triangles. If O_1 and O_2 are the centers of circles inscribed in triangles ACD and ABC, prove that lines DO_1 and CO_2 intersect and their point of intersection *K* along with points *A*, *C* and *D* lie on the same circle. (The problem is from an admission examination for Bulgarian universities, 1972)

Note: From the clause we infer $\sphericalangle CAB = \sphericalangle ACD = \alpha$, $\sphericalangle ABC = \sphericalangle CAD = \beta$ and $\sphericalangle ACB = \sphericalangle ADC = \gamma$ (figure 6). Since $\sphericalangle CDO_1 = \sphericalangle ACO_2 = \frac{\gamma}{2}$, we have $\sphericalangle O_1DC + \sphericalangle O_2CD = \frac{\gamma}{2} + \alpha + \frac{\gamma}{2} = \alpha + \gamma = 180° - \beta < 180°$, i.e. the lines DO_1 and CO_2 intersect in point *K*. For quadrangle *AKCD* the side *AK* is seen from vertices *D* and *C* at an angle $\frac{\gamma}{2}$, which is an application of condition E) of the theorem.

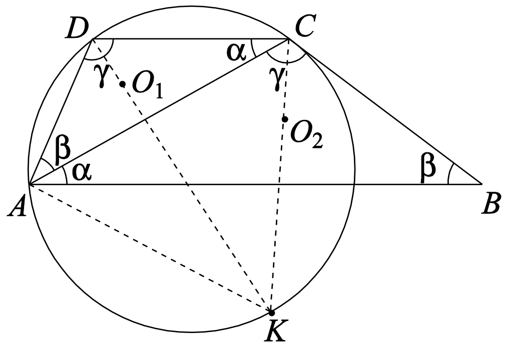


Figure 6.

Problem 7. An acute triangle *ABC* is given with orthocenter *H*. A circle with center the midpoint of *BC* passes through *H* and intersects the line *BC* in points A_1 and A_2. Similarly, a circle with center the midpoint of *AC* passes through *H*, intersects the line *AC* in points B_1 and B_2, and a circle with center the midpoint of *AB* passes through *H* and intersects the line *AB* in points C_1 and C_2. Prove that points A_1, A_2, B_1, B_2, C_1 and C_2 lie on the same circle. (The problem is from the 49th International Mathematics Olympiad, Spain, 2008)

Note: Let *O* be the center of the circumcircle around $\triangle ABC$, and *S* the midpoint of the side *BC* (figure 7). It is known that the perpendicular bisectors of A_1A_2, B_1B_2, and C_1C_2 pass through point *O*. This leads us to the thought that *O* is the center of the circle with points A_1, A_2, B_1, B_2, C_1 and C_2.

Since HS is a median in $\triangle BHC$, then $HS^2 = \frac{1}{4}\left(2BH^2 + 2CH^2 - BC^2\right)$. Using the known laws $BC = 2R\sin\sphericalangle A$, $AH = 2OS = 2R\cos\sphericalangle A$ and Pythagoras' theorem for $\triangle OSA_1$ we have

$$OA_1^2 = OS^2 + SA_1^2 = OS^2 + HS^2 = \frac{1}{4}AH^2 + \frac{1}{4}\left(2BH^2 + 2CH^2 - BC^2\right) =$$

$$= \frac{1}{2}\left(AH^2 + BH^2 + CH^2\right) - \frac{1}{4}\left(BC^2 + AH^2\right) = \frac{1}{2}\left(AH^2 + BH^2 + CH^2\right) - R^2.$$

The distances OB_1 and OC_1 can be calculated in the same way. Thus, we have condition A) of the theorem which is the proof we need.

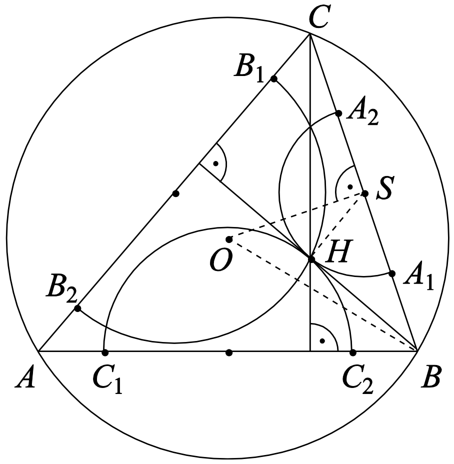


Figure 7.

GROUP II: Problems that are solved using more than one condition from the theorem.

Problem 8. The diagonals of parallelogram *ABCD* ($\sphericalangle A > 90°$) intersect in point *O*. Prove that point *O* and the feet of the perpendiculars from point *A* to lines *BC, CD* and

DB lie on the same circle. (The problem is from an Olympiad in Moscow, 10th grade, 1999)

Note: From $\sphericalangle APB = \sphericalangle AKB = 90^\circ$ and from condition D) we infer that points *P* and *K* are from a circle with diameter *AB* (figure 8). Since $\sphericalangle AMC + \sphericalangle CKA = 90^\circ + 90^\circ = 180^\circ$, then according to B) points *M* and *K* are from a circle with diameter *AC* and center *O*. We use inscribed angles in these two circles and prove that $\sphericalangle KPO + \sphericalangle OMK = 180^\circ$, which along with B) shows that points *P, K, M* and *O* are on the same circle.

