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CHEBYSHEVIAN COMPOSITIONS IN FOUR DIMENSIONAL SPACE WITH AN AFFINE CONNECTEDNESS WITHOUT A TORSION

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Abstract. Let A_4 be an affinely connected space without a torsion. Following [7], we define the affinors a_α^β and b_α^β, that define conjugate compositions $X \times \overline{X}_2$ and $Y \times \overline{Y}_2$ in A_4. We define a third composition $Z \times \overline{Z}_2$ with the help of the affinor $\widetilde{c}\,_\alpha^\beta = ic_\alpha^\beta$, $(i^2 = -1)$, where $c_\alpha^\beta = -a_\sigma^\beta b_\alpha^\sigma$. We have found a necessary and sufficient condition for any of the above composition to be a (ch-ch) composition. We have found the spaces A_4 that contain such compositions.We have shown that if the compositions $X \times \overline{X}_2$, $Y \times \overline{Y}_2$ and $Z \times \overline{Z}_2$ are of the kind $(ch - ch)$ then the space A_4 is affine.

Key words: affinely connected spaces, spaces of compositions, affinors of compositions
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1. Preliminary

Let A_N be a space with a symmetric affine connectedness without a torsion, defined by $\Gamma_{\alpha\beta}^\gamma$. Let us consider a composition $X_n \times X_m$ of two differentiable basic manifolds X_n and X_m $(n + m = N)$ in the space A_N. For every point of the space of compositions $A_N(X_n \times X_m)$ there are two positions of basic manifolds, which we denotes by $P(X_n)$ and $P(X_m)$.

The defining of a composition in the space A_N is equivalent to the defining of a field of an affinor a_α^β, that satisfies the condition [2], [3]

(1) $$a_\sigma^\beta\, a_\alpha^\sigma = \delta_\alpha^\beta.$$

The affinor a_α^β is called an affinor of the composition [2]. According to [3] and [5] the condition for integrability of the structure is

$$a_\beta^\sigma \nabla_{[\alpha}\ a_{\sigma]}^\nu - a_\alpha^\sigma \nabla_{[\beta}\ a_{\sigma]}^\nu = 0. \tag{2}$$

The projective affinors $\overset{n}{a}{}_\alpha^\sigma$ and $\overset{m}{a}{}_\alpha^\sigma$ [3], [4], defined by the equalities

$$\overset{n}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta + a_\alpha^\beta), \qquad \overset{m}{a}{}_\alpha^\beta = \frac{1}{2}(\delta_\alpha^\beta - a_\alpha^\beta)$$

satisfy the conditions

$$\overset{n}{a}{}_\alpha^\beta + \overset{m}{a}{}_\alpha^\beta = \delta_\alpha^\beta, \qquad \overset{n}{a}{}_\alpha^\beta - \overset{m}{a}{}_\alpha^\beta = a_\alpha^\beta.$$

For every vector $v^\alpha \in A_N(X_n \times X_m)$ we have

$$v^\alpha = \overset{n}{a}{}_\beta^\alpha\ v^\beta + \overset{m}{a}{}_\beta^\alpha\ v^\beta = \overset{n}{V}{}^\alpha + \overset{m}{V}{}^\alpha,$$

where $\overset{n}{V}{}^\alpha = \overset{n}{a}{}_\beta^\alpha\ v^\beta \in P(X_n)$ and $\overset{m}{V}{}^\alpha = \overset{m}{a}{}_\beta^\alpha\ v^\beta \in P(X_m)$ [4].

According to [3] the composition of the kind $(ch - ch)$, for which the positions $P(X_n)$ and $P(X_m)$ are parallelly translated along any line of X_m and X_n respectively, is characterized with the equality

$$\nabla_{[\alpha}\ a_{\beta]}^\sigma = 0. \tag{3}$$

2. Conjugate compositions in spaces A_4

Let A_4 be an space with affine connectedness without a torsion, defined by $\Gamma_{\alpha\beta}^\sigma$ $(\alpha, \beta, \sigma = 1,2,3,4)$. Let $\underset{1}{v}{}^\alpha, \underset{2}{v}{}^\alpha, \underset{3}{v}{}^\alpha, \underset{4}{v}{}^\alpha$ be independent vector fields in A_4. Following [7], we define the covectors $\overset{\sigma}{v}_\alpha$ by

$$\underset{\alpha}{v}{}^\beta\ \overset{\alpha}{v}_\sigma = \delta_\sigma^\beta \Longleftrightarrow \underset{\alpha}{v}{}^\beta\ \overset{\sigma}{v}_\beta = \delta_\alpha^\sigma. \tag{4}$$

According to [6], [7] we can define the affinors

$$a_\alpha^\beta = \underset{1}{v}{}^\beta \overset{1}{v}_\alpha + \underset{2}{v}{}^\beta \overset{2}{v}_\alpha - \underset{3}{v}{}^\beta \overset{3}{v}_\alpha - \underset{4}{v}{}^\beta \overset{4}{v}_\alpha, \tag{5}$$

that satisfy the equalities (1). The affinor (5) defines a composition $(X_n \times X_m)$ in A_4.

