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НЕЛИНЕЙНИ ДИСКРЕТНИ НЕРАВЕНСТВА С МАКСИМУМИ НА НЕИЗВЕСТНАТА СКАЛАРНА ФУНКЦИЯ

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NONLINEAR DIFFERENCE INEQUALITIES WITH MAXIMA OF THE UNKNOWN SCALAR FUNCTION

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Резюме. *В тази статия са разгледани някои нелинейни дискретни неравенства, които съдържат максималната стойност на неизвестната скаларна функция в предишен итервал от време с фиксирана дължина. Разгледаните неравенства представляват дискретни обобщения на класическото неравенство на Гронуол-Белман. Важността на тези дискретни неравенства се определя от широкото им приложение при качественото изследване на решенията на диференчни уравнения с максимуми и е илюстрирано чрез някои директни приложения.*

Ключови думи**:** *дискретни неравенства, диференчни уравнения с максимуми, оценки*

1. Introduction

In the recent years great attention has been paid to finite difference equations and their applications in modeling of real world problems (Agarwal 2000), (Elaydi 2005), (Wing-Sum, Qing-Hua, Pecaric 2008). At the same time there are many real world processes in which the present state depends significantly on its maximal value on a past time interval. Adequate mathematical models of these processes are so called difference equations with “maxima”. Meanwhile, this type of equations is not widely studied yet and there are only some isolated results (Luo, Bainov 2001), (Pachpatte 2002).

The development of the theory of difference equations with “maxima” requires solving of finite difference inequalities that involve the maximum value of the unknown function. The main purpose of the paper is solving of a new type of nonlinear discrete inequalities which contain the maximum over a past time interval. Some of the solved inequalities are applied to difference equations with “maxima” and bounds of their solutions are obtained.

2. Preliminary Notes and Definitions

Let $\mathbb{R}_+ = [0,+\infty)$, $\mathbb{Z}$ be the set of all integers, $h \geq 0$ be a given fixed integer and $a, b \in \mathbb{Z}$ be such that $a < b$. Denote by $\mathbb{Z}[a,b] = \{z \in \mathbb{Z} : a \leq z \leq b\}$. Throughout, in what follows for any function $Q:\mathbb{Z}[m,n]\to\mathbb{R}$, $m<n$, we shall assume that $\sum_{i=n}^{m} Q(i)=0$ and $\prod_{i=n}^{m} Q(i)=1$. Denote by $\mathcal{F}(M,N)$ the class of all functions defined on the set M to the set N.

We will introduce the following classes of functions.

Definition 1. *The function* $\omega\in C(\mathbb{R}_+,\mathbb{R}_+)$ *is said to be from the class* Ω *if it satisfies the following conditions:*

(i) ω *is a nondecreasing function;*
(ii) $\omega(u)>0$ *for* $u>0$*;*
(iii) $\int^{\infty}\frac{du}{\omega(u)}=\infty$.

Definition 2. *The function* $\psi\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ *is said to be from the class* Λ *if it satisfies the following conditions:*

(i) $\psi(u)$ *is an increasing function;*
(ii) $\psi(0)=0$ *and* $\lim_{u\to\infty}\psi(u)=\infty$.

3. Main Results

Theorem 1. *Let the following conditions be fulfilled:*

1. *The functions* $f_i, g_j\in\mathcal{F}(\mathbb{Z}[0,T],\mathbb{R}_+)$ *for* $i=1,2,\ldots,l$, $j=1,2,\ldots,m$.
2. *The function* $\phi\in\mathcal{F}(\mathbb{Z}[-h,0],\mathbb{R}_+)$.
3. *The function* $\psi\in\Lambda$.
4. *The functions* $\omega_i,\tilde{\omega}_j\in\Omega$, $i=1,2,\ldots,l$, $j=1,2,\ldots,m$.
5. *The function* $u\in\mathcal{F}(\mathbb{Z}[-h,T],\mathbb{R}_+)$ *satisfies the inequalities*