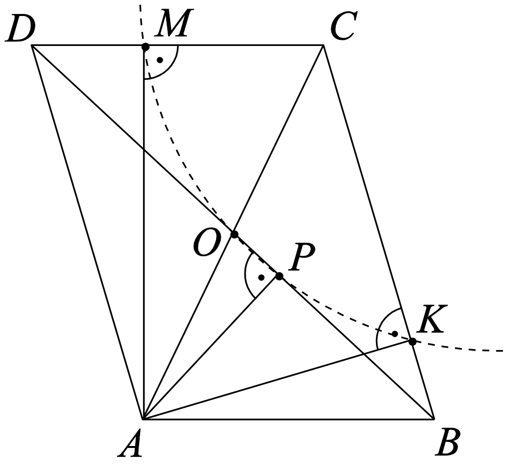


Figure 8.

Problem 9: In acute $\triangle ABC$ altitudes *AE* and *CD* ($E \in BC$, $D \in AB$) are built. On the side *AC* points *G* and *F* are chosen so that *GE*||*AB* and *DF*||*BC*. Prove that points *D*, *E*, *F* and *G* are on the same circle. (The problem is from an Olympiad in St. Petersburg, 10th grade, 1994)

Note: Depending on the place of the points *G* and *F* on the side *AC* the following cases are possible: point *G* is between *A* and *F* or point *F* is between *A* and *G*.

We will comment on the first case (figure 9).

Around quadrilateral *ADEC*, according to E), we can circumscribe a circle. Then $\sphericalangle CDE = \sphericalangle EAC$ and from the conditions for laterality, the following equalities

$\sphericalangle FDC = \sphericalangle DCE = \sphericalangle DAE$,

$\sphericalangle FDE = \sphericalangle FDC + \sphericalangle CDE = \sphericalangle DAE + \sphericalangle EAC = \sphericalangle DAC = \sphericalangle CGE$, so condition E) is met.

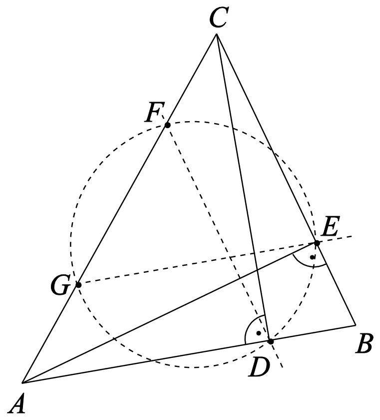


Figure 9.

Problem 10. In the acute $\triangle ABC$ the altitude AH ($H \in BC$) is built. Points K and L are feet of the perpendiculars from H to sides AB and AC. Prove that points B, C, L and K lie on the same circle. (The problem is from an Olympiad in St. Petersburg, 10th grade, 1996)

Note:

Solution I: The clause and condition B) from the theorem show that around the quadrilateral *AKHL* a circle can be circumscribed (figure 10a). Then $\sphericalangle LAH = \sphericalangle LKH$ and $\sphericalangle LCB + \sphericalangle LKB = 180°$, which along with B) proves the statement.

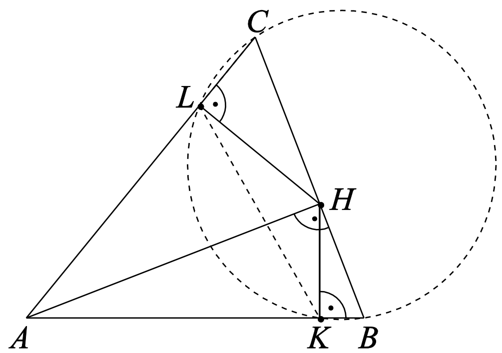


Figure 10 a.

Solution II: Segments *HK* and *HL* are parts of the altitudes of a triangle that we can reproduce. We draw the line *HK* until it intersects the line *AC* in point C_1 and the line *LH* until it intersects the line *AB* in point B_1 (figure 10 b)). Point *H* is the orthocenter of $\triangle AB_1C_1$. Therefore $AH \perp B_1C_1$, from where $BC \parallel B_1C_1$. Since $\sphericalangle C_1KB_1 = \sphericalangle B_1LC_1 = 90^\circ$, then from condition E) the points B_1, *K*, *L* and C_1 are on the same circle. From condition B) we find $\sphericalangle C_1LK + \sphericalangle C_1B_1K = 180^\circ$, from where $\sphericalangle CLK + \sphericalangle CBK = 180^\circ$ and after applying B) we find that points *K*, *B*, *C* and *L* lie on the same circle.

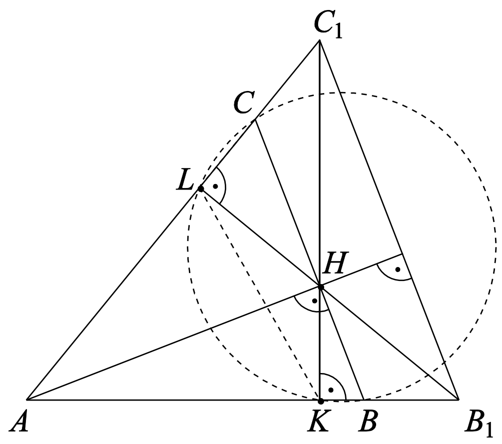


Figure 10 b.

The proposed system of problems is suitable for working with mathematically oriented students. The problems are from different Mathematics competitions. They allow the students freedom in making correct logical conclusions, showing good control of the studied subject matter and getting to new ideas.

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ДОКАЗВАНЕ НА ПРИНАДЛЕЖНОСТ НА ТОЧКИ ВЪРХУ ОКРЪЖНОСТ

Пенка Рангелова, Ивайло Старибратов

Резюме. Познатите от различните теми по математика необходими и достатъчни условия четири точки да принадлежат на една окръжност са припомнени чрез една теорема. С подходящи задачи е показано тяхното приложение. В първите задачи доказателството се получава с използване на едно от тези условия. За последните три задачи се налага използването на повече условия, за да се достигне до крайната цел.