According to [7] the projective affinors are

$$\overset{1}{a}{}^{\beta}_{\alpha} = \underset{1}{v}{}^{\beta}\overset{1}{v}_{\alpha} + \underset{2}{v}{}^{\beta}\overset{2}{v}_{\alpha}, \quad \overset{2}{a}{}^{\beta}_{\alpha} = \underset{3}{v}{}^{\beta}\overset{3}{v}_{\alpha} + \underset{4}{v}{}^{\beta}\overset{4}{v}_{\alpha}. \tag{6}$$

Following [7], let us choose for a coordinate net the net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v}\right)$ and consider the vectors

$$\underset{1}{w}{}^{\alpha} = \underset{1}{v}{}^{\alpha} + \underset{3}{v}{}^{\alpha}, \qquad \underset{2}{w}{}^{\alpha} = \underset{2}{v}{}^{\alpha} + \underset{4}{v}{}^{\alpha}, \qquad \underset{3}{w}{}^{\alpha} = \underset{1}{v}{}^{\alpha} - \underset{3}{v}{}^{\alpha}, \qquad \underset{4}{w}{}^{\alpha} = \underset{2}{v}{}^{\alpha} - \underset{4}{v}{}^{\alpha}. \tag{7}$$

We define the covectors $\overset{\alpha}{w}_{\sigma}$ by the equalities

$$\underset{\alpha}{w}{}^{\nu}\ \overset{\alpha}{w}_{\sigma} = \delta^{\nu}_{\sigma} \ \leftrightarrow\ \underset{\alpha}{w}{}^{\sigma}\ \overset{\beta}{w}_{\sigma} = \delta^{\beta}_{\alpha}. \tag{8}$$

By (4) and (8) hold the equalities

$$\begin{aligned} \overset{1}{w}_{\alpha} &= \frac{1}{2}\left(\overset{1}{v}_{\alpha} + \overset{3}{v}_{\alpha}\right), & \overset{2}{w}_{\alpha} &= \frac{1}{2}\left(\overset{2}{v}_{\alpha} + \overset{4}{v}_{\alpha}\right), \\ \overset{3}{w}_{\alpha} &= \frac{1}{2}\left(\overset{1}{v}_{\alpha} - \overset{3}{v}_{\alpha}\right), & \overset{1}{w}_{\alpha} &= \frac{1}{2}\left(\overset{2}{v}_{\alpha} - \overset{4}{v}_{\alpha}\right). \end{aligned} \tag{9}$$

Let consider the affinor

$$b^{\beta}_{\alpha} = \underset{1}{w}{}^{\beta}\overset{1}{w}_{\alpha} + \underset{2}{w}{}^{\beta}\overset{2}{w}_{\alpha} - \underset{3}{w}{}^{\beta}\overset{3}{w}_{\alpha} - \underset{4}{w}{}^{\beta}\overset{4}{w}_{\alpha}, \tag{10}$$

which according to [7] satisfies the equality $b^{\beta}_{\alpha}b^{\alpha}_{\sigma} = \delta^{\beta}_{\sigma}$.

Therefore the affinor (10) defines a composition $Y_2 \times \overline{Y}_2$ in A_4. By $P(Y_2)$ and $P(\overline{Y}_2)$ we denote the positions of this composition. By (4), (9) and (10) we obtain

$$b^{\beta}_{\alpha} = \underset{1}{v}{}^{\beta}\overset{3}{v}_{\alpha} + \underset{3}{v}{}^{\beta}\overset{1}{v}_{\alpha} + \underset{2}{v}{}^{\beta}\overset{4}{v}_{\alpha} + \underset{4}{v}{}^{\beta}\overset{2}{v}_{\alpha}. \tag{11}$$

According to [7] the compositions $X_2 \times \overline{X}_2$ and $Y_2 \times \overline{Y}_2$ are conjugate. Following [7], let us consider the affinor

$$c^{\beta}_{\sigma} = -a^{\beta}_{\alpha}b^{\alpha}_{\sigma}, \tag{12}$$

that satisfies the equality $c^{\beta}_{\sigma}c^{\sigma}_{\alpha} = -\delta^{\beta}_{\alpha}$. By (4), (5), (11) and (12) we get that

$$c^{\beta}_{\alpha} = \underset{3}{v}{}^{\beta}\overset{1}{v}_{\alpha} - \underset{1}{v}{}^{\beta}\overset{3}{v}_{\alpha} + \underset{4}{v}{}^{\beta}\overset{2}{v}_{\alpha} - \underset{2}{v}{}^{\beta}\overset{4}{v}_{\alpha}. \tag{13}$$

The affinors a_α^β and b_α^β are defining a hyperbolic composition, but the affinor c_α^β is defining an elliptic composition. The eigenvalue of the matrix (c_α^β) are

(14) $$\underset{1}{z}^\alpha = \underset{1}{v}^\alpha + i\underset{3}{v}^\alpha, \quad \underset{2}{z}^\alpha = \underset{2}{v}^\alpha + i\underset{4}{v}^\alpha, \quad \underset{3}{z}^\alpha = \underset{1}{v}^\alpha - i\underset{3}{v}^\alpha, \quad \underset{4}{z}^\alpha = \underset{2}{v}^\alpha - i\underset{4}{v}^\alpha,$$

where $i^2 = -1$. The affinor $\widetilde{c}_\alpha^\beta = i\ c_\alpha^\beta$ defines a composition $Z_2 \times \overline{Z}_2$ in A_4

3. Chebyshevian compositions in A_4

According to [8] we have the following derivative equations

(15) $$\nabla_\sigma \underset{\alpha}{v}^\beta = \underset{\alpha}{\overset{\nu}{T}}{}_\sigma\ \underset{\nu}{v}^\beta, \qquad \nabla_\sigma \overset{\alpha}{v}_\beta = -\underset{\nu}{\overset{\alpha}{T}}{}_\sigma\ \overset{\nu}{v}_\beta .$$

Let us consider the composition $X_2 \times \overline{X}_2$ and let us accept:

$\alpha,\ \beta,\ \gamma,\ \sigma,\ \nu \in \{1,\ 2,\ 3,\ 4\}; \quad i,\ j,\ k,\ s \in \{1,\ 2\}; \quad \bar{i},\ \bar{j},\ \bar{k},\ \bar{s} \in \{3,\ 4\}.$

Theorem 1. *The composition $X_2 \times \overline{X}_2$ is of the kind $(ch - ch)$ iff the coefficients of the derivative equations satisfy the conditions*

