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ON THE STRUCTURE OF THE FINITE-DIMENSIONAL COMMUTATIVE SEMISIMPLE ALGEBRAS OVER ALGEBRAICALLY CLOSED FIELD AND OVER THE FIELD OF THE REAL NUMBERS

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ВЪРХУ СТРУКТУРАТА НА КРАЙНОМЕРНИТЕ КОМУТАТИВНИ ПОЛУПРОСТИ АЛГЕБРИ НАД АЛГЕБРИЧНО ЗАТВОРЕНО ПОЛЕ И НАД ПОЛЕТО НА РЕАЛНИТЕ ЧИСЛА

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Резюме. *В статията се извежда критерий кога една крайномерна комутативна полупроста алгебра над алгебрично затворено поле* F *е изоморфна като* F*-алгебра на групова алгебра* FG *на крайна абелева група* G *като така се дава частично решение на Проблем 1 на Brauer. Изследва се структурата на крайномерните комутативни полупрости алгебри над полето* R *на реалните числа. Освен това се извежда необходимо и достатъчно условие една крайномерна комутативна алгебра над полето* R *да е изоморфна като* R*-алгебра на някоя групова алгебра.*

Ключови думи*: крайномерна комутативна алгебра; групова алгебра; изоморфизъм на алгебри; реална мощност на алгебра*

1. Introduction

In the present paper we examine the structure of the finite-dimensional commutative semisimple algebras over an algebraically closed field or over the field R. We give a criterion for a finite-dimensional commutative semisimple algebra over an algebraically closed field F to be isomorphic as an F-algebra to a group algebra FG of a finite abelian group G. Thus, we give a partial solution to Brauer's Problem 1 (Brauer 1963). We consider the structure of finite-dimensional commutative semisimple algebras over the field R and we describe it up to isomorphism. We define the concept real cardinality of a commutative semisimple algebra over R and we give a necessary and sufficient condition for such algebra to be isomorphic as an R-algebra to a group algebra RG of a finite abelian group G. Moreover, we find a necessary and sufficient condition for a finite-dimensional commutative algebra over R to be isomorphic as an R-algebra to some group algebra.

If G is a finite multiplicative abelian group, then we denote $G[2]=\{g \in G \mid g^2 = 1\}$ in the whole paper.

2. Structure of a finite-dimensional commutative semisimple algebras over an algebraically closed field

In the theory of group algebras the following fact which is a partial case of the result of (May 1971) is well known:

If G and $\overline{G}$ are torsion abelian groups and F is an algebraically closed field of characteristic 0, then the group algebras FG and $F\overline{G}$ are isomorphic as F-algebras if and only if $|G| = |\overline{G}|$.

We prove the following result:

Proposition. *Let F be an algebraically closed field and A be a commutative semisimple algebra over F with $\dim_F A = n$ ($n \in N$). Then A is isomorphic as an F-algebra to the group algebra FG of the abelian group G of order n.*

Proof. To the finite-dimensional commutative semisimple F-algebra A we apply the structural theorem of Wedderburn (Gluhov, Elizarov, Nechaev 2003, Pierce 1986, Lam 2001) and we get

$$A \cong M_{n_1}(F) \oplus M_{n_2}(F) \oplus \ldots \oplus M_{n_s}(F),$$

where $n_1^2 + n_2^2 + \ldots + n_s^2 = n$. Since A is a commutative algebra, then $M_{n_i}(F)$ is a commutative algebra for each $i = 1,2,\ldots,s$. Therefore, $n_i = 1$ for $i = 1,2,\ldots,s$, which leads to

$$A \cong F \oplus F \oplus \ldots \oplus F,$$

where the number of the direct addends is n.

On the other hand, according to (Passman 2011), if G is an abelian group of order n, then

$$FG \cong F \oplus F \oplus \ldots \oplus F,$$

where the number of direct addends is equal to the order of the group G. Therefore, A is isomorphic to the group algebra FG as an F-algebra.

Using this proposition in the case when F is the field C of the complex numbers, we give a partial solution to the following Brauer's Problem 1 (Brauer 1963): what are the possible complex group algebras of finite groups?

3. Structure of finite-dimensional commutative semisimple algebras over the field of the real numbers

There are a number of researches of the infinite-dimensional commutative semisimple algebras over the field R of the real numbers. Important results for such group algebras are obtained by Berman (Berman 1967) who finds a full system of invariants of a group algebra of infinitely countable torsion abelian group over the field R. Berman and Bogdan (Berman, Bogdan 1977) generalize this result for arbitrary infinite abelian groups. The normed multiplicative group of a group algebra of an abelian p-group over the field R is described by Mollov (Mollov 1984).

In this section we will examine the structure of the finite-dimensional commutative semisimple algebras over the field of the real numbers.

Theorem 1. *Let A be a finite-dimensional commutative semisimple algebra over the field R. Then*

$$A \cong R \oplus ... \oplus R \oplus C \oplus ... \oplus C. \quad (1)$$

Proof. Let $\dim_R A = n$ ($n \in N$). According to the structural theorem of Wedderburn (Gluhov, Elizarov, Nechaev 2003, Pierce 1986, Lam 2001) applied to the semisimple algebra A we get

$$A \cong M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus ... \oplus M_{n_s}(D_s), \quad (2)$$

where $\sum_{i=1}^{s} n_i^2 \dim_R D_i = n$ and D_i are algebras with a division over R for $i = 1,2,...,s$. Since A is a commutative algebra, then $M_{n_i}(D_i)$ are commutative algebras. Therefore, $n_i = 1$ for each $i = 1,2,...,s$ and by the theorem of Frobenius (Pontryagin 1986, Pontryagin 1987) it can be deduced that $D_i = R$ or $D_i = C$ for $i = 1,2,...,s$, i.e. (1) holds.

