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ON THE UNIT GROUPS OF COMMUTATIVE GROUP ALGEBRAS

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Abstract. Let RG be the group algebra of an abelian group G over a commutative indecomposable ring R with identity of prime characteristic p and $U(RG)$ be the unit group of RG. Warfield [5, Zbl. 0231.13004] introduces the concept KT-module M over a discrete valuation ring and invariants $W_{\alpha,q}(M)$, denoted by $h(\alpha, M)$, for an arbitrary limit ordinal α and prime q. Danchev [1, 2009, Zbl. 1183.16031] calculates the values $W_{\alpha,q}(U(RG))$, when the quotient group G_t/G_p is finite (Proposition 10), where G_t is the torsion subgroup of G and G_p is the p-component of G. In the present paper we establish that Proposition 10 is not valid and does not a sense, since for an arbitrary prime $r \neq p$ and $G = A \times B$, where A is a p-group and B is a cyclic group of order r^n, $n \in \mathbb{N}$, we obtain the contradiction that $W_{\alpha,q}(U(RG))$ is a fraction when ζ_{p^n} is a primitive p^n-th root of identity over R such that ζ_{r^n} is not a root of a polynomial over R of degree les than r^n.

Key words: Commutative indecomposable rings of prime characteristic, group of normalized units, invariants of KT-groups, KT-groups.

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Let p be a prime and α be an arbitrary limit ordinal. If M is a module over a ring then we denote the torsion submodule of $p^\alpha M$ by T_α. Warfield [5] introduces the invariants $h(\alpha, M)$ of KT-module M over a discrete valuation ring by

$$(1) \qquad h(\alpha, M) = dim(V_\alpha(M)), \quad V_\alpha(M) = p^\alpha M/(p^{\alpha+1}M + T_\alpha),$$

where $dimV_\alpha(M)$ is the dimension of $V_\alpha(M)$.

The concept a KT-group is a derivative concept of a KT-module. If A is a multiplicative KT-group, then the introduced invariants (1) for A is

$$h(\alpha, A) = r(A^{p^\alpha}/A^{p^{\alpha+1}}A_p^{p^\alpha})$$

where $r(A)$ is the rank of the group A and α is a limit ordinal. In the end of the paper [5] Warfield considers T^*-modules M and invariants $g(e, M)$ of these modules which now, in the contemporary scientific mathematical literature, are called Warfield modules and Warfield invariants, respectively. We note, that the class of the Warfield modules is broader and it concludes the class of the KT-modules. By this reason $h(\alpha, M)$ are not called "Warfield invariants" though they are introduced by Warfield, since $h(\alpha, M)$ are invariants of a subclass of the class of Warfield modules. Note that Fuchs [2] does not call them "Warfield invariants" and he only refers to these invariants as invariants given by Warfield. Besides in [5], one can find, for example, a discussion for the Warfield modules in [3].

Let $U(RG)$ be the unit group in RG and let $V(RG)$ be the group of the normalized units in RG. We note, that in [1] Danchev considers the invariants $h(\alpha, U(RG))$ denoted by $W_{\alpha,p}(U(RG))$. Therefore, $W_{\alpha,p}(U(RG))$ must not to be called Warfield invariants. In the titles of the paper [1] and of some other articles Danchev uses incorrect the consept "Warfield invariants". These titles and the obtained results insert a fallacy in the readers, since they do not refer to Warfield invariants $g(e, U(RG))$. They are refered to the invariants $h(\alpha, U(RG))$ of the KT-groups.

Now we shall see that Proposition 10 of [1] does not have a sense. Proposition 10 is the following.

Preposition 10. [1] *Suppose R is an indecomposable ring of prime characteristic p and G is an abelian group such that G_t/G_p is finite. Then, for each ordinal number α and prime q,*

$$W_{\alpha,q}(U(RG)) = \sum_{d/|G_t/G_p|} \sum_{a(d)} W_{\alpha,q}(R[\zeta_d]^*) + \sum a(d) \cdot W_{\alpha,q}(G/\coprod_{l \neq p} G_l)$$

where $a(d) = |\{g \in G_t/G_p : order(g) = d\}|/[R[\zeta_d] : R]$.