(16) $$\underset{\bar{j}}{\overset{i}{T}}{}_\alpha\ \underset{k}{v}^\alpha = 0, \qquad \underset{j}{\overset{\bar{i}}{T}}{}_\alpha\ \underset{\bar{k}}{v}^\alpha = 0.$$

Proof. By the equalities (5) and (15) we have

(17) $$\begin{aligned} \nabla_\sigma a_\alpha^\beta = {} & \underset{1}{\overset{\nu}{T}}{}_\sigma\ \underset{\nu}{v}^\beta\ \overset{1}{v}_\alpha - \underset{\nu}{\overset{1}{T}}{}_\sigma\ \underset{1}{v}^\beta\ \overset{\nu}{v}_\alpha + \underset{2}{\overset{\nu}{T}}{}_\sigma\ \underset{\nu}{v}^\beta\ \overset{2}{v}_\alpha - \underset{\nu}{\overset{2}{T}}{}_\sigma\ \underset{2}{v}^\beta\ \overset{\nu}{v}_\alpha \\ & - \underset{3}{\overset{\nu}{T}}{}_\sigma\ \underset{\nu}{v}^\beta\ \overset{3}{v}_\alpha + \underset{\nu}{\overset{3}{T}}{}_\sigma\ \underset{3}{v}^\beta\ \overset{\nu}{v}_\alpha - \underset{4}{\overset{\nu}{T}}{}_\sigma\ \underset{\nu}{v}^\beta\ \overset{4}{v}_\alpha + \underset{\nu}{\overset{4}{T}}{}_\sigma\ \underset{4}{v}^\beta\ \overset{\nu}{v}_\alpha. \end{aligned}$$

It follows by (3) and (17) that the composition $X_2 \times \overline{X}_2$ is of the kind $(ch - ch)$ iff the equality

(18) $$\begin{aligned} & \underset{1}{\overset{\nu}{T}}{}_{[\sigma}\ \overset{1}{v}_{\alpha]}\ \underset{\nu}{v}^\beta - \underset{\nu}{\overset{1}{T}}{}_{[\sigma}\ \overset{\nu}{v}_{\alpha]}\ \underset{1}{v}^\beta + \underset{2}{\overset{\nu}{T}}{}_{[\sigma}\ \overset{2}{v}_{\alpha]}\ \underset{\nu}{v}^\beta - \underset{\nu}{\overset{2}{T}}{}_{[\sigma}\ \overset{\nu}{v}_{\alpha]}\ \underset{2}{v}^\beta \\ & - \underset{3}{\overset{\nu}{T}}{}_{[\sigma}\ \overset{3}{v}_{\alpha]}\ \underset{\nu}{v}^\beta + \underset{\nu}{\overset{3}{T}}{}_{[\sigma}\ \overset{\nu}{v}_{\alpha]}\ \underset{3}{v}^\beta - \underset{4}{\overset{\nu}{T}}{}_{[\sigma}\ \overset{4}{v}_{\alpha]}\ \underset{\nu}{v}^\beta + \underset{\nu}{\overset{4}{T}}{}_{[\sigma}\ \overset{\nu}{v}_{\alpha]}\ \underset{4}{v}^\beta = 0 \end{aligned}$$

holds. The equality (18) is equivalent to the following equalities

(19) $$\begin{aligned} & \underset{3}{\overset{1}{T}}{}_{[\sigma}\ \overset{3}{v}_{\alpha]} + \underset{4}{\overset{1}{T}}{}_{[\sigma}\ \overset{4}{v}_{\alpha]} = 0, \qquad \underset{3}{\overset{2}{T}}{}_{[\sigma}\ \overset{3}{v}_{\alpha]} + \underset{4}{\overset{2}{T}}{}_{[\sigma}\ \overset{4}{v}_{\alpha]} = 0, \\ & \underset{1}{\overset{3}{T}}{}_{[\sigma}\ \overset{1}{v}_{\alpha]} + \underset{2}{\overset{3}{T}}{}_{[\sigma}\ \overset{2}{v}_{\alpha]} = 0, \qquad \underset{1}{\overset{4}{T}}{}_{[\sigma}\ \overset{1}{v}_{\alpha]} + \underset{2}{\overset{4}{T}}{}_{[\sigma}\ \overset{2}{v}_{\alpha]} = 0, \end{aligned}$$

because the vectors $\underset{\alpha}{v}^{\beta}$ are independent. After a contraction of the equalities (19) with $\underset{\nu}{v}^{\sigma}$ and $\underset{\beta}{v}^{\alpha}$ and taking into account (4) we get (16).

Theorem 2. *If the net* $\left(\underset{1}{v},\ \underset{2}{v},\ \underset{3}{v},\ \underset{4}{v}\right)$ *is chosen as a coordinate one, then the composition* $X_2 \times \overline{X}_2$ *of the kind* $(ch - ch)$ *is characterized by the following equalities for the coefficients of the derivative equations*

$$\underset{j}{\overset{\bar{i}}{T}}{}_{\bar{k}} = 0, \qquad \underset{\bar{j}}{\overset{i}{T}}{}_{k} = 0 \tag{20}$$

and by the following equalities for the coefficients of connectedness

$$\Gamma^{\alpha}_{13} = \Gamma^{\alpha}_{14} = \Gamma^{\alpha}_{23} = \Gamma^{\alpha}_{24} = 0\ . \tag{21}$$

Proof. Let choose the net $\left(\underset{1}{v},\ \underset{2}{v},\ \underset{3}{v},\ \underset{4}{v}\right)$ for a coordinate. Then

$$\underset{1}{v}^{\alpha}(1,\,0,\,0,\,0), \quad \underset{2}{v}^{\alpha}(0,\,1,\,0,\,0), \quad \underset{3}{v}^{\alpha}(0,\,0,\,1,\,0), \quad \underset{4}{v}^{\alpha}(0,\,0,\,0,\,1). \tag{22}$$

By equality (15) we have $\partial_{\sigma}\underset{\alpha}{v}^{\beta} + \Gamma^{\beta}_{\sigma\nu}\,\underset{\alpha}{v}^{\nu} = \underset{\alpha}{\overset{\nu}{T}}{}_{\sigma}\,\underset{\nu}{v}^{\beta}$ and after using (22) we obtain

$$\Gamma^{\beta}_{\sigma\nu} = \underset{\alpha}{\overset{\nu}{T}}{}_{\sigma}. \tag{23}$$

Equalities (20) follow from (16) and (22). Then from (20) and (23) we get the equalities (21).

