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EXISTENCE OF $L_P(\varphi, \psi)$-SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH GENERALIZED DICHOTOMY IN A BANACH SPACE

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Abstract. A generalization of the well known dichotomies for a class of homogeneous linear differential equations in an arbitrary Banach space is used. By the help of them are found sufficient conditions for the existence of $L_p(\varphi, \psi)$-solutions of the nonhomogeneous equation.

Key words: Ordinary Differential Equations, Generalized Dichotomy, $L_p(\varphi, \psi)$-solutions
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1. Introduction

The notion of exponential and ordinary dichotomy is fundamental in the qualitative theory of ordinary differential equations. It is considered in detail for example in the monographs [1]-[2],[7]-[9]. In the given paper we use a (M, N, R) dichotomy, introduced in [6] and considered in [4],[5], which is a generalization of all dichotomies known by the authors.

We study the existence of solutions in the spaces $L_p(\varphi, \psi)$ of nonhomogeneous linear differential equations with generalized dichotomy of the corresponding homogeneous equation. In the paper are found sufficient conditions for the existence of such $L_p(\varphi, \psi)$-solutions.

2. Problem statement

Let X is an arbitrary Banach space with norm $|.|$ and identity I and let $\mathbb{R}_+ = [0, \infty)$. Let $L(X)$ is the space of all linear bounded operators acting in X with the norm $||.||$.

We consider the nonhomogeneous linear equation

$$\frac{du}{dt} = A(t)u + f(t) \tag{1}$$

where $A(t) \in L(X), t \in \mathbb{R}_+$.

By $V(t)$ we will denote the Cauchy operator of

$$\frac{du}{dt} = A(t)u \tag{2}$$

In this paper we will use the (M, N, R)-dichotomy, introduced in [4] with following properties and definitions.

Let $R(t) : X \to X \quad (t \in J)$ is an arbitrary bounded operator.

Lemma 1. [4] *The function*

$$u(t) = \int_0^t V(t)R(s)V^{-1}(s)f(s)ds - \int_t^\infty V(t)(I - R(s))V^{-1}(s)f(s)ds \tag{3}$$

is a solution of the equation (1) if the integrals in (3) exist.

Following conditions are introduced
(H1). $| V(t)R(s)V^{-1}(s)z | \leq M(t, s, z), t \geq s, z \in X$
(H2). $| V(t)(I - R(s))V^{-1}(s)z | \leq N(t, s, z), t < s, z \in X$

Definition 1. [4] *We call the equation (2) be a (M, N, R) - dichotomous if the conditions (H1), (H2) are fulfilled.*

Remark 1. [4] *Let $R(t) = P$,where $P : X \to X$ is a projector.*
For

$$M(t, s, z) = K_1 e^{-\int_s^t \delta_1(\tau)\mathrm{d}\tau} |z| \ (t \geq s, z \in X)$$

$$N(t, s, z) = K_2 e^{-\int_t^s \delta_1(\tau)\mathrm{d}\tau} |z| \ (s > t, z \in X)$$

where K_1, K_2 are positive constants and δ_1, δ_2 are continuous real-valued functions on $\mathbb{R}_+$, we obtain the exponential dichotomy of [8].

For $\delta_i(t) = const$ $(t \in \mathbb{R}_+,\ i = 1,2)$ we obtain the exponential dichotomy of [1],[2],[7]. In the case $\delta_i(t) = 0$ $(t \in \mathbb{R}_+,\ i = 1,2)$ we obtain the ordinary dichotomy of [1],[7].

For

$$M(t,s,z) = Kh(t)h^{-1}(s)|z| \ (t \geq s \geq 0,, z \in X)$$

$$N(t,s,z) = Kk(t)k^{-1}(s)|z| \ (0 \leq t \leq s, z \in X)$$

where K is a positive constant and $h,k : \mathbb{R}_+ \to \mathbb{R}_+$ are two continuous functions, we obtain the dichotomy of [9].

Let the right hand part of (H1) and (H2) has the form

$$\begin{cases} M(t,s,z) = \varphi_1(t)\varphi_2(s) \mid z \mid,\ (t \geq s),\ z \in X \\ N(t,s,z) = \psi_1(t)\psi_2(s) \mid z \mid,\ (t < s),\ z \in X \end{cases} \tag{4}$$

where $\varphi_1(t), \varphi_2(t), \psi_1(t), \psi_2(t)$ are positive scalar functions.