$$\psi\left(u(n)\right)\le k+\sum_{i=1}^{l}\sum_{s=0}^{n-1} f_i(s)u^p(s)\omega_i\left(u(s)\right)+\sum_{j=1}^{m}\sum_{s=0}^{n-1} g_j(s)u^p(s)\tilde{\omega}_j\left(\max_{\xi\in\mathbb{Z}[s-h,s]} u(\xi)\right)\quad \text{for } n\in\mathbb{Z}[0,T], \tag{1}$$

$$u(n)\le\phi(n)\qquad \text{for } n\in\mathbb{Z}[-h,0], \tag{2}$$

where the constants $p\ge 0$ *and* $\mathrm{k}>0$.

Then for $n\in\mathbb{Z}[0,\alpha_1]$ *the inequality*

$$u(n)\le\psi^{-1}\left\{\Psi^{-1}\left[W^{-1}\left(W\left(\Psi\left(M\right)\right)+A(n)\right)\right]\right\} \tag{3}$$

holds, where Ψ^{-1} *and* W^{-1} *are the inverse functions of*

$$\Psi(r)=\int_{r_0}^{r}\frac{ds}{\left[\psi^{-1}(s)\right]^p},\quad 0<r_0<M,\ r_0\le r, \tag{4}$$

$$W(r)=\int_{r_1}^{r}\frac{ds}{\overline{w}\left[\psi^{-1}\left(\Psi^{-1}(s)\right)\right]},\ \ 0<r_1<\Psi(M),\ \ r_1\le r, \tag{5}$$

$$\overline{w}(n)=\max\left(\max_{1\le i\le l}\omega_i(n),\max_{1\le j\le m}\tilde{\omega}_j(n)\right), \tag{6}$$

$$A(n)=\sum_{i=1}^{l}\sum_{s=0}^{n-1}f_i(s)+\sum_{j=1}^{m}\sum_{s=0}^{n-1}g_j(s), \tag{7}$$

$$M=\max\left(k,\psi\left(\max_{s\in\mathbb{Z}[-h,0]}\phi(s)\right)\right)>0, \tag{8}$$

$$\begin{aligned}\alpha_1=\sup\Big\{\tau\in\mathbb{Z}[0,T]:\ &\left(W\left(\Psi(M)\right)+A(n)\right)\in\mathrm{Dom}\,(W^{-1}),\\ &W^{-1}\left(W\left(\Psi(M)\right)+A(n)\right)\in\mathrm{Dom}\,(\Psi^{-1})\\ &\text{and } \Psi^{-1}\left[W^{-1}\left(W\left(\Psi(M)\right)+A(n)\right)\right]\in\mathrm{Dom}\,(\psi^{-1})\text{ for all } n\in\mathbb{Z}[0,\tau]\Big\}.\end{aligned}$$

Proof. Define a nondecreasing function $v:\mathbb{Z}[-h,T]\to[M,\infty)$ by the equalities

$$v(n)=\begin{cases}M+\sum_{i=1}^{l}\sum_{s=0}^{n-1}f_i(s)u^p(s)\omega_i\left(u(s)\right),\\ \quad+\sum_{j=1}^{m}\sum_{s=0}^{n-1}g_j(s)u^p(s)\tilde{\omega}_j\left(\max_{\xi\in\mathbb{Z}[s-h,s]}u(\xi)\right) & \text{for } n\in\mathbb{Z}[0,T],\\ M & \text{for } n\in\mathbb{Z}[-h,0].\end{cases}$$

From the definitions of $v(n)$ and M, and condition 3 of Theorem 1 it follows

$$u(n)\le\psi^{-1}\left(v(n)\right)\qquad\text{for } n\in\mathbb{Z}[-h,T], \tag{9}$$

$$\max_{\xi\in\mathbb{Z}[s-h,s]}u(\xi)\le\max_{\xi\in\mathbb{Z}[s-h,s]}\psi^{-1}\left(v(\xi)\right)=\psi^{-1}\left(v(s)\right),\qquad s\in\mathbb{Z}[0,T]. \tag{10}$$

