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# CONTINUABILITY OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND UNLIMITED NUMBER OF IMPULSIVE EFFECTS

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# ПРОДЪЛЖИМОСТ НА РЕШЕНИЯТА НА ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ С ПРОМЕНЛИВА СТРУКТУРА И НЕОГРАНИЧЕН БРОЙ ИМПУЛСНИ ВЪЗДЕЙСТВИЯ

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***Резюме.*** *Изследват се нелинейни неавтономни системи диференциални уравнения с променлива структура и импулсни въздействия. Предполага се, че смяната на структурата и импулсните въздействия се осъществяват едновременно в така наречените превключващи моменти. Тези моменти зависят от решението, т.е. те са променливи. Определят се с помощта на предварително зададени превключващи множества, съответни на десните страни на системата. Множествата са разположени във фазовото пространство. Точно в тези превключващи моменти траекторията среща превключващите множества. Възможно е при някои решения (това означава при някои начални точки) съответните на решението превключващи моменти да притежават точка на сгъстяване. Следователно, в тези случаи решението на съответната начална задача не е продължимо до безкрайност. В работата са получени достатъчни условия за неограничено нарастване на моментите на превключване за почти всички решения, т.е за почти всички начални точки.*

***Ключови думи:*** *неавтономни нелинейни диференциални уравнения, импулсни ефекти, превключващи моменти*

The object of research in the paper is the following initial problem for nonlinear non-autonomous systems of ordinary differential equations with variable structure and impulses in non fixed moments and nonlinear switching functions:

$$\frac{dx}{dt} = f_i\left(t, x\right), \quad \varphi_i\left(x\left(t\right)\right) \neq 0, \ t_{i-1} < t < t_i, \tag{1}$$

$$\varphi_i\left(x\left(t_i\right)\right) = 0, \ i = 1, 2, \ldots, \tag{2}$$

$$x\left(t_i + 0\right) = x\left(t_i\right) + I_i\left(x\left(t_i\right)\right), \tag{3}$$

$$x\left(t_0\right) = x_0, \tag{4}$$

where the functions $f_i : R^+ \times D \to R^n$, $f_i = \left( f_i^1, f_i^2, \ldots f_i^n \right)$; the phase space $D$ is non empty domain in $R^n$; the functions $\varphi_i : D \to R$; the functions $I_i : D \to R^n$; $\left( Id + I_i \right) : D \to D$. An identity in $R^n$ is denoted by $Id$, i.e. $Id\left(x\right) = x$; an initial point $\left(t_0, x_0\right) \in R^+ \times D$, $\varphi_1\left(x_0\right) \neq 0$.

The solution of the considered initial problem is a piecewise continuous function. It is continuous on the left at each point of the interval domain, including the moments $t_1, t_2, \ldots$, named moments of switching. The functions $I_i$, $i = 1, 2, \ldots$, are impulsive and $\varphi_i$, $i = 1, 2, \ldots$, are switching functions.

The following notations are used:

- $f = \{ f_1, f_2, \ldots \}$, $\varphi = \{ \varphi_1, \varphi_2, \ldots \}$, $I = \{ I_1, I_2, \ldots \}$;
- $x\left(t; t_0, x_0\right)$ is a solution of problem (1) - (4);
- The sets $\Phi_i = \left\{ x \in D;\ \varphi_i\left(x\right) = 0 \right\}$, $i = 1, 2, \ldots$ are named switching hypersurfaces of the initial problem which is investigated;
- The function $I_0\left(x\right) = 0$ for $x \in D$;
- The function $x_i\left(t; t_0, x_0\right)$ is a solution of the problem with constant structure and without impulses;

$$\frac{dx}{dt} = f_i\left(t, x\right),\ x\left(t_0\right) = x_0,\ i = 1, 2, \ldots; \tag{5}$$

- The curve $\gamma\left(t_0, x_0\right) = \left\{ x\left(t; t_0, x_0\right),\ t \in J\left(t_0, x_0, f\right) \right\}$ is a trajectory of the studied problem, where $J\left(t_0, x_0, f\right)$ is the maximum interval of existence of this solution;
- The curve $\gamma_i\left(t_0, x_0\right) = \left\{ x_i\left(t; t_0, x_0\right),\ t \in J\left(t_0, x_0, f_i\right) \right\}$ is a trajectory of problem (5), where $J\left(t_0, x_0, f_i\right)$ is the maximum interval of existence of the solution, $i = 1, 2, \ldots$;
- $\|.\|$ and $\langle .,. \rangle$ are the Euclidean norm and the scalar product in $R^n$, respectively.