Definition 2. *We call the equation (2) be a (φ,ψ,R) - dichotomous if it is (M,N,R) - dichotomous and the conditions (H1), (H2) have the form (4).*

Let the equation (2) is a (φ,ψ,R) - dichotomous. We consider the following Banach spaces :

$$L_p(\varphi) = \{g(.) : \mathbb{R}_+ \to X : \sup_{t\in\mathbb{R}_+} \ \varphi_1(t)\int_0^t \varphi_2(s)|g(s)|^p ds < \infty\}$$

with the norm

$$|g|_{L_p(\varphi)} = \sup_{t\in\mathbb{R}_+} \ (\varphi_1(t)\int_0^t \varphi_2(s)|g(s)|^p ds)^{\frac{1}{p}},$$

$$L_p(\psi) = \{g(.) : \mathbb{R}_+ \to X : \sup_{t\in\mathbb{R}_+} \ \psi_1(t)\int_t^\infty \psi_2(s)|g(s)|^p ds < \infty\}$$

with the norm

$$|g|_{L_p(\psi)} = \sup_{t\in\mathbb{R}_+} \ (\psi_1(t)\int_t^\infty \psi_2(s)|g(s)|^p ds)^{\frac{1}{p}}$$

and

$$L_\infty = \{g(.) : \mathbb{R}_+ \to X : \sup_{t\in\mathbb{R}_+} |g(t)| < \infty\}$$

with the norm

$$|g|_{L_\infty} = \sup_{t\in\mathbb{R}_+} |g(t)|.$$

We introduce the following conditions:

$$\text{(H3)} \qquad \sup_{t\in\mathbb{R}_+} \varphi_1(t)\int_0^t \varphi_2(s)ds < M_1$$

$$\text{(H4)} \qquad \sup_{t\in\mathbb{R}_+} \psi_1(t)\int_t^\infty \psi_2(s)ds < M_2$$

3. Main results

Lemma 2. *Let the following conditions are fulfilled:*
1. The operator-function $A(t)$ is continuous for $t \in \mathbb{R}_+$.
2. Conditions (H3) and (H4) hold.
3. The linear differential equation (2) is (φ, ψ, R) - dichotomous.
Then for any function $f \in L_p(\varphi) \cap L_p(\psi)$ the linear nonhomogeneous equation (1) has a bounded solution $u(t)$ $(t \in \mathbb{R}_+)$ for which the following formula is valid

$$(5) \qquad u(t) = \int_0^t V(t)R(s)V^{-1}(s)f(s)ds - \int_t^\infty V(t)(I - R(s))V^{-1}(s)f(s)ds$$

Proof. The formula (5) follows from Lemma 1. We shall estimate the norm of the integrals in (5).

Let $q = \frac{p}{p-1}$. We use Holder's inequality. For the norm of the integrals we obtain

$$\begin{aligned} |\int_0^t V(t)R(s)V^{-1}(s)f(s)ds| &\le \int_0^t ||V(t)R(s)V^{-1}(s)||\ |f(s)|ds \le \\ &\le \int_0^t \varphi_1(t)\varphi_2(s)|f(s)|ds \le \\ &\le \{\int_0^t \varphi_1(t)\varphi_2(s)ds\}^{\frac{1}{q}} \int_0^t \varphi_1(t)\varphi_2(s)|f(s)|^p ds^{\frac{1}{p}} \le \\ &\le M_1{}^{\frac{1}{q}}|f|_{L_p(\varphi)} \end{aligned}$$

and

$$|\int_t^\infty V(t)(I-R(s))V^{-1}(s)f(s)ds| \le \int_t^\infty ||V(t)(I-R(s))V^{-1}(s)||\ |f(s)|ds \le$$
$$\le \int_t^\infty \psi_1(t)\psi_2(s)|f(s)|ds \le$$
$$\le \{\int_t^\infty \psi_1(t)\psi_2(s)ds\}^{\frac{1}{q}} \int_t^\infty \psi_1(t)\psi_2(s)|f(s)|^p ds^{\frac{1}{p}} \le$$
$$\le {M_2}^{\frac{1}{q}}|f|_{L_p(\psi)}$$

□

Definition 3. *The solution $u(t)$ $(t\in\mathbb{R}_+)$ of the linear nonhomogeneous equation (1) is said to be a $L_p(\varphi,\psi)$-solution, if $u\in L_p(\varphi)\cap L_p(\psi)\cap L_\infty$.*

For $\xi\in L_p(\varphi)\cap L_p(\psi)$ we introduce the norm

$$|\xi|_{L_p(\varphi)\cap L_p(\psi)} = max\{|\xi|_{L_p(\varphi)}, |\xi|_{L_p(\psi)}\}$$

Lemma 3. *Let the following conditions are fulfilled:*
1. The operator-function $A(t)$ is continuous for $t\in\mathbb{R}_+$.
2. Conditions (H3) and (H4) hold.
3. The linear differential equation (2) is (φ,ψ,R) - dichotomous.
Then the operator G, defined by the formula

$$Gf(t) = \int_0^t V(t)R(s)V^{-1}(s)f(s)ds - \int_t^\infty V(t)(I-R(s))V^{-1}(s)f(s)ds \tag{6}$$

maps $L_p(\varphi)\cap L_p(\psi)$ into $L_p(\varphi)\cap L_p(\psi)\cap L_\infty$ and the following estimates are valid