The equalities (21) are obtained in [3], when the coordinates are adaptive with the composition $X_2 \times \overline{X}_2$. This happens so, because the chosen coordinate net raise adaptive with the composition coordinates.

Theorem 3. *The composition* $Y_2 \times \overline{Y}_2$ *is of the kind* $(ch - ch)$ *iff the coefficients of the derivative equations satisfy the conditions*

$$\begin{aligned}
&\left(\underset{3}{\overset{1}{T}}{}_{\sigma} - \underset{1}{\overset{3}{T}}{}_{\sigma}\right)\underset{2}{v}^{\sigma} = \left(\underset{4}{\overset{1}{T}}{}_{\sigma} - \underset{2}{\overset{3}{T}}{}_{\sigma}\right)\underset{1}{v}^{\sigma}, \quad \left(\underset{3}{\overset{3}{T}}{}_{\sigma} - \underset{1}{\overset{1}{T}}{}_{\sigma}\right)\underset{1}{v}^{\sigma} = \left(\underset{1}{\overset{3}{T}}{}_{\sigma} - \underset{3}{\overset{1}{T}}{}_{\sigma}\right)\underset{3}{v}^{\sigma},\\
&\left(\underset{4}{\overset{3}{T}}{}_{\sigma} - \underset{2}{\overset{1}{T}}{}_{\sigma}\right)\underset{1}{v}^{\sigma} = \left(\underset{1}{\overset{3}{T}}{}_{\sigma} - \underset{3}{\overset{1}{T}}{}_{\sigma}\right)\underset{4}{v}^{\sigma}, \quad \left(\underset{3}{\overset{3}{T}}{}_{\sigma} - \underset{1}{\overset{1}{T}}{}_{\sigma}\right)\underset{2}{v}^{\sigma} = \left(\underset{2}{\overset{3}{T}}{}_{\sigma} - \underset{4}{\overset{1}{T}}{}_{\sigma}\right)\underset{3}{v}^{\sigma},\\
&\left(\underset{4}{\overset{1}{T}}{}_{\sigma} - \underset{2}{\overset{3}{T}}{}_{\sigma}\right)\underset{4}{v}^{\sigma} = \left(\underset{2}{\overset{1}{T}}{}_{\sigma} - \underset{2}{\overset{3}{T}}{}_{\sigma}\right)\underset{2}{v}^{\sigma}, \quad \left(\underset{1}{\overset{1}{T}}{}_{\sigma} - \underset{3}{\overset{3}{T}}{}_{\sigma}\right)\underset{4}{v}^{\sigma} = \left(\underset{2}{\overset{1}{T}}{}_{\sigma} - \underset{4}{\overset{3}{T}}{}_{\sigma}\right)\underset{3}{v}^{\sigma}.
\end{aligned} \tag{24}$$

Proof. Because of the equalities (11) and (15) we have

$$
\begin{aligned}
\nabla_\sigma b^\beta_\alpha = {} & \overset{\nu}{\underset{1}{T}}{}_\sigma \underset{\nu}{v}{}^\beta \overset{3}{v}{}_\alpha - \overset{3}{\underset{\nu}{T}}{}_\sigma \underset{1}{v}{}^\beta \overset{\nu}{v}{}_\alpha + \overset{\nu}{\underset{3}{T}}{}_\sigma \underset{\nu}{v}{}^\beta \overset{1}{v}{}_\alpha - \overset{1}{\underset{\nu}{T}}{}_\sigma \underset{3}{v}{}^\beta \overset{\nu}{v}{}_\alpha \\
& + \overset{\nu}{\underset{2}{T}}{}_\sigma \underset{\nu}{v}{}^\beta \overset{4}{v}{}_\alpha - \overset{4}{\underset{\nu}{T}}{}_\sigma \underset{2}{v}{}^\beta \overset{\nu}{v}{}_\alpha + \overset{\nu}{\underset{4}{T}}{}_\sigma \underset{\nu}{v}{}^\beta \overset{2}{v}{}_\alpha - \overset{2}{\underset{\nu}{T}}{}_\sigma \underset{4}{v}{}^\beta \overset{\nu}{v}{}_\alpha .
\end{aligned} \tag{25}
$$

From (3) and (25) it follows that the composition $Y_2 \times \overline{Y}_2$ is of the kind $(ch - ch)$ iff the equalities

$$
\begin{aligned}
& \overset{\nu}{\underset{1}{T}}{}_{[\sigma} \overset{3}{v}{}_{\alpha]} \underset{\nu}{v}{}^\beta - \overset{3}{\underset{\nu}{T}}{}_{[\sigma} \overset{\nu}{v}{}_{\alpha]} \underset{1}{v}{}^\beta + \overset{\nu}{\underset{3}{T}}{}_{[\sigma} \overset{1}{v}{}_{\alpha]} \underset{\nu}{v}{}^\beta - \overset{1}{\underset{\nu}{T}}{}_{[\sigma} \overset{\nu}{v}{}_{\alpha]} \underset{3}{v}{}^\beta \\
& + \overset{\nu}{\underset{2}{T}}{}_{[\sigma} \overset{4}{v}{}_{\alpha]} \underset{\nu}{v}{}^\beta - \overset{4}{\underset{\nu}{T}}{}_{[\sigma} \overset{\nu}{v}{}_{\alpha]} \underset{2}{v}{}^\beta + \overset{\nu}{\underset{4}{T}}{}_{[\sigma} \overset{2}{v}{}_{\alpha]} \underset{\nu}{v}{}^\beta - \overset{2}{\underset{\nu}{T}}{}_{[\sigma} \overset{\nu}{v}{}_{\alpha]} \underset{4}{v}{}^\beta = 0
\end{aligned} \tag{26}
$$

hold.

