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ISOMORPHISM OF COMMUTATIVE GROUP ALGEBRAS OF FINITE ABELIAN P-GROUPS

Todor Zh. Mollov and Nako A. Nachev

Abstract. Let R be a direct product of commutative indecomposable rings with identities and let G be a finite abelian p-group. In the present paper we give a complete system of invariants of the group algebra RG of G over R when p is an invertible element in R. These investigations extend some classical results of Berman (Zbl 0050.25504 and Zbl 0080.02102), Sehgal (Zbl 0209.05804) and Karpilovsky (Zbl 0526.20004) as well as a result of Mollov (Zbl 0655.16004).

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1. Introduction

Let RG be the group algebra of an abelian group G over a commutative ring R with identity. Denote by G_p the p-component of G and by tG the torsion subgroup of G.

The investigation of the unit group of the group algebra RG represents a high interest. However, the question for the isomorphism of two group algebras as R-algebras is more important. It can be formulated in the following way: if G is a group, H is any group and R is a ring with identity then find necessary and sufficient conditions for the isomorphism $RG \cong RH$ as R-algebras, that is find a full system of invariants of RG in the terms of R and G which determines RG up to isomorphism. Further we will not say always that the considered above isomorphism of algebras is an isomorphism of R-algebras.

The important of the isomorphism problem is underlined in many algebraic forums. This problem is raised in 1947 year in the Michigan algebraic conference from R. Trell. More specially for the group algebras of the crystallographic groups this problem is raised in the session of AMS in 1979 year in Washington from Farkash [13].

The investigation of the isomorphism problem begins with the unpublished thesis of Higman [15] who proves that if G and H are arbitrary abelian groups, then $\mathbb{Z}H \cong \mathbb{Z}G$ implies $H \cong G$, that is the group algebra $\mathbb{Z}G$ determines the isomorphism class of G. The first proof of this result which appears in the literature gives Cohn [10].

For infinite abelian p-groups we note the fundamental article of Berman [3] which solves the isomorphism problem when G is a countable abelian p-group. May [21] solves this problem for an arbitrary abelian group G if K is an algebraic closed field such that the characteristic of K does not divide the orders of the torsion elements of G and shows that a full system of invariants of RG is $|G/tG|$ and $|tG|$. Berman and Bogdan [4] find a full system of invariants of the group algebra $\mathbb{R}G$ of an abelian group G.

Let ε_i be a primitive p^i-root of the identity. Berman and Mollov [6] give a full system of invariants of the group algebra KG, when K is a field of characteristic different from the prime p and either (i) G is an abelian p-group and the first Ulm factor G/G^1 is a direct sum of cyclic groups or (ii) G is p-mixed and the degree $(K(\varepsilon_1, \varepsilon_2, \dots) : K) < \infty$, that is K is a field of the second kind with respect to p. Nachev and Mollov [30] give a new simple form of the above result (i). Ullery [33] call R-favourable an abelian group G if whenever a prime p is a unit in R, then $G_p = 1$ and proves that if the ring R is indecomposable of characteristic 0 and G is a R-favoarable group then the group algebra RG defines the isomorphic class of G. Let $\mathbb{Z}[1/p]$ be the ring of every rationals which denominators are degrees of a prime p. Nachev [27] proves that if G and H are arbitrary abelian p-groups, then $\mathbb{Z}[1/p]G \cong \mathbb{Z}[1/p]H$ as $\mathbb{Z}[1/p]$-algebras if and only if $\mathbb{Q}G \cong \mathbb{Q}H$ as $\mathbb{Q}$-algebras. This theorem implies that the solution of the indicated conjecture is reduced to the isomorphism over the field $\mathbb{Q}$ of the rationals. Nachev [28] considers special field K of characteristic 0 which is called primarily neat field and proves that if G is a torsion 2-divisible abelian group and H is an arbitrary group, then the isomorphism $KG \cong KH$ as K-algebras holds if and only if H is a torsion 2-divisible abelian group and the K-algebra isomorphism $KG_p \cong KH_p$ holds for every prime integer p. This result is an analogue of a theorem of Perlis and Walker [31].