The following conditions are introduced:

H1. The functions $f_i \in C\left[R^+ \times D, R^n\right]$ and the constants $C_{f_i} > 0$ exist such that

$$\left(\ \forall (t, x) \in R^+ \times D\ \right) \Rightarrow \left\| f_i(t, x) \right\| \le C_{f_i},\ i = 1, 2, \ldots.$$

H2. The functions $\varphi_i \in C^1\left[D, R\right]$ and the constants $C_{grad\varphi_i} > 0$ exist such that

$$\left(\forall x \in D\right) \Rightarrow \left\| grad\varphi_i\left(x\right) \right\| \le C_{grad\varphi_i},\ i = 1, 2, \ldots.$$

H3. The functions $I_i \in C\left[\Phi_i, R^n\right]$ and $\left(Id + I_i\right) : \Phi_i \to D$, $i = 1, 2, \ldots$.

H4. The following inequalities are valid:

$$\varphi_i\left(\left(Id + I_{i-1}\right)(x)\right)\left\langle grad\varphi_i(x), f_i(t, x)\right\rangle < 0, (t, x) \in R^+ \times D,\ i = 1, 2, \ldots,$$

where $I_0\left(x\right) = 0,\ x \in D$.

H5. There are constants $C_{\langle grad\varphi_i, f_i \rangle} > 0$ such that

$$\left(\ \forall (t, x) \in R^+ \times D\ \right) \Rightarrow \left| \left\langle grad\varphi(x), f_i(t, x) \right\rangle \right| \ge C_{\langle grad\varphi, f_i \rangle}, i = 1, 2, \ldots.$$

H6. For any point $\left(t_0, x_0\right) \in R^+ \times D$ and for each $i = 1, 2, \ldots$, a solution of initial problem (5) exists and it is unique for $t \ge t_0$.

H7. There are constants $C_{\varphi_{i+1}\left(Id + I_i\right)} > 0$ such that

$$\left(\forall x \in \Phi_i\right) \Rightarrow \left| \varphi_{i+1}\left(\left(Id + I_i\right)\left(x\right)\right) \right| \ge C_{\varphi_{i+1}\left(Id + I_i\right)},\ i = 1, 2, \ldots.$$

H8. The series $\sum_{j=1}^{\infty} \frac{C_{\varphi_j(Id+I_{j-1})}}{C_{grad\varphi_j}.C_{f_j}}$ are divergent.

The next theorems are valid:

**Theorem 1.** *Let the conditions H1 and H6 be fulfilled. Then the solution of problem (1) - (4) exists and it is unique for any* $t \geq t_0$.

**Theorem 2.** *Let the conditions H1÷ H6 be fulfilled. Then the trajectory of problem (1) - (4) meets each one of the hypersurfaces* $\Phi_i$, $i = 1, 2, \ldots$.

**Proof.** Firstly, we show that the trajectory $\gamma_1(t_0, x_0)$ of the initial problem with fixed structure and without impulses

$$\frac{dx}{dt} = f_1(t, x), \quad x(t_0) = x_0$$

meets the hypersurface $\Phi_1$. Recall that

$$(Id + I_0)(x) = x, \quad x \in D.$$

Then, using condition H4 (for $i = 1$), it follows that one of both cases is satisfied:

Case 1. $\varphi_1(x) < 0$, $x \in D$ and $\langle grad\varphi_1(x), f_1(t, x)\rangle > 0$, $(t, x) \in R^+ \times D$;

Case 2. $\varphi_1(x) > 0$, $x \in D$ and $\langle grad\varphi_1(x), f_1(t, x)\rangle < 0$, $(t, x) \in R^+ \times D$.