In particular, if R is perfect,

$$W_{\alpha,p}(U(RG)) = \sum_{d/|G_t/G_p|} a(d) \cdot W_{\alpha,p}(G/\coprod_{l \neq p} G_l).$$

In Proposition 10 the numbers p and q are fixed primes. Let r be a prime distinct from p and let G be an abelian group such that $G = A \times B$, when A is an arbitrary abelian p-group and $B =< h >$ is a cyclic group of order r^n, $n \in N$. Suppose, that ζ_{p^n} is a primitive p^n-th root of identity over R such that ζ_{r^n} is not a root of a polynomial over R of degree les than r^n. For example, we can take $\zeta_{r^n} = h \in B$. The element h is considered as an element of the group algebra RB and in RB it is a root of a polynomial $x^n - 1$ and h is not a root of a polynomial of a degree less than r^n. Therefore, $[R[\zeta_{r^n}] : R] = r^n$. We note that ζ_{r^n} satisfies the condition which is indicated in [1]. We note also that the number of the elements of order r^n in the cyclic group $< h >$ is $\varphi(r^n) < r^n$. Therefore, in the expression of $W_{\alpha,q}(U(RG))$ in Preposition 10 participates the number

$$a(r^n) = \varphi(r^n)/[R[\zeta_{r^n}] : R] = \varphi(r^n)/r^n.$$

Hence we see that $a(r^n)$ is a fraction, that is we obtain the contradiction that $W_{\alpha,q}(U(RG))$ is a fraction. Consequently, Proposition 10 of [1] does not have a sense and the result of this proposition is invalid.

We note the well known decomposition $U(RG) = R^* \times V(RG)$. Therefore, the study of $U(RG)$ is reduced to the study of $V(RG)$. The ring R is called p-perfect if $R^p = R$, where $R^p = \{r^p | r \in R\}$. We note that in Theorem 9 of [1] Danchev calculates the invariants $W_{\alpha,q}(U(RG))$, when G is a p-mixed abelian group, that is when $G_t = G_p$, R is a commutative ring with identity of prime characteristic p and q is a prime. For an arbitrary abelian group G and a commutative indecomposable p-perfect ring R with identity of prime characteristic p the invariants $W_{\alpha,p}(V(RG))$, denoted by $h(\alpha, V(RG))$, are calculated by Mollov and Nachev [4].

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ВЪРХУ МУЛТИПЛИКАТИВНИТЕ ГРУПИ НА КОМУТАТИВНИ ГРУПОВИ АЛГЕБРИ

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Резюме. Нека RG е груповата алгебра на абелева група G над комутативен неразложим пръстен R с единица и проста характеристика p и $U(RG)$ е мултипликативната група на RG. Уорфилд [5 Zbl. 0231.13004] въвежда понятието KT-модул M над дискретно нормиран пръстен и инварианти $W_{\alpha,q}(M)$, означени с $h(\alpha, M)$, за произволно ординално число α и просто число q. Данчев [1, Zbl 1183.16031] изчислява стойностите $W_{\alpha,q}(U(RG))$, когато фактор-групата G_t/G_p е крайна (Предложение 10), където G_t е периодичната подгрупа на G и G_p е p-компонентата на G. В настоящата статия установяваме, че Предложение 10 е невалидно и няма смисъл, тъй като за произволно просто $r \neq p$ и $G = A \times B$, където A е p-група и B е циклична група от ред r^n, $n \in \mathbb{N}$, получаваме противоречието,че $W_{\alpha,q}(U(RG))$ е дробно число, когато ζ_{p^n} е примитивен p^n-ти корен на единицата, така че ζ_{r^n} не е корен на полином над R от степен по-малка от r^n.