We recall that the group algebra RG is called modular if the characteristic of the ring R is a prime number p. In connection with the isomorphism problem for modular group algebras there exists the following wide known conjecture from "Group rings" of Zalesky and Mikhalev [34, conjecture 9.4, page 61]: if K is a field of prime characteristic p, G is a p-group and H is an arbitrary group, then $KG \cong KH$ as K-algebras if and only if $G \cong H$, that is the group algebra RG determines the isomorphism class of G. This conjecture is not proved even in a partial case when G is an abelian p-group. In this direction we mark the paper [5] of Berman and Mollov which solves the isomorphism problem, that is the above conjecture, for the group algebra RG when G is a direct sum of cyclic p-groups and R is a ring of a prime characteristic p. We note also that W. May [20] shows that if R is a ring of a prime characteristic p and G and H are arbitrary groups groups, then the isomorphism $RG \cong RH$ as R-algebras implies that (i) G_p and H_p have the same Ulm-Kaplansky invariants, (ii) $dG_p \cong dH_p$ and (iii) $G/tG \cong H/tH$ where dG_p is the maximal divisible subgroup of G_p. We mark that the results (i) and (ii) are obtained independently from Berman and Mollov [5], when $G = G_p$. In the paper [22] of May the isomorphism problem is solved for modular group algrebra RG when G is a p-local Warfield group and in particular case when G is a simply presented abelian p-group.

There are few results for the isomorphism problem for modular group algebras over a field K of prime characteristic p when G is a mixed abelian group such that the torsion subgroup tG of G is a p-group, i.e. G is p-mixed. In this direction we note that Hill and Ullery [16] solve the isomorphism problem when $r_0(G) = 1$ and tG is a simply presented p-group with $l(G) < \Omega+\omega$, where $r_0(G)$ is the torsion free rank of G, lG is the lenght of G, ω is the first infinite ordinal and Ω is the first uncountable ordinal. We note also the following corollaries of Hill and Ullery [16, Corollaries 5.7 and 5.8], which are slightly modified. Let tG be a simply presented p-group and either (i) $l(tG) < \Omega$ or (ii) $r_0(G)$ is countable and $l(tG) < \Omega + \omega$. If H is any group such that $KG \cong KH$ as K-algebras, then $tG \cong tH$ and $G \times T \cong H \times T$ for some simply presented p-group T. In the paper [23] of May, Mollov and Nachev an abelian group G is called special p-mixed if G is p-mixed and there exists a subgroup F of G such that $F \supseteq tG$ and the following conditions are fulfilled: (i) F/tG is a free subgroup of G/tG, (ii) $G/F = t(G/F)$ and (iii) $(G/F)_p$ is identity. In the indicated paper [23] May, Mollov and Nachev decide the isomorphism problem for a field of a prime characteristic p when G is a special p-mixed abelian group and G_p is a reduced totally projective p-group.

For the isomorphism problem we shall consider specially the case when G is a finite abelian group. Jennings [17], Deskins [12] and Coleman [11] prove that if G is a finite abelian p-group and F is a field of characteristic p, then the group algebra FG determines the isomorphism class of G. We note that now this result is known as theorem of Deskins. Perlis and Walker [31] prove that if G is a finite abelian group and F is a field such that the characteristic of F does not divide the order of G then for any group H it holds (i) $FH \cong FG$ if and only if $FH_p \cong FG_p$ for every prime p and (ii) $\mathbb{Q}H \cong \mathbb{Q}G$ if and only if $H \cong G$. Note that the proof of the result (i) is incorrect and it is corrected in addition in the papers of Cohen [9], Bautista [8] and Berman and Rossa [7]. If G is a finite abelian p-group and F is a field such that the characteristic of F is different from p, then Berman [1, 2] finds a complete system of invariants of the group algebra FG in the terms of F-conjugate classes of the group G. Sehgal [32] proves that if G is a finite abelian p-group and $\mathbb{Q}_p$ is the field of the p-adic numbers, then the group algebra $\mathbb{Q}_pG$ determines the isomorphism class of G. If G is a finite abelian group and F is a field, then Karpilovsky [18 and 19, Theorem12.31] generalizes the above result of Berman giving a full system of invariants of the group algebra FG in the terms of F-conjugate classes of the group G. If G is a finite abelian group and F is a field, such that the characteristic of F does not divide the order of G, then Mollov [25] finds a complete system of invariants of the group algebra FG in the terms of the group G and the field F, using the concept a spectrum of F with respect to an arbitrary prime number p (see Mollov [24]).