From (1), (9), (10) and the definition of the function $\overline{w}(n)$ we get for $n\in\mathbb{Z}[0,T]$

$$\begin{aligned}v(n)\le M&+\sum_{i=1}^{l}\sum_{s=0}^{n-1}f_i(s)\left(\psi^{-1}\left(v(s)\right)\right)^p\overline{w}\left(\psi^{-1}\left(v(s)\right)\right)\\ &+\sum_{j=1}^{m}\sum_{s=0}^{n-1}g_j(s)\left(\psi^{-1}\left(v(s)\right)\right)^p\overline{w}\left(\psi^{-1}\left(v(s)\right)\right):=K(n).\end{aligned} \tag{11}$$

Note that $K(n)$ is a nondecreasing function and the inequality $v(n)\le K(n)$ holds for $n\in\mathbb{Z}[0,T]$. Therefore,

$$\Delta K(n)\le\left(\psi^{-1}\left(K(n)\right)\right)^p\left[\sum_{i=1}^{l}f_i(n)\overline{w}\left(\psi^{-1}\left(K(n)\right)\right)+\sum_{j=1}^{m}g_j(n)\overline{w}\left(\psi^{-1}\left(K(n)\right)\right)\right]. \tag{12}$$

According to the mean value theorem we get for some $\xi\in(K(n),K(n+1))$

$$\Delta\Psi\left(K(n)\right)=\Psi\left(K(n+1)\right)-\Psi\left(K(n)\right)=\Psi'(\xi)\Delta K(n)=\frac{\Delta K(n)}{\left(\psi^{-1}(\xi)\right)^p}. \tag{13}$$

Then from (12) and (13) it follows

$$\Delta\Psi\left(K(n)\right) \le \sum_{i=1}^{l} f_i(n)\overline{w}\left(\psi^{-1}\left(K(n)\right)\right) + \sum_{j=1}^{m} g_j(n)\overline{w}\left(\psi^{-1}\left(K(n)\right)\right). \quad (14)$$

Summing up inequality (14) from 0 to $n-1$, where $n \in \mathbb{Z}[0,T]$, we get

$$\sum_{s=0}^{n-1}\Delta\Psi\left(K(s)\right) \le \sum_{i=1}^{l}\sum_{s=0}^{n-1} f_i(s)\overline{w}\left(\psi^{-1}\left(K(s)\right)\right) + \sum_{j=1}^{m}\sum_{s=0}^{n-1} g_j(s)\overline{w}\left(\psi^{-1}\left(K(s)\right)\right). \quad (15)$$

From (15) we obtain

$$\Psi\left(K(n)\right) \le \Psi(M) + \sum_{i=1}^{l}\sum_{s=0}^{n-1} f_i(s)\overline{w}\left(\psi^{-1}\left(K(s)\right)\right) + \sum_{j=1}^{m}\sum_{s=0}^{n-1} g_j(s)\overline{w}\left(\psi^{-1}\left(K(s)\right)\right) := K_1(n). \quad (16)$$

Note the function $K_1(n)$ is nondecreasing, $K_1(0) = \Psi(M)$ and the inequalities $v(n) \le K(n) \le \Psi^{-1}\left(K_1(n)\right)$ holds for $n \in \mathbb{Z}[0,\alpha_1]$. Therefore, for any $n \in \mathbb{Z}[0,\alpha_1]$ we get

$$\Delta K_1(n) \le \overline{w}\left[\psi^{-1}\left(\Psi^{-1}\left(K_1(n)\right)\right)\right]\left(\sum_{i=1}^{l} f_i(n) + \sum_{j=1}^{m} g_j(n)\right). \quad (17)$$

According to the mean value theorem for the function W it follows for some $\xi \in (K_1(n), K_1(n+1))$

$$\Delta W\left(K_1(n)\right) = W\left(K_1(n+1)\right) - W\left(K_1(n)\right) = W'\left(\xi\right)\Delta K_1(n) = \frac{\Delta K_1(n)}{\overline{w}[\psi^{-1}(\Psi^{-1}(\xi))]}. \quad (18)$$