Note 1. Obviously the cardinality of the algebra A and the number of direct addends R in (1) determine A up to isomorphism.

Definition. Let A be a commutative semisimple algebra over the field R and $\dim_R A = n$ ($n \in N$). We call the number r_A of the direct addends R in the decomposition (1) a *real cardinality of* A.

Theorem 2. *Let* A *be a finite-dimensional commutative semisimple algebra over the field* R, *and* G *be a finite abelian group. Then the algebra* A *is isomorphic as an* R*-algebra to the group algebra* RG *if and only if* $\dim_R A = |G|$ *and the real cardinality* r_A *of* A *is equal to* $|G[2]|$.

Proof. *Necessity.* Let A be isomorphic as R-algebra to the group algebra RG. Then $\dim_R A = \dim_R RG = |G|$. We shall prove that the real cardinality r_A of A (i.e. the real cardinality r_{RG} of RG) is equal to $|G[2]|$. The group algebra RG by the condition of the theorem is semisimple. Then $RG \cong \dot{\sum} RGe_\chi$, where e_χ are different minimum idempotents of RG, which correspond to the characters χ of the group G. The real cardinality r_{RG} of RG is equal to the number of those characters $\chi : G \to R^*$ with the property $g\chi = \pm 1$ for each $g \in G$. Let $G = \langle g_1 \rangle \times ... \times \langle g_s \rangle \times H$ is the decomposition of G in direct product of primary groups where $\langle g_i \rangle$ are cyclic 2-groups ($i = 1,...,s$) and 2 does not divide $|H|$, i.e. $|G[2]| = 2^s$. For the direct factor H there is a single character χ_0 with the mentioned properties, namely $h\chi_0 = 1$ for each $h \in H$. For each of the direct factors $\langle g_i \rangle$ there are two different such characters χ_{i0} and χ_{i1}, namely $g_i\chi_{i0} = 1$ and $g_i\chi_{i1} = -1$. Therefore, the number of all characters χ of G with the property $g\chi = \pm 1$ for each $g \in G$ is $2^s = |G[2]|$. Since the case $G = H$ is trivial, then the proof of the necessity is complete.

Sufficiency. Let $\dim_R A = |G|$ and the real cardinality r_A of A is equal to $|G[2]|$. In order to prove that A is isomorphic as R-algebra to the group algebra RG it is enough, according to Theorem 1, to prove that $\dim_R A = \dim_R RG$ and the real cardinalities of the two algebras are equal, i.e. $r_A = r_{RG}$. The first condition, i.e. $\dim_R A = \dim_R RG$, can be obtained from $\dim_R RG = |G|$. The second condition holds, since in the necessity we proved that $r_{RG} = |G[2]|$.

Note 2. Let G and $\overline{G}$ be finite abelian groups. We can give by using the condition of Theorem 2 the following necessary and sufficient condition for an isomorphism of the group algebras RG and $R\overline{G}$:

The group algebras RG and $R\overline{G}$ of the finite abelian groups G and $\overline{G}$ over the field R are isomorphic as R-algebras if and only if $|G| = |\overline{G}|$ and $|G[2]| = |\overline{G}[2]|$.

The last result is a partial case of the result of Berman and Bogdan (Berman, Bogdan 1977).

Theorem 3. *Let A be a finite-dimensional commutative algebra over the field R. Then A is isomorphic as an R-algebra to some group algebra over R if and only if the following conditions are satisfied:*

- *(i) A is semisimple algebra;*
- *(ii) $r_A = 2^t$, where t is non-negative integer;*
- *(iii) r_A divides $\dim_R A$.*

Proof. *Necessity.* Let A be isomorphic as an R-algebra to the group algebra RG for some group G. Since A is finite-dimensional and commutative, then G is a finite abelian group. The algebra RG by the theorem of Maschke (Pierce 1986, van der Waerden 1990, Lang 2002) is semisimple which implies that A is semisimple, i.e. (i) is fulfilled.

The equality $r_A = |G[2]|$ is fulfilled. Consequently, $r_A = 2^t$ for some non-negative integer t. In this way (ii) is proved.

Since $|G[2]|$ divides $|G|$, where $|G| = \dim_R A$, and according to Theorem 2 $r_A = |G[2]|$ holds, then r_A divides $\dim_R A$, i.e. (iii) is fulfilled. The necessity is proved.

Sufficiency. Let the conditions (i), (ii) and (iii) hold. The condition (i) and Theorem 1 imply that the decomposition (1) holds, i.e.

$$A \cong R \oplus ... \oplus R \oplus C \oplus ... \oplus C,$$

where, by (ii), the real cardinality of A is $r_A = 2^t$. We denote by $n = \dim_R A$. Let G be an arbitrary abelian group of order n whose 2-component is decomposed in direct product of t cyclic groups. The existence of such group when $t \geq 1$ is given by conditions (ii) and (iii). In the case $t = 0$ we get $n = 1 + 2c_A$, where c_A is the number of the direct addends C in the decomposition (1) of A. As n is an odd integer, then each abelian group G of order n satisfies the condition for the 2-component. When we apply Theorem 2 to A and RG we get that $A \cong RG$ as R-algebras. The proof of the sufficiency is completed.

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