We will discuss first case. The other one is considered analogously. For the convenience of recording, we introduce the function

$$\phi_1(t) = \varphi_1(x_1(t; t_0, x_0)) = \varphi_1(x_1^1(t; t_0, x_0), x_1^2(t; t_0, x_0), \ldots, x_1^n(t; t_0, x_0)),$$

which is defined for $t \in J(t_0, x_0, f_1) = [t_0, \infty)$. We have

$$\phi_1(t_0) = \varphi_1(x_1(t_0; t_0, x_0)) = \varphi_1(x_0) < 0.$$

Under condition H5, it is fulfilled

$$\begin{aligned}
\frac{d}{dt}\phi_1(t) = {} & \frac{\partial}{\partial x^1}\varphi_1(x_1(t; t_0, x_0))\frac{d}{dt}x_1^1(t; t_0, x_0) \\
& + \frac{\partial}{\partial x^2}\varphi_1(x_1(t; t_0, x_0))\frac{d}{dt}x_1^2(t; t_0, x_0) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
& + \frac{\partial}{\partial x^n}\varphi_1(x_1(t; t_0, x_0))\frac{d}{dt}x_1^n(t; t_0, x_0) \\
= {} & \frac{\partial}{\partial x^1}\varphi_1(x_1(t; t_0, x_0))\, f_1^1(t, x_1(t; t_0, x_0)) \\
& + \frac{\partial}{\partial x^2}\varphi_1(x_1(t; t_0, x_0))\, f_1^2(t, x_1(t; t_0, x_0)) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
& + \frac{\partial}{\partial x^n}\varphi_1(x_1(t; t_0, x_0))\, f_1^n(t, x_1(t; t_0, x_0)) \\
= {} & \langle grad\varphi_1(x_1(t; t_0, x_0)),\ f_1(t, x_1(t; t_0, x_0))\rangle \\
= {} & \left|\langle grad\varphi_1(x_1(t; t_0, x_0)),\ f_1(t, x_1(t; t_0, x_0))\rangle\right| \\
\geq {} & C_{\langle grad\varphi_1, f_1\rangle} = const > 0.
\end{aligned}$$

By the fact

$$\phi_1(t_0) < 0 \quad \text{and} \quad \frac{d}{dt}\phi_1(t) = const > 0, \ \ t > t_0,$$

it follows that there exists a point $t_1 > t_0$ such that

$$\varphi_1\left(x_1\left(t_1;t_0,x_0\right)\right) = \phi_1\left(t_1\right) = 0.$$

i.e. at the moment $t_1$, trajectory $\gamma_1(t_0,x_0)$ meets the hypersurface $\Phi_1$. Since

$$\gamma(t_0,x_0) \equiv \gamma_1(t_0,x_0) \ \ for \ \ t_0 \le t \le t_1,$$

we conclude that the trajectory of the problem with variable structure and impulses also meets the hypersurface $\Phi_1$ at moment $t_1$.

Assume that, the trajectory of problem (1) - (4) meets successively the hypersurfaces $\Phi_1, \Phi_2, ..., \Phi_i$ at the moments $t_1, t_2, ..., t_i$, respectively. We will show that the trajectory

$$\gamma_{i+1}\left(t_i, x\left(t_i;t_0,x_0\right) + I_i\left(x\left(t_i;t_0,x_0\right)\right)\right) \equiv \gamma_{i+1}\left(t_i, x\left(t_i+0;t_0,x_0\right)\right)$$

of the problem with fixed structure and without impulses

$$\frac{dx}{dt} = f_{i+1}(t,x), \quad x(t_i) = x\left(t_i;t_0,x_0\right) + I_i\left(x\left(t_i;t_0,x_0\right)\right) = x\left(t_i+0;t_0,x_0\right)$$

for $t \in J\left(t_i, x\left(t_i+0;t_0,x_0\right), f_{i+1}\right) = [t_i, \infty)$ meets the hypersurface $\Phi_{i+1}$. Under condition H4, the functions

$$\varphi_{i+1}\left(\left(Id + I_i\right)(x)\right) \ \text{ and } \ \left\langle grad\varphi_{i+1}(x),\, f_{i+1}(t,x)\right\rangle$$

do not cancel in their domains and for any point $(t,x) \in R^+ \times D$, they have opposite signs. Without loss of generality, we suppose that the next inequalities are valid:

$$\varphi_{i+1}\left(\left(Id+I_i\right)(x)\right) < 0, \ \ x \in D \ \text{ and } \left\langle grad\varphi_{i+1}(x), f_{i+1}(t,x)\right\rangle > 0, (t,x) \in R^+ \times D. \tag{6}$$