Taking into account that the vectors $\underset{\alpha}{v}{}^\beta$ are independent we find that equality (26) is equivalent to the equalities

$$
\begin{aligned}
& \overset{1}{\underset{1}{T}}{}_{[\sigma} \overset{3}{v}{}_{\alpha]} - \overset{3}{\underset{\nu}{T}}{}_{[\sigma} \overset{\nu}{v}{}_{\alpha]} + \overset{1}{\underset{3}{T}}{}_{[\sigma} \overset{1}{v}{}_{\alpha]} + \overset{1}{\underset{2}{T}}{}_{[\sigma} \overset{4}{v}{}_{\alpha]} + \overset{1}{\underset{4}{T}}{}_{[\sigma} \overset{2}{v}{}_{\alpha]} = 0 \\
& \overset{2}{\underset{1}{T}}{}_{[\sigma} \overset{3}{v}{}_{\alpha]} + \overset{2}{\underset{3}{T}}{}_{[\sigma} \overset{1}{v}{}_{\alpha]} + \overset{2}{\underset{2}{T}}{}_{[\sigma} \overset{4}{v}{}_{\alpha]} - \overset{4}{\underset{\nu}{T}}{}_{[\sigma} \overset{\nu}{v}{}_{\alpha]} + \overset{2}{\underset{4}{T}}{}_{[\sigma} \overset{2}{v}{}_{\alpha]} = 0 \\
& \overset{3}{\underset{1}{T}}{}_{[\sigma} \overset{3}{v}{}_{\alpha]} + \overset{3}{\underset{3}{T}}{}_{[\sigma} \overset{1}{v}{}_{\alpha]} - \overset{1}{\underset{\nu}{T}}{}_{[\sigma} \overset{\nu}{v}{}_{\alpha]} + \overset{3}{\underset{2}{T}}{}_{[\sigma} \overset{4}{v}{}_{\alpha]} + \overset{3}{\underset{4}{T}}{}_{[\sigma} \overset{2}{v}{}_{\alpha]} = 0 \\
& \overset{4}{\underset{1}{T}}{}_{[\sigma} \overset{3}{v}{}_{\alpha]} + \overset{4}{\underset{3}{T}}{}_{[\sigma} \overset{1}{v}{}_{\alpha]} + \overset{4}{\underset{2}{T}}{}_{[\sigma} \overset{4}{v}{}_{\alpha]} + \overset{4}{\underset{4}{T}}{}_{[\sigma} \overset{2}{v}{}_{\alpha]} - \overset{2}{\underset{\nu}{T}}{}_{[\sigma} \overset{\nu}{v}{}_{\alpha]} = 0.
\end{aligned} \tag{27}
$$

After a contraction of the equalities (27) with $\underset{\nu}{v}{}^\sigma$ and $\underset{\beta}{v}{}^\alpha$ and using (4) we get (24).

Theorem 4 *If the net* $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v}\right)$ *is chosen as a coordinate one then the composition* $Y_2 \times \overline{Y}_2$ *of the kind* $(ch - ch)$ *is characterized by the following equalities for the coefficients of the derivative equations*

$$
\overset{\alpha}{\underset{1}{T}}{}_1 = \overset{\alpha}{\underset{3}{T}}{}_3 , \qquad \overset{\alpha}{\underset{2}{T}}{}_2 = \overset{\alpha}{\underset{4}{T}}{}_4 , \qquad \overset{\alpha}{\underset{2}{T}}{}_1 = \overset{\alpha}{\underset{3}{T}}{}_4 , \qquad \overset{\alpha}{\underset{3}{T}}{}_2 = \overset{\alpha}{\underset{4}{T}}{}_1 \tag{28}
$$

and with the following equalities of the coefficients of the connectedness

$$\Gamma^{\alpha}_{11} = \Gamma^{\alpha}_{33}\,, \qquad \Gamma^{\alpha}_{22} = \Gamma^{\alpha}_{44}\,, \qquad \Gamma^{\alpha}_{12} = \Gamma^{\alpha}_{34}\,, \qquad \Gamma^{\alpha}_{23} = \Gamma^{\alpha}_{14}\,. \tag{29}$$

Proof. Let us choose the net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v}\right)$ for a coordinate net. The equalities in (28) follow from (22) and (24), and the equalities in (29) follow from (23) and (28).

Theorem 5 *The composition* $Z_2 \times \overline{Z}_2$ *is of the kind* $(ch - ch)$ *iff the coefficients of the derivative equations satisfy the conditions*

