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# ЗА РАЗПРЕДЕЛЕНИЯТА НА НУЛИТЕ НА ДВЕ УРАВНЕНИЯ НА ТРЪБОПРОВОД

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# ON THE DISTRIBUTIONS OF THE ZEROS OF TWO EQUATIONS OF A PIPELINE

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***Резюме.*** *В тази статия дискутираме осцилационното поведение на две уравнения на тръбопровод, които са известни от публикациите на Вельмисов, Гърневска и Милушева. Тези автори са поставили различни гранични условия. Граничните задачи са изследвани чрез метода на Гальоркин. Моделиращите уравнения на тръбопровод са хомогенни. Тук ще разгледаме разпределението на нулите на техни нехомогенни обобщения. Настоящата статия е естествена крайна точка на цикъла от наши предишни тематично свързани резултати за споменатите по-горе уравнения.*

***Ключови думи:*** *осцилация, тръбопровод, функционално-диференциално уравнение*

## 1. Introduction

The papers of Vel'misov, Garnefska, and Milusheva concerned with the dynamical stability of a pipeline, stated at the origin of many publication of the author (see the cited literature). In particular, in (Petrova, 2003), as we know, have been made for the first time simultaneous consideration of oscillation theory and Galerkin's method. More precisely, there we found sufficient conditions for oscillation of these solutions of the respective equations, which were obtained only by Galerkin's method. Traditionally, there are sufficient conditions for oscillation of all solutions of the investigated equations, but not these one, derived by a some method.

Vel'misov, Garnefska, and Milusheva considered two equations of a pipeline. The present paper is concerned mainly with the linear one:

$$\begin{aligned} L_1(w) \equiv M w_{t^2}(x,t) + D w_{x^4}(x,t) + \xi w_{x^4 t}(x,t-\tau_1) - \eta w_{x^2 t^2}(x,t) + \alpha w_{xt}(x,t) \\ + p w_{x^2}(x,t) + \gamma w_t(x,t-\tau_2) + \beta w(x,t-\tau_3) = 0,\ x \in (0,l),\ \ t > 0, \end{aligned} \tag{1}$$

where $l > 0, M > 0, D > 0, \xi \ge 0, \eta \ge 0, \alpha \ge 0, p \ge 0, \gamma \ge 0, \beta > 0, \tau_1 \ge 0, \tau_2 \ge 0, \tau_3 \ge 0$ are constants. The second equation of a pipeline is a non-linear one:

$$L_2(w) \equiv L_1(w) + \theta[w_t(x,t) + Vw_x(x,t)] + \alpha_{30} w^3(x,t-\tau_3) + \\ + \alpha_{21} w^2(x,t-\tau_3) w_t(x,t-\tau_2) + \alpha_{03} w_t^3(x,t-\tau_3) = 0,\ x \in (0,l),\ t > 0, \tag{2}$$

where the above assumptions have been saved and additionally we assume that all the new coefficients are nonnegative constants.

Here we shall explore the distribution of the zeros of the following generalizations of (1) and (2), respectively,

$$L_1(w) = f(x,t),\ x \in (0,l),\ t > 0, \tag{3}$$

and

$$L_2(w) = f(x,t),\ x \in (0,l),\ t > 0, \tag{4}$$

where $f(x,t) \in C([0,l], R_+; R)$. It is very difficult to establish sufficient conditions for oscillation of all sufficiently smooth solutions of the equations (1) − (4). We shall obtain sufficient conditions for oscillation of the solutions of (1) − (4), which are of type

$$w(x,t) = w_1(t) g_1(x), \tag{5}$$

where $g_1(x)$ is the first function among the basis of functions $\{g_n(x)\}_{n=1}^{\infty}$, forming a complete system of functions on $[0,l]$ whose form is determined by the type of fixing of the endpoints of the pipeline. Vel'misov, Garnefska, and Milusheva treated both equations (1) and (2) under different boundary conditions at x = 0 or x = l. We shall be concerned with two of them:

$$w = w_x = 0 \quad (\textit{rigid fixing}) \text{ and } w = \widetilde{M} = 0 \quad (\textit{hinged fixing}),$$

where $\widetilde{M} = Dw_{x^2}(x,t) + \xi w_{x^2 t}(x,t)$. In the article (Vel'misov, Garnefska, and Milusheva, 2001) have been stated the boundary conditions:

$$a).\quad w(0,t) = w_x(0,t) = 0,\quad w(l,t) = w_x(l,t) = 0;$$

$$b).\quad w(0,t) = w_x(0,t) = 0,\quad w(l,t) = \widetilde{M}(l,t) = 0;$$

$$c).\quad w(0,t) = \widetilde{M}(0,t) = 0,\quad w(l,t) = w_x(l,t) = 0;$$

$$d).\quad w(0,t) = \widetilde{M}(0,t) = 0,\quad w(l,t) = \widetilde{M}(l,t) = 0.$$