In the present paper we give a full system of invariants of the group algebra RG when G is a finite abelian p-group, R is a direct product of commutative indecomposable rings with identity and p is an invertible element in R.

2. Some concepts and preliminary results

Let G be an abelian group and let R be a commutative ring with identity. Denote by $U(RG)$ the multiplicative group of RG, by R^* the multiplicative group of R and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$.

Recall some well known definitions. Namely, a ring R is called indecomposable, if it cannot be decomposed into a direct sum of two or more nontrivial ideals of R. This definition is equivalent to the following condition: the ring R does not have nontrivial idempotents (different from 0 and 1). We say that an abelian group G has *exponent* n if $G^n = 1$ and n is the least natural with this property.

Let G be a finite abelian group of exponent n and $n \in R^*$, that is n is an invertible element in R. For every divisor d of n we denote $\lambda(d) = \mu(d)\nu(d)$, where $\mu(d)$ is the number of the cyclic subgroups of G of order d and $\nu(d)$ is the number of the monic irreducible divisor of the cyclotomic polynomial $\Phi_d(x)$

over R. Then the number $\lambda(d)$, where d/n, is called *a d-number of the group algebra RG* (see Moloov and Nachev [26])

We shall use the signs $\dot{\sum}$ for a mark of direct sums of algebras. Let $\coprod_n G$ denote the coproduct of n copies of G, where $n \in \mathbb{N}$. If $n,\ m \in \mathbb{Z}$, then n/m will denote n divides m.

The abelian group terminology is in agreement with the books [14] of Fuchs.

If α is an algebraic element over R, α is a root of the polynomial $f(x) \in R[x]$ and $f(x)$ is a polynomial of the least degree with this property, then $f(x)$ is called *a minimal polynomial of α over R*. An algebraic element α over R is called *an integral algebraic element over R* if there exists a minimal polynomial of α over R which is monic. This definition differs from the definition in the theory of the algebraic numbers and in the field theory since an element α can be a root of a monic polynomial over R and α can have not a minimal polynomial over R which is monic.

Let R_i be a commutative indecomposable ring with identity. We denote by ε_d *an integral algebraic element over R_i which is a root of a monic irreducible divisor of $\Phi_d(x)$ over R_i*, that is ε_d is a root of a monic irreducible divisor $\varphi(x)$ of $\Phi_d(x)$ such that $\varphi(x)$ is the monic minimal polynomial of ε_d over R_i (see Mollov and Nachev [26]).

The following result for group algebras of finite abelian groups, which generalizes results of Berman [2] and Perlis and Walker [31], is proved by Mollov and Nachev [26].

Theorem 2.1. *Let G be a finite abelian group of exponent n and let R be a commutative ring with identity which is a direct product of m indecomposable rings R_i (e.g. R is a noetherian), $m,\ i \in \mathbb{N}$.*

Then

$$RG \cong \dot{\sum_{i=1}^{m}} R_i G.$$

If n is an invertible element in R, then

$$R_i G \cong \dot{\sum_{d/n}} \lambda_i(d) R_i[\varepsilon_d],$$

where $\lambda_i(d)$ is a d-number of the group ring R_iG.