$$\begin{gathered}
\left(\overset{3}{\underset{1}{T}}_\sigma + \overset{1}{\underset{3}{T}}_\sigma\right)\underset{2}{v}^\sigma = \left(\overset{3}{\underset{2}{T}}_\sigma + \overset{1}{\underset{4}{T}}_\sigma\right)\underset{1}{v}^\sigma, \quad \left(\overset{1}{\underset{3}{T}}_\sigma + \overset{3}{\underset{1}{T}}_\sigma\right)\underset{3}{v}^\sigma = \left(\overset{3}{\underset{3}{T}}_\sigma - \overset{1}{\underset{1}{T}}_\sigma\right)\underset{1}{v}^\sigma,\\
\left(\overset{1}{\underset{3}{T}}_\sigma + \overset{3}{\underset{1}{T}}_\sigma\right)\underset{4}{v}^\sigma = \left(\overset{3}{\underset{4}{T}}_\sigma - \overset{1}{\underset{2}{T}}_\sigma\right)\underset{1}{v}^\sigma, \quad \left(\overset{3}{\underset{3}{T}}_\sigma - \overset{1}{\underset{1}{T}}_\sigma\right)\underset{2}{v}^\sigma = \left(\overset{3}{\underset{2}{T}}_\sigma + \overset{1}{\underset{4}{T}}_\sigma\right)\underset{3}{v}^\sigma,\\
\left(\overset{3}{\underset{2}{T}}_\sigma + \overset{1}{\underset{4}{T}}_\sigma\right)\underset{4}{v}^\sigma = \left(\overset{3}{\underset{4}{T}}_\sigma - \overset{1}{\underset{2}{T}}_\sigma\right)\underset{2}{v}^\sigma, \quad \left(\overset{3}{\underset{3}{T}}_\sigma - \overset{1}{\underset{1}{T}}_\sigma\right)\underset{4}{v}^\sigma = \left(\overset{3}{\underset{4}{T}}_\sigma - \overset{1}{\underset{2}{T}}_\sigma\right)\underset{3}{v}^\sigma.
\end{gathered} \tag{30}$$

Proof. By the equalities (13) and (15) we obtain

$$\begin{aligned}
\nabla_\sigma \widetilde{c}\,^{\beta}_{\alpha} = i\Big(&\overset{\nu}{\underset{3}{T}}_\sigma \underset{\nu}{v}^\beta \overset{1}{v}_\alpha - \overset{1}{\underset{\nu}{T}}_\sigma \underset{3}{v}^\beta \overset{\nu}{v}_\alpha - \overset{\nu}{\underset{1}{T}}_\sigma \underset{\nu}{v}^\beta \overset{3}{v}_\alpha + \overset{3}{\underset{\nu}{T}}_\sigma \underset{1}{v}^\beta \overset{\nu}{v}_\alpha\\
&+\overset{\nu}{\underset{4}{T}}_\sigma \underset{\nu}{v}^\beta \overset{2}{v}_\alpha - \overset{2}{\underset{\nu}{T}}_\sigma \underset{4}{v}^\beta \overset{\nu}{v}_\alpha - \overset{\nu}{\underset{2}{T}}_\sigma \underset{\nu}{v}^\beta \overset{4}{v}_\alpha + \overset{4}{\underset{\nu}{T}}_\sigma \underset{2}{v}^\beta \overset{\nu}{v}_\alpha\Big).
\end{aligned} \tag{31}$$

Therefore, by (3) and (31) it follows that the composition $Z \times \overline{Z}_2$ is of the kind $(ch - ch)$ iff the equality

$$\begin{aligned}
&\overset{\nu}{\underset{3}{T}}_{[\sigma} \overset{1}{v}_{\alpha]} \underset{\nu}{v}^\beta - \overset{1}{\underset{\nu}{T}}_{[\sigma} \overset{\nu}{v}_{\alpha]} \underset{3}{v}^\beta - \overset{\nu}{\underset{1}{T}}_{[\sigma} \overset{3}{v}_{\alpha]} \underset{\nu}{v}^\beta + \overset{3}{\underset{\nu}{T}}_{[\sigma} \overset{\nu}{v}_{\alpha]} \underset{1}{v}^\beta\\
\text{[6pt]}\quad &+ \overset{\nu}{\underset{4}{T}}_{[\sigma} \overset{2}{v}_{\alpha]} \underset{\nu}{v}^\beta - \overset{2}{\underset{\nu}{T}}_{[\sigma} \overset{\nu}{v}_{\alpha]} \underset{4}{v}^\beta - \overset{\nu}{\underset{2}{T}}_{[\sigma} \overset{4}{v}_{\alpha]} \underset{\nu}{v}^\beta + \overset{4}{\underset{\nu}{T}}_{[\sigma} \overset{\nu}{v}_{\alpha]} \underset{2}{v}^\beta = 0.
\end{aligned} \tag{32}$$

holds. Taking into account that the vectors $\underset{\alpha}{v}^\beta$ are independent, we can rewrite (32) in the following form

$$
(33)\qquad
\begin{aligned}
&\overset{1}{\underset{3}{T}}{}_{[\sigma}\overset{1}{v}{}_{\alpha]} - \overset{1}{\underset{1}{T}}{}_{[\sigma}\overset{3}{v}{}_{\alpha]} + \overset{3}{\underset{\nu}{T}}{}_{[\sigma}\overset{\nu}{v}{}_{\alpha]} + \overset{1}{\underset{4}{T}}{}_{[\sigma}\overset{2}{v}{}_{\alpha]} - \overset{1}{\underset{2}{T}}{}_{[\sigma}\overset{4}{v}{}_{\alpha]} = 0,\\
&\overset{2}{\underset{3}{T}}{}_{[\sigma}\overset{1}{v}{}_{\alpha]} - \overset{2}{\underset{1}{T}}{}_{[\sigma}\overset{3}{v}{}_{\alpha]} + \overset{2}{\underset{4}{T}}{}_{[\sigma}\overset{2}{v}{}_{\alpha]} - \overset{2}{\underset{2}{T}}{}_{[\sigma}\overset{4}{v}{}_{\alpha]} + \overset{4}{\underset{\nu}{T}}{}_{[\sigma}\overset{\nu}{v}{}_{\alpha]} = 0,\\
&\overset{3}{\underset{3}{T}}{}_{[\sigma}\overset{1}{v}{}_{\alpha]} - \overset{1}{\underset{\nu}{T}}{}_{[\sigma}\overset{\nu}{v}{}_{\alpha]} - \overset{3}{\underset{1}{T}}{}_{[\sigma}\overset{3}{v}{}_{\alpha]} + \overset{3}{\underset{4}{T}}{}_{[\sigma}\overset{2}{v}{}_{\alpha]} - \overset{3}{\underset{2}{T}}{}_{[\sigma}\overset{4}{v}{}_{\alpha]} = 0,\\
&\overset{4}{\underset{3}{T}}{}_{[\sigma}\overset{1}{v}{}_{\alpha]} - \overset{4}{\underset{1}{T}}{}_{[\sigma}\overset{3}{v}{}_{\alpha]} + \overset{4}{\underset{4}{T}}{}_{[\sigma}\overset{2}{v}{}_{\alpha]} - \overset{2}{\underset{\nu}{T}}{}_{[\sigma}\overset{\nu}{v}{}_{\alpha]} - \overset{4}{\underset{2}{T}}{}_{[\sigma}\overset{4}{v}{}_{\alpha]} = 0,
\end{aligned}
$$