In fact, in (Vel'misov, Garnefska, and Milusheva, 2001) the following nine subcases of (1) have been formulated:

$$I.\quad \tau_1 = \tau_2 = 0,\quad \tau_3 = h > 0;\quad II.\quad \tau_1 = h > 0,\quad \tau_2 = \tau_3 = 0;$$

$$III.\quad \tau_1 = \tau_3 = 0,\quad \tau_2 = h > 0;\quad IV.\quad \tau_1 = 0,\quad \tau_2 = \tau_3 = h > 0;$$

$$V.\quad \tau_2 = 0,\quad \tau_1 = \tau_3 = h > 0;\quad VI.\quad \tau_3 = 0,\quad \tau_1 = \tau_2 = h > 0;$$

$$VII.\quad \tau_3 = 0,\quad \tau_1 \neq \tau_2, \tau_1 > 0, \tau_2 > 0;\quad VIII.\quad \tau_1 = \tau_2 = \tau_3 = h > 0;$$

$$IX.\quad \tau_1 > 0, \tau_2 > 0, \tau_3 > 0,\quad \tau_1 \neq \tau_2 \neq \tau_3 \neq \tau_1,$$

where $h$ is a positive constant. These subcases correspond to the sufficient conditions for the asymptotic stability of the null solution ($x = 0$, $y = 0$) of the respective system.

We refer the reader to (Petrova, 2008) for the different roles of the delays and for the main reasons for the present arguments. We shall prove sufficient conditions for oscillation of the solutions of the problems (2), a) − d) and (4), a) − d), which are of the type (5) in the next two situations: case ***I***. and the present new one:

$$X.\quad \tau_1 = \tau_2 = \tau_3 = 0.$$

Then we shall make the respective conclusions for the problems (1), a) − d) as well as for (2), a) − d).

## 2. Preliminary results

We save all the assumptions from the previous section. Everything in the present one is related to Galerkin's method.

**Lemma 1.** *Let the function* $w(x,t)$ *of the type (5) be a solution of the problems (3), a) − d). Then the function* $w_1(t)$ *satisfies the equation*

$$Az''(t)+Bz(t)+Ez'(t-\tau_1)+Fz'(t-\tau_2)+Gz(t-\tau_3)=F_1(t), \tag{6}$$

*where*

$$A=\int_0^l[Mg_1(x)-\eta g_1''(x)g_1(x)]dx;\ A>0; \tag{7}$$

$$B=\int_0^l[Dg_1^{IV}(x)+pg_1''(x)]g_1(x)dx;\ B\in R; \tag{8}$$

$$E=\xi\int_0^l g_1^{IV}(x)g_1(x)dx\geq 0;\ F=\gamma\int_0^l g_1^2(x)dx\geq 0;\ G=\beta\int_0^l g_1^2(x)dx\geq 0; \tag{9}$$

$$F_1(t)=\int_0^l f(x,t)g_1(x)dx. \tag{10}$$

**Corollary 1.** *Let the function* $w(x,t)$ *of the type (5) be a solution of the problems (1), a) − d). Then the function* $w_1(t)$ *satisfies the equation*

$$Az''(t)+Bz(t)+Ez'(t-\tau_1)+Fz'(t-\tau_2)+Gz(t-\tau_3)=0, \tag{11}$$

*where* (7) − (9) *hold again.*

**Lemma 2.** *Let us suppose that (7) − (10) be fulfilled and let* $\alpha=0$. *If the function* $w(x,t)$ *of the type (5) satisfies the problems (4), a) − d), then the function* $w_1(t)$ *is a solution of the equation*

$$Az''(t)+Ez'(t-\tau_1)+Bz(t)+Fz'(t-\tau_2)+Gz(t-\tau_3)+Q_1z^3(t-\tau_3)+$$
$$Q_2z^2(t-\tau_3)z'(t-\tau_2)+Q_3\left(z'(t-\tau_2)\right)^3+\theta^*z'(t)=F_1(t), \tag{12}$$

*where we have additionally that*

$$Q_1=\alpha_{30}\int_0^l g_1^4(x)dx\geq 0;\ Q_2=\alpha_{21}\int_0^l g_1^4(x)dx\geq 0; \tag{13}$$

$$Q_3=\alpha_{03}\int_0^l g_1^4(x)dx\geq 0;\ \theta^*=\theta\int_0^l g_1^2(x)dx\geq 0. \tag{14}$$

**Corollary 2.** *Let* $\alpha=0$ *and let (7) − (9) be satisfied together with (13) and (14). If the function* $w(x,t)$ *of the type (5) is a solution of the problems (4), a) − d), then the function* $w_1(t)$ *satisfies the equation*

$$Az''(t)+Ez'(t-\tau_1)+Bz(t)+Fz'(t-\tau_2)+Gz(t-\tau_3)+Q_1z^3(t-\tau_3)+$$
$$Q_2z^2(t-\tau_3)z'(t-\tau_2)+Q_3\left(z'(t-\tau_2)\right)^3+\theta^*z'(t)=0. \tag{15}$$