Therefore,

$$U(R_i G) \cong \coprod_{d/n} \coprod_{\lambda_i(d)} R_i[\varepsilon_d]^*.$$

The following statement is a direct corollary from the result of May [20]. However, for the completeness of the presentation we shall give its proof.

Proposition 2.2. *Let G be a finite abelian p-group and let R be a commutative ring with identity such that p is not invertible in R. If H is any group then $RH \cong RG$ as R-algebras if and only if $H \cong G$.*

Proof. Let $RH \cong RG$ as R-algebras for some group H. Obviously, H is a finite abelian p-group, since RH is a commutative algebra and $|H| = |G|$. We have, by result of May [20], that the corresponding Ulm-Kaplansky invariants of H and G coincide. Since H and G are finite abelian p-groups, then $H \cong G$.

Conversely, if $H \cong G$, then obviously $RH \cong RG$ as R-algebras.

□

The following assertion which generalizes the result of Deskins is a direct corollary of Proposition 2.2. We shall give an independent proof of this statement.

Corollary 2.3. *Let G be a finite abelian p-group and let R be a commutative ring with identity of characteristic p. If H is any group then $RH \cong RG$ as R-algebras if and only if $H \cong G$.*

Proof. Let $RH \cong RG$ as R-algebras. Then, as in Proposition 2.2, we see that H is a finite abelian p-group. We have, by a well known formula, $R^{p^n}H^{p^n} \cong R^{p^n}G^{p^n}$ for every $n \in \mathbb{N}_0$. Hence, $|H^{p^n}| = |G^{p^n}|$ holds for every $n \in \mathbb{N}_0$ and as it is not hard to see, $H \cong G$ is fulfilled.

□

3. Main results

In this section we give a complete system of invariants of the group algebra RG when G is a finite abelian p-group and R is a commutative ring with identity which is a direct product of indecomposable rings such that p is an invertible element in R.

Let R_i, $i \in I$, be a system of rings and let G be an arbitrary group. If $a \in \left(\prod_{i\in I} R_i\right) G$, then

$$a = \sum_{g\in G_a} a_g g, \quad a_g \in \prod_{i\in I} R_i, \tag{3.1}$$

where G_a is a finite subset of G. We note that G_a and the system $\{a_g | g \in G_a\}$ are defined identically from the element a. Besides, $a_g = (\dots, a_{gi}, \dots)$ where $a_{gi} \in R_i$ for every $i \in I$ and every $g \in G_a$. We define a map

$$(3.2) \qquad \varphi : \left(\prod_{i\in I} R_i\right) G \to \prod_{i\in I}(R_i G)$$

by

$$(3.3) \qquad \varphi(a) = \Big(\dots , \sum_{g\in G_a} a_{gi} g \, , \dots \Big).$$

It is not hard to see that φ is a natural injective homomorphism of R-algebras.

The following result is announced in the Dissertation of Nachev [29].

Proposition 3.1. *The homomorphism (3.2) of R-algebras, defined by equality (3.3), is an isomorphism of R-algebras if and only if either I is a finite set or G is a finite group.*

Proof. We shall prove that if either I is a finite set or G is a finite group, then φ is a surjective homomorphism, that is φ is an isomorphism of R-algebras.

Let I be a finite set, $I = \{1, 2, \dots , n\}$ and let $z \in \prod\limits_{i\in I}(R_i G)$. Then

$$z = (z_1, z_2, \dots , z_n), \quad z_i \in R_i G, \quad i = 1, 2, \dots , n.$$

We put $G_z = \bigcup_{i=1}^n G_i$, where G_i is the support of z_i. Then

$$(3.4) \qquad z_i = \sum_{g\in G_z} z_{gi} g, \quad z_{gi} \in R_i, \quad i = 1, 2, \dots , n,$$

where $z_{gi} = 0$ if $g \notin G_i$. Formula (3.4) has a sense, since G_z is a finite set of G. Now we put

$$x = \sum_{g\in G_z} z_g g, \quad z_g = (z_{g1}, z_{g2}, \dots , z_{gn}).$$

Then, by (3.3), we have $\varphi(x) = z$, that is φ is a surjective homomorphism.