and after a contraction of the above equalities with $\underset{\nu}{v}^{\sigma}$ and $\underset{\beta}{v}^{\alpha}$ and using (4) we get (30).

Theorem 6. *If the net* $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v}\right)$ *is chosen as a coordinate one then the composition* $Z_2 \times \overline{Z}_2$ *from the kind* $(ch-ch)$ *is characterized by the following equalities for the coefficients of the derivative equations:*

$$
(34)\qquad \overset{\alpha}{\underset{1}{T}}{}_1 = -\overset{\alpha}{\underset{3}{T}}{}_3\,, \quad \alpha 2 = -\overset{\alpha}{\underset{4}{T}}{}_4\,, \quad \overset{\alpha}{\underset{2}{T}}{}_1 = -\overset{\alpha}{\underset{3}{T}}{}_4\,, \quad \overset{\alpha}{\underset{3}{T}}{}_2 = \overset{\alpha}{\underset{4}{T}}{}_1
$$

and with the following equalities of the coefficients of the connectedness:

$$
(35)\qquad \Gamma^{\alpha}_{11} = -\Gamma^{\alpha}_{33}\,, \quad \Gamma^{\alpha}_{22} = -\Gamma^{\alpha}_{44}\,, \quad \Gamma^{\alpha}_{12} = -\Gamma^{\alpha}_{34}\,, \quad \Gamma^{\alpha}_{23} = \Gamma^{\alpha}_{14}\,.
$$

Proof. Let choose the net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v}\right)$ for a coordinate net. Then the equalities (34) follow by (22) and (30), and the equalities (35) follow by (23) and (34).

Corollary 1. *If the conjugate compositions* $X_2 \times \overline{X}_2$ *and* $Y_2 \times \overline{Y}_2$ *are simultaneously of the kind* $(ch-ch)$ *then in the parameters of the coordinate net* $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v}\right)$ *the coefficients of the connectedness* $\Gamma^{\sigma}_{\alpha\beta}$ *satisfy the conditions*

$$
(36)\qquad
\begin{gathered}
\Gamma^{\alpha}_{11} = \Gamma^{\alpha}_{33}\,, \quad \Gamma^{\alpha}_{22} = \Gamma^{\alpha}_{44}\,, \quad \Gamma^{\alpha}_{12} = \Gamma^{\alpha}_{34}\,,\\
\Gamma^{\alpha}_{13} = \Gamma^{\alpha}_{14} = \Gamma^{\alpha}_{23} = \Gamma^{\alpha}_{24} = 0\ .
\end{gathered}
$$

Corollary 2. *If the compositions $X_2 \times \overline{X}_2$ and $Z_2 \times \overline{Z}_2$ are simultaneously of the kind $(ch-ch)$, then in the parameters of the coordinate net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v}\right)$ the coefficients of the connectedness $\Gamma^{\sigma}_{\alpha\beta}$ satisfy the conditions*

$$\Gamma^{\alpha}_{11} = -\Gamma^{\alpha}_{33}, \quad \Gamma^{\alpha}_{22} = -\Gamma^{\alpha}_{44}, \quad \Gamma^{\alpha}_{12} = -\Gamma^{\alpha}_{34}, \tag{37}$$
$$\Gamma^{\alpha}_{13} = \Gamma^{\alpha}_{14} = \Gamma^{\alpha}_{23} = \Gamma^{\alpha}_{24} = 0 \ .$$

The conditions (36) and (37) can be written in the form

$$\Gamma^{\alpha}_{11} = \varepsilon\,\Gamma^{\alpha}_{33}, \qquad \Gamma^{\alpha}_{22} = \varepsilon\,\Gamma^{\alpha}_{44}, \qquad \Gamma^{\alpha}_{12} = \varepsilon\Gamma^{\alpha}_{34}, \tag{38}$$
$$\Gamma^{\alpha}_{13} = \Gamma^{\alpha}_{14} = \Gamma^{\alpha}_{23} = \Gamma^{\alpha}_{24} = 0, \ \text{ where } \varepsilon = \pm 1.$$

Using [1] and (38), we find the following representation for the components of the tensor of the curvature

$$R_{123\cdot}{}^{\alpha} = R_{124\cdot}{}^{\alpha} = R_{341\cdot}{}^{\alpha} = R_{342\cdot}{}^{\alpha} = 0; \tag{39}$$

$$R_{3ij\cdot}{}^{\alpha} = R_{3ji\cdot}{}^{\alpha} = \partial_3\Gamma^{\alpha}_{ij} + \varepsilon\left(\Gamma^{\alpha}_{11}\Gamma^{3}_{ij} + \Gamma^{\alpha}_{12}\Gamma^{4}_{ij}\right),$$

$$R_{4ij\cdot}{}^{\alpha} = R_{4ji\cdot}{}^{\alpha} = \partial_4\Gamma^{\alpha}_{ij} + \varepsilon\left(\Gamma^{\alpha}_{22}\Gamma^{4}_{ij} + \Gamma^{\alpha}_{12}\Gamma^{3}_{ij}\right),$$