We consider the function $\phi_{i+1} : [t_{i,}\infty) \to R$, which is defined by the equalities

$$\phi_{i+1}(t) = \varphi_{i+1}\left(x_{i+1}\left(t;t_i, x\left(t_i;t_0,x_0\right) + I_i\left(x\left(t_i;t_0,x_0\right)\right)\right)\right) \tag{7}$$

$$= \varphi_{i+1}\left(x_{i+1}\left(t;t_i, x\left(t_i+0;t_0,x_0\right)\right)\right).$$

We have

$$\phi_{i+1}(t_i) = \varphi_{i+1}\left(x_{i+1}\left(t_i;t_i, x\left(t_i+0;t_0,x_0\right)\right)\right) \tag{8}$$

$$= \varphi_{i+1}\left(x\left(t_i+0;t_0,x_0\right)\right)$$

$$= \varphi_{i+1}\left(x\left(t_i;t_0,x_0\right) + I_i\left(x\left(t_i;t_0,x_0\right)\right)\right)$$

$$= \varphi_{i+1}\left(\left(Id+I_i\right)\left(x\left(t_i;t_0,x_0\right)\right)\right) < 0.$$

For $t > t_i$ it is satisfied

$$\frac{d}{dt}\phi_{i+1}(t) \tag{9}$$

$$= \frac{d}{dt}\varphi_{i+1}\left(x_{i+1}\left(t;t_i,x\left(t_i+0;t_0,x_0\right)\right)\right)$$

$$= \left\langle grad\varphi_{i+1}\left(x_{i+1}\left(t;t_i,x\left(t_i+0;t_0,x_0\right)\right)\right),\ f_{i+1}\left(t, x_{i+1}\left(t;t_i,x\left(t_i+0;t_0,x_0\right)\right)\right)\right\rangle$$

$$= \left|\left\langle grad\varphi_{i+1}\left(x_{i+1}\left(t;t_i,x\left(t_i+0;t_0,x_0\right)\right)\right),\ f_{i+1}\left(t, x_{i+1}\left(t;t_i,x\left(t_i+0;t_0,x_0\right)\right)\right)\right\rangle\right|$$

$$\ge C_{\langle grad\varphi_{i+1}, f_{i+1}\rangle} = const > 0.$$

From (8) and (9) it follows that there exists a point $t_{i+1} > t_i$ such that

$$\phi_{i+1}(t_{i+1}) = 0 \quad \Leftrightarrow \quad \varphi_{i+1}\left(x_{i+1}\left(t_{i+1}; t_i, x\left(t_i + 0; t_0, x_0\right)\right)\right) = 0 .$$

The last equality means that the trajectory $\gamma_{i+1}\left(t_i, x\left(t_i + 0; t_0, x_0\right)\right)$ meets the hypersurface $\Phi_{i+1}$ at the moment $t_{i+1}$. As

$$\gamma(t_0, x_0) \equiv \gamma_{i+1}\left(t_i, x\left(t_i + 0; t_0, x_0\right)\right) \textit{ for } \; t_i < t \le t_{i+1},$$

we find that the trajectory $\gamma(t_0, x_0)$ of problem (1) - (4) also meets the hypersurface $\Phi_{i+1}$. The proof follows by induction.