Now let G be a finite group and let again $z \in \prod\limits_{i\in I}(R_i G)$. Then we have

$$z = (\dots , z_i \, , \dots), \quad z_i \in R_i G.$$

Since G is finite, then

$$z_i = \sum_{g\in G} z_{gi} g, \quad z_{gi} \in R_i, \quad i \in I.$$

If we put

$$y = \sum_{g \in G} z_g g, \quad z_g = (\,\dots\,, z_{gi}, \,\dots), $$

then (3.3) implies $\varphi(y) = z$, that is φ is a surjective homomorphism.

In the end we shall prove that if I and G are infinite, then the homomorphism φ is not surjective. Indeed, then I can be represented in the kind $I = I_1 \bigcup I_2$ where I_1 is countably infinite and I_2 is its complement in I. We choose also an infinite countable subset M of G which elements g_i are indexed by naturals. Now we consider the element $r = (\,\dots\,, r_i, \,\dots) \in \prod_{i \in I} (R_i G)$ which is constructed by the following way:

$$r_i = \begin{cases} g_i, & \text{if} \quad i \in I_1; \\ 0, & \text{if} \quad i \in I_2. \end{cases}$$

We shall prove that r does not have preimage. Suppose the contrary. Let $\varphi(a) = r$. Then a has a form (3.1). Since G_a is finite, then there exists an element $g_i \in M$ such that $g_i \notin G_a$. Then, by (3.3), for the indicated i we have

$$\sum_{g \in G_a} a_{gi} g = g_i.$$

In the right part of this equality the element g_i participates with a coefficient 1 and in the left part this coefficient is 0, since $g_a \notin G_a$, which is a contradiction.

Therefore, the homomorphism φ is not surjective.

□

For every $i \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ we shall use also the designation $\varepsilon_{p^i} = \varepsilon(i)$. As in the paper of Mollov [24] we give the following definition.

Definition. We define *a spectrum* $s_p(R)$ *of the ring* R *with respect to the prime* p by the following way:

$$s_p(R) = \Big\{ i \in \mathbb{N}_0 \Big| R[\varepsilon(i)] \neq R[\varepsilon(i+1)] \Big\}.$$

If $s_p(R) \neq \varnothing$, then we put $s_p(R) = \{i_1, i_2, \,\dots \mid i_1 < i_2 < \,\dots\}$.

Further G will be a finite non-identity abelian p-group of an exponent p^n, $n \in \mathbb{N}$, R a commutative ring with identity and p an invertible element in R.

Lemma 3.2. *Exactly one of the following conditions holds:*
(a) *either* $s_p(R) = \varnothing$ *or* $s_p(R) \neq \varnothing$ *but* $n \leq i_1$*;*
(b) $s_p(R) \neq \varnothing$ *and there is* $s \in \mathbb{N}$*, such that* $i_s < n \leq i_{s+1}$*;*
(c) $s_p(R) \neq \varnothing, s_p(R) = \{i_1, i_2, \dots, i_s\}$ *is a finite set and* $n > i_s$*.*

Proof. Suppose that the condition (a) is not fulfilled. Then $s_p(R) \neq \varnothing$ and $n > i_1$. Therefore, either (i) there exists $s \in \mathbb{N}$, such that $i_s < n \leq i_{s+1}$, that is the case (b) holds or (ii) s with the indicated property does not exist. In the case (ii) $s_p(R) = \{i_1, i_2, \dots, i_s\}$ is a finite set and $n > i_s$, that is the case (c) holds.