$$|Gf|_{L_\infty} \le ({M_1}^{\frac{1}{q}} + {M_2}^{\frac{1}{q}})\ |f|_{L_p(\varphi)\cap L_p(\psi)} \tag{7}$$

$$|Gf|_{L_p(\varphi)} \le 2^{\frac{1}{q}} M_1 (1 + (\frac{M_2}{M_1})^{\frac{1}{q}})\ |f|_{L_p(\varphi)\cap L_p(\psi)} \tag{8}$$

$$|Gf|_{L_p(\psi)} \le 2^{\frac{1}{q}} M_2 (1 + (\frac{M_1}{M_2})^{\frac{1}{q}})\ |f|_{L_p(\varphi)\cap L_p(\psi)} \tag{9}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $f \in L_p(\varphi) \cap L_p(\psi)$. From Lemma 2. follows the estimate

$$|Gf(t)| \le M_1^{\frac{1}{q}} |f|_{L_p(\varphi)} + M_2^{\frac{1}{q}} |f|_{L_p(\psi)} \qquad (t \in \mathbb{R}_+).$$

Then

$$|Gf|_{L_\infty} \le (M_1^{\frac{1}{q}} + M_2^{\frac{1}{q}})\ max\{|f|_{L_p(\varphi)}, |f|_{L_p(\psi)}\}.$$

Hence (7) holds.

Now we shall prove, that $Gf \in L_p(\varphi)$. From (7) and condition (H3) we obtain

$$\begin{aligned}
|Gf|_{L_p(\varphi)} &= \sup_{t\in\mathbb{R}_+} \left(\varphi_1(t) \int_0^t \varphi_2(s)|Gf(s)|^p ds\right)^{\frac{1}{p}} \le \\
&\le \sup_{t\in\mathbb{R}_+} \left(\varphi_1(t) \int_0^t \varphi_2(s)\left((M_1^{\frac{1}{q}} + M_2^{\frac{1}{q}})\right)^p |f|^p_{L_p(\varphi)\cap L_p(\psi)} ds\right)^{\frac{1}{p}} \le \\
&\le 2^{\frac{1}{q}} (M_1^{\frac{p}{q}} + M_2^{\frac{p}{q}})^{\frac{1}{p}} |f|_{L_p(\varphi)\cap L_p(\psi)} \sup_{t\in\mathbb{R}_+} \left(\varphi_1(t) \int_0^t \varphi_2(s) ds\right)^{\frac{1}{p}} \le \\
&\le 2^{\frac{1}{q}} M_1^{\frac{1}{p}} (M_1^{\frac{p}{q}} + M_2^{\frac{p}{q}})^{\frac{1}{p}} |f|_{L_p(\varphi)\cap L_p(\psi)} \le \\
&\le 2^{\frac{1}{q}} M_1^{\frac{1}{p}} (M_1^{\frac{1}{q}} + M_2^{\frac{1}{q}}) |f|_{L_p(\varphi)\cap L_p(\psi)} = \\
&= 2^{\frac{1}{q}} M_1 \left(1 + \left(\frac{M_2}{M_1}\right)^{\frac{1}{q}}\right) |f|_{L_p(\varphi)\cap L_p(\psi)}
\end{aligned}$$

Hence the inequality (8) holds.

The proof of the inequality (9) is analogously. □

Theorem 1. *Let the following conditions are fulfilled:*

1. The operator-function $A(t)$ is continuous for $t \in \mathbb{R}_+$.

2. Conditions (H3) and (H4) hold.

3. The linear differential equation (2) is (φ, ψ, R) - dichotomous.

Then for any function $f \in L_p(\varphi) \cap L_p(\psi)$ the linear nonhomogeneous equation (1) has a $L_p(\varphi, \psi)$-solution.

Proof. Let $f \in L_p(\varphi) \cap L_p(\psi)$. Then the equality (5) has the form

$$u(t) = Gf(t) \tag{10}$$

where G is the operator, defined by (6). From (10), Lemma 2., Lemma 3. and condition 2 of Theorem 1. follows, that the solution of the linear nonhomogeneous equation (1) lies in the space $L_p(\varphi) \cap L_p(\psi) \cap L_\infty$, i.e. the equation (1) has a $L_p(\varphi, \psi)$-solution. □

Remark 2. *The case, when the linear operator $A(t)$ is unbounded, the equation is impulsive and for the linear equation is not presumed dichotomous is considered in [3].*

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СЪЩЕСТВУВАНЕ НА $L_P(\varphi, \psi)$-РЕШЕНИЯ НА ЛИНЕЙНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ С ОБОБЩЕНА ДИХОТОМИЯ В БАНАХОВО ПРОСТРАНСТВО

Христо Кискинов, Степан Костадинов

Резюме. Използвано е обобщение на добре известните дихотомии за клас от хомогенни линейни диференциални уравнения в произволно Банахово пространство. С негова помощ са намерени достатъчни условия за съществуване на $L_p(\varphi, \psi)$-решения на съответното нехомогенно уравнение.