Therefore,

$$\Delta W\left(K_1(n)\right) \le \sum_{i=1}^{l} f_i(n) + \sum_{j=1}^{m} g_j(n). \quad (19)$$

Summing up inequality (19) from 0 to $n-1$, where $n \in \mathbb{Z}[0,\alpha_1]$, we obtain

$$W\left(K_1(n)\right) \le W\left(\Psi(M)\right) + A(n), \quad (20)$$

where the function $A(n)$ is defined by equality (7).

Since W^{-1} is an increasing function from inequalities $u(n) \le \psi^{-1}\left(v(n)\right) \le \psi^{-1}\left(K(n)\right) \le \psi^{-1}\left(\Psi^{-1}\left(K_1(n)\right)\right)$ and (20) we obtain inequality (3).

□

Inequalities (1), (2) can have a different type of solution which is given in the following result:

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then for* $n \in \mathbb{Z}[0,\alpha_2]$ *the inequality*

$$u(n) \le \psi^{-1}\left\{\Psi_1^{-1}\left(\Psi_1(M) + A(n)\right)\right\} \quad (21)$$

holds, where $\overline{w}(n)$, $A(n)$ *and* M *are defined by* (6), (7) *and* (8), *respectively,* Ψ_1^{-1} *is the inverse function of*

$$\Psi_1(r) = \int_{r_2}^{r} \frac{ds}{\left[\psi^{-1}(s)\right]^p \overline{w}\left[\psi^{-1}(s)\right]}, \quad 0 < r_2 < M, \; r_2 \le r, \quad (22)$$

$$\alpha_2 = \sup\Big\{\tau \in \mathbb{Z}[0,T]:\ \Psi_1(M) + A(n) \in \mathrm{Dom}\left(\Psi_1^{-1}\right)$$
$$\text{and } \Psi_1^{-1}\Big(\Psi_1(M) + A(n)\Big) \in \mathrm{Dom}\left(\psi^{-1}\right) \ \text{ for all } \ n \in \mathbb{Z}[0,\tau]\Big\}.$$

Proof. Following the proof of Theorem 1 we obtain inequality (12). Then from the monotonicity of the functions ψ^{-1} and $K(n)$ it follows

$$\Delta K(n) \le \left(\psi^{-1}\left(K(n)\right)\right)^p \,\overline{w}\left(\psi^{-1}\left(K(n)\right)\right)\left[\sum_{i=1}^{l} f_i(n) + \sum_{j=1}^{m} g_j(n)\right]. \tag{23}$$

According to the mean value theorem we get for some $\xi \in (K(n), K(n+1))$

$$\begin{aligned}\Delta\Psi_1\left(K(n)\right) &= \Psi_1\left(K(n+1)\right) - \Psi_1\left(K(n)\right)\\ &= \Psi_1'(\xi)\Delta K(n)\\ &\le \frac{\Delta K(n)}{\left(\psi^{-1}\left(K(n)\right)\right)^p \,\overline{w}\left(\psi^{-1}\left(K(n)\right)\right)}.\end{aligned} \tag{24}$$

Therefore,

$$\Delta\Psi_1\left(K(n)\right) \le \sum_{i=1}^{l} f_i(n) + \sum_{j=1}^{m} g_j(n). \tag{25}$$

Sum inequality (25) from 0 to $n-1$, where $n \in \mathbb{Z}[0,T]$, and obtain

$$\Psi_1\left(K(n)\right) \le \Psi_1\left(M\right) + A(n), \tag{26}$$

where the function $A(n)$ is defined by equality (7).

Since Ψ_1^{-1} is an increasing function from inequalities $u(n) \le \psi^{-1}\left(v(n)\right) \le \psi^{-1}\left(\Psi_1^{-1}\left(K(n)\right)\right)$ and (26) we obtain the required inequality (21).