$$R_{i33\cdot}{}^{\alpha} = \varepsilon\left(\partial_i\Gamma^{\alpha}_{11} + \Gamma^{\alpha}_{ii}\Gamma^{i}_{11} + \Gamma^{\alpha}_{12}\Gamma^{j}_{ii}\right), \quad i \neq j,$$

$$R_{i44\cdot}{}^{\alpha} = \varepsilon\left(\partial_i\Gamma^{\alpha}_{22} + \Gamma^{\alpha}_{ii}\Gamma^{i}_{22} + \Gamma^{\alpha}_{12}\Gamma^{j}_{22}\right), \quad i \neq j, \tag{40}$$

$$R_{134\cdot}{}^{\alpha} = R_{143\cdot}{}^{\alpha} = \varepsilon\left(\partial_i\Gamma^{\alpha}_{12} + \Gamma^{\alpha}_{ii}\Gamma^{i}_{12}\right) + \Gamma^{\alpha}_{12}\Gamma^{j}_{12}, \quad i \neq j,$$

$$R_{ijj\cdot}{}^{\alpha} = \partial_i\Gamma^{\alpha}_{jj} - \partial_j\Gamma^{\alpha}_{12} + \Gamma^{\alpha}_{ii}\Gamma^{i}_{jj} + \Gamma^{\alpha}_{12}\Gamma^{j}_{jj} - R\Gamma^{\alpha}_{12}\Gamma^{i}_{12} - \Gamma^{\alpha}_{jj}\Gamma^{j}_{12}, \quad i \neq j,$$

$$R_{344\cdot}{}^{\alpha} = \varepsilon(\partial_3\Gamma^{\alpha}_{22} - \partial_4\Gamma^{\alpha}_{12}) + \Gamma^{\alpha}_{11}\Gamma^{3}_{22} + \Gamma^{\alpha}_{12}\Gamma^{4}_{22} - \Gamma^{\alpha}_{12}\Gamma^{3}_{12} - \Gamma^{\alpha}_{22}\Gamma^{4}_{12},$$

$$R_{433\cdot}{}^{\alpha} = \varepsilon\left(\partial_4\Gamma^{\alpha}_{11} - \partial_3\Gamma^{\alpha}_{12}\right) + \Gamma^{\alpha}_{22}\Gamma^{4}_{11} + \Gamma^{\alpha}_{12}\Gamma^{3}_{11} - \Gamma^{\alpha}_{12}\Gamma^{4}_{12} - \Gamma^{\alpha}_{11}\Gamma^{3}_{12}.$$

Hence,

$$R_{\bar{\imath}[ij]\cdot}{}^{\alpha} = 0, \qquad R_{i[\bar{\imath}\;\bar{\jmath}]\cdot}{}^{\alpha} = 0.$$

Corollary 3. *If the compositions $Y_2 \times \overline{Y}_2$ and $Z_2 \times \overline{Z}_2$ are simultaneously of the kind $(ch - ch)$ then in the parameters of the coordinate net $\left(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \underset{4}{v}\right)$ the coefficients of the connectedness $\Gamma^{\sigma}_{\alpha\beta}$ satisfy the conditions*

$$\Gamma^{i}_{\bar{i}\ \bar{j}} = 0\,, \qquad \Gamma^{\bar{i}}_{ij} = 0 \tag{41}$$

and

$$\Gamma^{k}_{ij} = 0\,, \qquad \Gamma^{\bar{k}}_{\bar{i}\ \bar{j}} = 0\,, \qquad \Gamma^{\alpha}_{23} = \Gamma^{\alpha}_{14}. \tag{42}$$

In this case, using [1], (41), (42) we find $R_{ijk\cdot}^{\ \ \ \ \alpha} = 0$ and $R_{\bar{i}\ \bar{j}\ \bar{k}\cdot}^{\ \ \ \ \ \ \alpha} = 0$. From [3] and (41) it follows that the composition $X_2 \times \overline{X}_2$ is of the kind $(g - g)$.

Corollary 4. *If the compositions $X_2 \times \overline{X}_2$, $Y_2 \times \overline{Y}_2$ and $Z_2 \times \overline{Z}_2$ are simultaneously of the kind $(ch - ch)$, then the space A_N is an affine space.*

Proof. If the compositions $X_2 \times \overline{X}_2$, $Y_2 \times \overline{Y}_2$ and $Z_2 \times \overline{Z}_2$ are simultaneously of the kind $(ch - ch)$, then by (36), (37), (41), (42) we obtain that $\Gamma^{\sigma}_{\alpha\beta} = 0$, and consequently A_N is an affine space.

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ЧЕБИШЕВИ КОМПОЗИЦИИ В ЧЕТИРИМЕРНО ПРОСТРАНСТВО С АФИННА СВЪРЗАНОСТ БЕЗ ТОРЗИЯ

Муса Айети

Резюме. Нека е дадено пространство с афинна свързаност A_4 без торзия. Следвайки [7] се определят афинори a_α^β и b_α^β, които определят спрегнати композиции $X \times \overline{X}_2$ и $Y \times \overline{Y}_2$ в A_4. С помощта на афинора $\widetilde{c}\,_\alpha^\beta = ic_\alpha^\beta$ ($i^2 = -1$), където $c_\alpha^\beta = -a_\sigma^\beta b_\alpha^\sigma$, се определя трета композиция $Z \times \overline{Z}_2$. Намерени са необходими и достатъчни условия при които всяка от тези композиции е от вида $(ch - ch)$. Определени са пространствата A_4, които съдържат такива композиции. Доказано е, че ако композициите $X \times \overline{X}_2$, $Y \times \overline{Y}_2$ и $Z \times \overline{Z}_2$ са от вида $(ch - ch)$, то пространството A_4 е афинно.