□

Definition. We define *a spectrum* $s_G(R)$ *of the ring* R *with respect to the group* G by the following way:

(a) if either (i) $s_p(R) = \varnothing$ or (ii) $s_p(R) \neq \varnothing$ but $n \leq i_1$, then we put $s_G(R) = \{-1\}$;

(b) if $s_p(R) \neq \varnothing$ and there is $s \in \mathbb{N}$ such that $i_s < n \leq i_{s+1}$, then we put $s_G(R) = \{i_1, i_2, \dots, i_s, i_{s+1}\}$;

(c) if $s_p(R) \neq \varnothing, s_p(R) = \{i_1, i_2, \dots, i_s\}$ is a finite set and $n > i_s$, then we put $s_G(R) = \{i_1, i_2, \dots, i_s, \infty\}$.

Lemma 3.3. *If* $R[\varepsilon(i)] \neq R[\varepsilon(i+1)]$*, then* $R[\varepsilon(i)]$ *is not isomorphic of* $R[\varepsilon(i+1)]$ *as* R*-algebras.*

Proof. Suppose the contrary that $R[\varepsilon(i)] \cong R[\varepsilon(i+1)]$. Therefore, there exists an isomorphism $\varphi : R[\varepsilon(i+1)] \to R[\varepsilon(i)]$. Let $f(x)$ be the minimal polynomial of $\varepsilon(i+1)$ over R. By the above isomorphism $\varepsilon(i+1)$ will be mapped in an element $\lambda \in R[\varepsilon(i)]$ and λ is a root of $f(x)$. However, $f(x)$ does not have a root in $R[\varepsilon(i)]$. This is a contradiction.

□

If $k \in \mathbb{N}$, then further we denote $kR = R \oplus \dots \oplus R$, where the number of the addends is k. In the following theorem $\lambda(p^i)$, $i = 0, 1, \dots, n$, are the d-numbers of the group algebra RG when G has an exponent p^n.

Theorem 3.4. *Let* G *be a non-identity finite abelian* p*-group of an exponent* p^n*,* R *a commutative indecomposable ring with identity and* p *an invertible element in* R*. Then*

(a) *if* $s_G(R) = \{-1\}$*, then*

$$RG \cong \theta(-1)R, \quad \theta(-1) = \lambda(1) + \lambda(p) + \dots + \lambda(p^n); \tag{3.5}$$

(*b*) *if* $s_G(R) = \{i_1,\, i_2,\, \dots,\, i_s,\, i_{s+1}\}$, *then*

$$RG \cong \theta(i_1)R[\varepsilon(i_1)] \,\oplus\, \theta(i_2)R[\varepsilon(i_2)] \,\oplus\, \dots \,\oplus\, \theta(i_{s+1})R[\varepsilon(i_{s+1})], \tag{3.6}$$

where

$$\begin{aligned} &\theta(i_1) = \lambda(1) + \lambda(p) + \dots + \lambda(p^{i_1}), \\ &\theta(i_k) = \lambda(p^{i_{k-1}+1}) + \lambda(p^{i_{k-1}+2}) + \dots + \lambda(p^{i_k}), \quad k = 2, 3, \dots, s, \\ &\theta(i_{s+1}) = \lambda(p^{i_s+1}) + \lambda(p^{i_s+2}) + \dots + \lambda(p^n); \end{aligned} \tag{3.7}$$

(*c*) *if* $s_G(R) = \{i_1,\, i_2,\, \dots,\, i_s,\, \infty\}$, *then*

$$RG \cong \theta(i_1)R[\varepsilon(i_1)] \,\oplus\, \dots \,\oplus\, \theta(i_s)R[\varepsilon(i_s)] \,\oplus\, \theta(i_s+1)R[\varepsilon(i_s+1)], \tag{3.8}$$

where $\theta(i_1), \dots, \theta(i_s), \theta(i_s+1)$ *coincide with the numbers* $\theta(i_1), \dots, \theta(i_s), \theta(i_{s+1})$ *from (3.7), respectively.*

For the group algebra RG *exactly one of the indicated cases* (*a*), (*b*) *and* (*c*) *holds.*

Proof. Since p is an invertible element in R, then Theorem 2.1 implies

$$RG \cong \lambda(1)R[\varepsilon_1] \,\oplus\, \lambda(p)R[\varepsilon_p] \,\oplus\, \dots \,\oplus\, \lambda(p^n)R[\varepsilon_{p^n}]. \tag{3.9}$$