**Remark 1.** *The assumption that* $\alpha=0$ *in Lemma 2 and Corollary 2 is an essential one for the applications of the results of (Petrova, 2007).*

**Remark 2.** *The assumptions that*

$$A>0\quad and\quad B>0 \tag{16}$$

*are related to applications of the results of (Petrova, 2007) again. In fact, in accordance with (7) and (8), we suppose additionally that* $B > 0$, *which is an absolutely possible situation in proper mathematical models.*

**Corollary 3.** *(see (Petrova, 2007)) Let the constants* $A$, $B$, $\theta$, $\{\beta_k\}_{k=1}^n$ *and* $\{\sigma_k\}_{k=1}^n$ *satisfy (16) and*

$$\beta_k \ge 0, \quad \sigma_k \ge 0, \quad \forall\, k = \overline{1,n}, n \in N, \quad \sigma = \max_{1\le k\le n} \sigma_k \tag{17}$$

*and* $\theta$ *be a real constant such that*

$$4AB > \theta^2 \quad and \quad L_1 = \frac{\sqrt{4AB-\theta^2}}{2A} > 0. \tag{18}$$

*If there is a number* $s \ge \rho$ *such that*

$$\int_s^{s+\pi/L_1} F(t) e^{\frac{\theta t}{2A}} \sin L_1(t-s)dt \le 0, \tag{19}$$

*then the inequality*

$$Az''(t) + \theta z'(t) + \sum_{k=1}^{n} \beta_k z(t-\sigma_k) + Bz(t) \le F(t) \tag{20}$$

*has no positive solution in* $(s-\sigma, s+\pi/L_1\,]$.

**Remark 3.** *The above Corollary remains true, if just* $s \in R$.

**Remark 4.** *If the direction of the inequality (19) is opposite, then we have an analogue of Corollary 3 about the absence of negative solution of the inequality, whose direction is the opposite one.*

For important partial cases of Corollary 3 see (Yoshida,1987) and (Yoshida,1990).
The function

$$\Phi_1(s) = \int_s^{s+\pi/L_1} F(t) e^{\frac{\theta t}{2A}} \sin L_1(t-s)dt \tag{21}$$

plays an essential role.

## 3. Main results

**Theorem 1.** *Let* $I.$, *(16) as well as (7) − (10) be fulfilled together with*

$$4AB > (E+F)^2 \quad and \quad L_2 = \frac{\sqrt{4AB-(E+F)^2}}{2A}. \tag{22}$$

*Further, let*

$$\Phi_2(s) = \int_s^{s+\pi/L_2} F_1(t) e^{\frac{(E+F)t}{2A}} \sin L_2(t-s)dt \tag{23}$$

*and let there exist a sequence* $\{s_n\}_{n=1}^\infty$ *such that*

$$\lim_{n\to\infty} s_n = \infty \quad and \quad \Phi_2(s_n) = 0. \tag{24}$$

*Then every solution* $w(x,t)$ *of the problem (3), a) − d), which is of the form (5), has at least one zero in the set* $(0,l)\times(s_n-\tau_3, s_n+\pi/L_2\,]$, $\forall\, n \in N$.

**Theorem 2.** *Let* $I.$, *(16), (7) − (10) as well as (13), (13) hold, where we suppose additionally that*

$$\alpha = \alpha_{21} = \alpha_{03} = 0.$$

*Further, let*

$$4AB > (E + F + \theta^*)^2 \quad and \quad L_3 = \frac{\sqrt{4AB - (E + F + \theta^*)^2}}{2A} \tag{25}$$

*as well as*

$$\Phi_3(s) = \int_s^{s+\pi/L_3} F_1(t) e^{\frac{(E+F+\theta^*)t}{2A}} \sin L_3(t - s) dt. \tag{26}$$

*If there exists a sequence* $\{u_n\}_{n=1}^{\infty}$ *such that*

$$\lim_{n\to\infty} u_n = \infty \quad and \quad \Phi_3(u_n) = 0, \tag{27}$$

*then every solution* $w(x,t)$ *of the problem (4), a) − d), which is of the form (5), has at least one zero in the set* $(0,l)\times(u_n - \tau_3, u_n + \pi/L_3], \forall\, n \in N$

We point out that Theorems 1 and 2 would remain true, if we replace $I.$ by $X.$, but the families of sets with at least one zero of the solution of the form (5) would be $\{(0,l)\times(s_n, s_n + \pi / L_2)\}_{n=1}^{\infty}$ and $\{(0,l)\times(u_n, u_n + \pi / L_3)\}_{n=1}^{\infty}$. Finally, all the described above distribution of the zeros remain true for the respective homogenous subcases of (3) and (4), respectively.

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