□

4. Applications

Example: Consider the following difference equation with "maxima"

$$\Delta v(n) = F\left(n, v(n), \max_{s\in\mathbb{Z}[n-h,n]} v(s)\right) \quad \text{for } \ n \in \mathbb{Z}[0,T], \tag{27}$$

with the initial condition

$$v(n) = \varphi(n) \qquad\qquad \text{for } \ n \in \mathbb{Z}[-h,0], \tag{28}$$

where $\Delta v(n) = v(n+1) - v(n),\ v(n) \in \mathbb{R},\ \varphi : \mathbb{Z}[-h,0] \to \mathbb{R},\ F : \mathbb{Z}[0,T] \times \mathbb{R}^2 \to \mathbb{R}.$

Theorem 3. (*Upper bound*). *Let the following conditions be fulfilled:*

1. *The function* $F \in \mathcal{F}(\mathbb{Z}[0,T] \times \mathbb{R}^2, \mathbb{R})$ *satisfies for* $n \in \mathbb{Z}[0,T]$ *and* $x, y \in \mathbb{R}$ *the following condition*

$$\left|F\left(n, x, y\right)\right| \le \left|x\right|^q \left[R(n)\left|x\right|^t + Q(n)\left|y\right|^t\right],$$

where the functions $R, Q \in \mathcal{F}(\mathbb{Z}[0,T], \mathbb{R}_+)$ *and the constants* $q,\ t$ *are such that* $0 \le q < 1$, $0 \le t < 1$ *and* $1 - q - t > 0$.

2. *The function* $\varphi \in \mathcal{F}(\mathbb{Z}[-h,0], \mathbb{R})$.
3. *The IVP* (27), (28) *has at least one solution, defined for* $n \in \mathbb{Z}[-h,T]$.

Then for $n\in\mathbb{Z}[0,T]$ *the solution of the IVP* (27), (28) *satisfies the inequality*

$$|v(n)|\le \sqrt[1-q-t]{M^{1-q-t}+(1-q-t)\sum_{s=0}^{n-1}\left[R(s)+Q(s)\right]}\,, \qquad (29)$$

where $M=\max_{s\in\mathbb{Z}[-h,0]}|\varphi(s)|$.

Proof. From condition 1 of Theorem 3 for the norm of the solution $v(n)$ of IVP (27), (28) it follows

$$|v(n)|\le M+\sum_{s=0}^{n-1}R(s)|v(s)|^q\left(|v(s)|\right)^t+\sum_{s=0}^{n-1}Q(s)\left|v(s)\right|^q\left(\max_{\xi\in\mathbb{Z}[s-h,s]}\left|v(\xi)\right|\right)^t,\quad n\in\mathbb{Z}[0,T], \qquad (30)$$

$$|v(n)|\le M, \qquad n\in\mathbb{Z}[-h,0]. \qquad (31)$$

Set $|v(n)|=V(n)$ for $n\in\mathbb{Z}[-h,T]$. According to Theorem 2 from (30), (31) for $u(n)=V(n)$, $l\equiv m\equiv 1$, $\mathrm{p}=\mathrm{q}$, $f(n)\equiv R(n)$, $g(n)\equiv R(n)$, $n\in\mathbb{Z}[0,T]$, $\psi(V)\equiv V$, $\omega(V)\equiv\tilde{\omega}(V)=V^t$, $\Psi_1(r)=\int_0^r\frac{ds}{s^{q+t}}=\frac{r^{1-q-t}}{1-q-t}$, $\Psi_1^{-1}(r)=\left[(1-q-t)r\right]^{\frac{1}{1-q-t}}$, $\mathrm{Dom}(\Psi_1^{-1})=\mathbb{R}_+$ we obtain for $n\in\mathbb{Z}[0,T]$

$$V(n)\le\sqrt[1-q-t]{M^{1-q-t}+(1-q-t)\sum_{s=0}^{n-1}\left[R(s)+Q(s)\right]}\,. \qquad (32)$$

From inequality (32) and the definition of the function $V(n)$ we obtain the required inequality (29).

□

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