(*a*) Let $s_G(R) = \{-1\}$. Then either (*i*) $s_p(R) = \varnothing$ or (*ii*) $s_p(R) \neq \varnothing$ but $n \leq i_1$. In the case (*i*) $R = R[\varepsilon(1)] = R[\varepsilon(2)] = \dots$ hold and in the case (*ii*) $R = R[\varepsilon(1)] = \cdots = R[\varepsilon(i_1)]$ are fulfilled. Therefore, in the cases (*i*) and (*ii*) we have $R = R[\varepsilon(1)] = \cdots = R[\varepsilon(n)]$ and (3.9) obtains the form (3.5).

(*b*) Let $s_G(R) = \{i_1, i_2, \dots, i_s, i_{s+1}\}$. Then $s_p(R) \neq \varnothing$, $i_s < n \leq i_{s+1}$ and (3.9), by Lemma 3.3, obtains the form (3.6), where the numbers $\theta(j)$ are determined by (3.7).

(*c*) Let $s_G(R) = \{i_1, i_2, \dots, i_s, \infty\}$. Then $s_p(R) \neq \varnothing$, $s_p(R) = \{i_1, i_2, \dots, i_s\}$ is a finite set and $n > i_s$. Hence, by Lemma 3.3, $R[\varepsilon(s)] \neq R[\varepsilon(s+1)] = R[\varepsilon(s+2)] = \dots$ and the decomposition (3.9), by Lemma 3.3, obtains the form (3.8), where the numbers $\theta(j)$ are indicated in the case (*c*) of the theorem.

Lemma 3.2 implies that for the group algebra RG exactly one of the indicated cases (*a*), (*b*) and (*c*) holds.

□

We note that the proved theorem for group algebras over rings continues results of Berman [2], Perlis and Walker [31] and Mollov [25] giving the number of the repetitions of the addends in the decomposition of the group algebra RG in a direct sum of extensions of the ring R.

By using of the cases (a), (b) and (c) in the formulation of Theorem 3.4 we can give the following definition.

Definition. Let G be a finite non-identity abelian p-group of an exponent p^n, R a commutative indecomposable ring with identity and p an invertible element in R. *Characteristic numbers* $\theta(j)$ *of the ring* R *with respect to the group* G are called the following numbers:

(a) $\theta(-1) = \lambda(1) + \lambda(p) + \ldots + \lambda(p^n)$, if $s_G(R) = \{-1\}$;

(b) $\theta(i_1), \theta(i_2), \ldots, \theta(i_{s+1})$ from formula (3.7), if $s_G(R) = \{i_1, i_2, \ldots, i_s, i_{s+1}\}$;

(c) $\theta(i_1), \theta(i_2), \ldots, \theta(i_s)$ from formula (3.7) and $\theta(i_s + 1) = \lambda(p^{i_s+1}) + \lambda(p^{i_s+2}) + \ldots + \lambda(p^n)$, if $s_G(R) = \{i_1, i_2, \ldots, i_s, \infty\}$.

In the following result we denote by $s_H(R)$ the spectrum of the ring R with respect to a finite abelian p-group H.

Theorem 3.5. *Suppose G is a finite non-identity abelian p-group, R is a commutative indecomposable ring with identity and p is an invertible element in R. If H is any group, then $RH \cong RG$ as R-algebras if and only if H is a finite abelian p-group such that (i) $s_H(R) = s_G(R)$ and (ii) $\theta'(j) = \theta(j)$ for every j, where $\theta'(j)$ are the characteristic numbers of the ring R with respect to H.*