□

**Theorem 3.** *Let the conditions H1÷H7 be fulfilled. Then the next estimates are valid:*

$$t_{i+1} - t_i \ge \frac{C_{\varphi_{i+1}(Id+I_i)}}{C_{grad\varphi_{i+1}} . C_{f_{i+1}}}, \; i = 1, 2, \dots .$$

**Proof.** Again, without loss of generality, we suppose that the inequalities (6) are valid. For the function $\phi_{i+1}$, which is defined by (7), we have

$$\begin{aligned} &\phi_{i+1}(t_{i+1}) - \phi_{i+1}(t_i) \qquad (10)\\ &= \varphi_{i+1}\left(x_{i+1}\left(t_{i+1}; t_i, x\left(t_i + 0; t_0, x_0\right)\right)\right) - \varphi_{i+1}\left(x_{i+1}\left(t_i + 0; t_i, x\left(t_i + 0; t_0, x_0\right)\right)\right) \\ &= 0 - \varphi_{i+1}\left(x\left(t_i; t_0, x_0\right) + I_i\left(x\left(t_i; t_0, x_0\right)\right)\right) \\ &= -\varphi_{i+1}\left(\left(Id + I_i\right)\left(x\left(t_i; t_0, x_0\right)\right)\right) \\ &= \left|\varphi_{i+1}\left(\left(Id + I_i\right)\left(x\left(t_i; t_0, x_0\right)\right)\right)\right| \\ &\ge C_{\varphi_{i+1}(Id+I_i)} . \end{aligned}$$

On the other hand, there exists a point $\tau$, where $t_i < \tau < t_{i+1}$, such that

$$\begin{aligned} &\phi_{i+1}(t_{i+1}) - \phi_{i+1}(t_i) \\ &= \frac{d}{dt}\phi(\tau)(t_{i+1} - t_i) = \frac{d}{dt}\varphi_{i+1}\left(x\left(\tau; t_0, x_0\right)\right).(t_{i+1} - t_i) \\ &= \left(\frac{\partial}{\partial x_1}\varphi_{i+1}\left(x\left(\tau; t_0, x_0\right)\right) f_{i+1}^1\left(\tau, x\left(\tau; t_0, x_0\right)\right)\right. \\ &\quad + \frac{\partial}{\partial x_2}\varphi_{i+1}\left(x\left(\tau; t_0, x_0\right)\right) f_{i+1}^2\left(\tau, x\left(\tau; t_0, x_0\right)\right) \\ &\quad + \dots\dots + \\ &\quad \left. + \frac{\partial}{\partial x_n}\varphi_{i+1}\left(x\left(\tau; t_0, x_0\right)\right) f_{i+1}^n\left(\tau, x\left(\tau; t_0, x_0\right)\right)\right).(t_{i+1} - t_i) \\ &= \left\langle grad\varphi_{i+1}\left(x\left(\tau; t_0, x_0\right)\right), \; f_{i+1}\left(\tau, x\left(\tau; t_0, x_0\right)\right)\right\rangle.(t_{i+1} - t_i) \\ &\le \left\|grad\varphi_{i+1}\left(x\left(\tau; t_0, x_0\right)\right)\right\|.\left\|f_{i+1}\left(\tau, x\left(\tau; t_0, x_0\right)\right)\right\|.(t_{i+1} - t_i) \\ &\le C_{grad\varphi_{i+1}} . C_{f_{i+1}} . (t_{i+1} - t_i). \end{aligned}$$

From the inequality above we obtain

$$t_{i+1} - t_i \ge \frac{1}{C_{grad\varphi_{i+1}} . C_{f_{i+1}}}\left(\phi(t_{i+1}) - \phi(t_i)\right),$$

from where, bearing in mind inequality (10), we obtain the required estimate.

□

**Theorem 4.** *Let the conditions H1÷H8 be fulfilled. Then the solution of problem (1) - (4) is defined for* $t \geq t_0$.

**Proof.** The statement follows from the fact that

$$\lim_{i\to\infty} t_i = \lim_{i\to\infty}\left(\left(t_i - t_{i-1}\right) + \left(t_{i-1} - t_{i-2}\right) + ... + \left(t_1 - t_0\right) + t_0\right)$$
$$\geq \lim_{i\to\infty}\sum_{j=1}^{i}\Delta_j + t_0 = \sum_{j=1}^{\infty}\Delta_j + t_0$$
$$= \sum_{j=1}^{\infty}\frac{C_{\varphi_j\left(Id+I_{j-1}\right)}}{C_{grad\varphi_j}.C_{f_j}} + t_0 = \infty.$$

□

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