Proof. Necessity. Let $RH \cong RG$ as R-algebras for some group H. Suppose that the finite abelian p-group G has an exponent p^n. Obviously, H is a finite abelian p-group of some exponent $p^{n'}$, since RH is a commutative algebra and $|H| = |G|$. It is not hard to see, because of the decomposition of $RH \cong RG$ as R-algebras, giving in Theorem 3.4 and Lemma 3.3, that if for $s_G(R)$ one of the conditions (a), (b) or (c) of the definition of $s_G(R)$ holds, then for $s_H(R)$ the same corresponding condition is fulfilled. Therefore, $s_H(R) = s_G(R)$, that is the condition (i) holds. Since RH and RG have the same corresponding decompositions from the kind (3.5), (3.6) and (3.8) with the same corresponding conditions for $\theta'(j)$ and $\theta(j)$ (in the expressing of $\theta'(j)$ participate the d-numbers $\lambda'(p^i)$ of RH and n'), then $RH \cong RG$ as R-algebras implies $\theta'(j) = \theta(j)$ for every j, that is the condition (ii) holds.

Sufficiency. Let G and H be finite abelian p-groups of exponents p^n and $p^{n'}$, respectively, and let the conditions (i) and (ii) of the theorem are fulfilled. Then, by Theorem 3.4, RH and RG have the same corresponding decompositions from the kind (3.5), (3.6) and (3.8) with the same corresponding conditions for $\theta'(j)$ and $\theta(j)$. This decompositions and $\theta'(j) = \theta(j)$ implies $RH \cong RG$ as R-algebras. The theorem is proved.

□

The following theorem gives a complete solution of the isomorphism problem for group algebras of finite abelian p-groups over a commutative ring R with identity which is a direct product of indecomposable rings and p is an invertible element in R.

Theorem 3.6. *Suppose G is a non-identity finite abelian p-group, $R = \prod_{i\in I} R_i$, every R_i is a commutative indecomposable ring with identity and p is an invertible element in R. If H is any group, then $RH \cong RG$ as R-algebras if and only if H is a finite abelian p-group such that (i) $s_H(R_i) = s_G(R_i)$ for every $i \in I$ and (ii) $\theta'_i(j) = \theta_i(j)$ for every $i \in I$ and every j, where $\theta'_i(j)$ and $\theta_i(j)$ are the characteristic numbers of the ring R_i with respect to H and G, respectively.*

Proof. If (a) $RH \cong RG$ as R-algebras holds, then, as in Theorem 3.5, we see that H is a finite abelian p-group. Therefore, we shall suppose this condition for H. Besides the condition (a) we consider the following conditions: (b) $\prod_{i\in I}(R_iH) \cong \prod_{i\in I}(R_iG)$ as R-algebras; (c) $R_iH \cong R_iG$ as R_i-algebras for every $i \in I$ and (d) the conditions (i) and (ii) of the theorem are fulfilled. The condition (a), by Proposition 3.1, is equivalent to the condition (b), (b) and (c) are obviously equivalent and (c) and (d) are equivalent by Theorem 3.5. Consequently, (a) and (d) are equivalent. The proof is completed.

□

The proved theorem extends some classical results of Berman [1, 2], Sehgal [32] and Karpilovsky [18] as well as a result of Mollov [25].

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Department of Algebra,
University of Plovdiv
4000 Plovdiv, Bulgaria
e-mail: mollov@uni-plovdiv.bg, nachev@uni-plovdiv.bg

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ИЗОМОРФИЗЪМ НА КОМУТАТИВНИ ГРУПОВИ АЛГЕБРИ НА КРАЙНИ АБЕЛЕВИ P-ГРУПИ

Тодор Ж. Моллов и Нако А. Начев

Резюме. Нека R е директно произведение на комутативни неразложими пръстени с единици и G е крайна абелева p-група. В представената статия се дава пълна система инварианти на груповата алгебра RG на G над R, когато p е обратим елемент в R.

Тези изследвания продължават някои класически резултати на Берман (Zbl 0050.25504 и Zbl 0080.02102), Сегал (Zbl 0209.05804) и Карпиловски (Zbl 0526.20004), както и един резултат на Моллов (Zbl 0655.16004).